Conformal geometry of curves and surfaces (Lausanne, avril 2001)

R. Langevin

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(very preliminary version)

It contains long quotations of [La-O'H]; it uses also quotations of a text of mine published in Brazil [La].

Contents

1 introduction

For a metric study of curves and surfaces in the euclidean space $\mathbb{R}^3$, we use distances, isometries. The integral geometry uses planes and lines.

If we want to keep track only of angular informations, switching then to conformal geometry, we can as well deal with $\mathbb{R}^3$ or S^3, as the stereographic projection from S^3 to $\mathbb{R}^3$ is a conformal map. The group $\mathcal{M}$ of conformal transformations of $\mathbb{R}^3$ is larger than the group of isometries (it is of dimension 10). An element $g \in \mathcal{M}$ distort the "shape" of a curve or of a surface, but not too much. Spheres are mapped to spheres (or sometimes planes), circles to circles (or sometimes lines).

One of the goals of this serie of lectures is to give a conformal statement akin to the Fary-Fenchel-Milnor statement ([Far] [Fe2], [Mil1]) :

Theorem 1.0.1 *If a curve K is a non trivial knot, then it total curvature $\int_K |k(x(s))|ds$, $|k(x)|$ the curvature of K at a point x, s an arc-length parameter, satisfies:*

$$\int_K |k(x(s))|ds > 4\pi$$

Before getting to that, we need to study in detail spheres of $\mathbb{R}^3$ or S^3.

2 The space of spheres in various dimensions and meaningful subspaces.

2.1 The space of spheres

Let L be the Lorentz quadratic form defined on the (n+2)-dimensional space E by:

$$L(x_1, x_2, ..., x_{n+2}) = (x_1)^2 + (x_2)^2 + ... + (x_{n+1})^2 - (x_{n+2})^2$$

We will use only Lorentz spaces of dimensions 3, 4 and 5, corresponding to $0-$spheres in S^1, circles in S^2 and spheres in S^3.

We will call *light cone* the isotropic cone of L. We note also L the associated bilinear form, and call L-orthogonal vectors a, b such that $L(a, b) = 0$.

Let us prove that the set $\mathcal{S}$ of oriented $(n-1)$-spheres of the sphere S^n admits a bijection with the set of points of the quadric Λ of equation $L = 1$.

The points at ∞ of the light cone form two n- dimensional spheres. We retain the "positive" one S^n_∞, that is the points at ∞ of the light cone in the upper half space $x_{n+2} > 0$.

Definition 2.1.1 *A vector of E is called space-like if $L(v) > 0$. It is called time-like if $L(v) < 0$. A line is called space-like (resp time-like) if it contains a space-like (resp time-like) vector.*

Any space-like line L intersects the quadric Λ in exactly two points. The hyperplane L-orthogonal to a space-like line L (notation $L^\perp$), intersects the light cone transversly and therefore intersects its positive sphere at ∞ in a sphere Σ_L. This gives a correspondance between the set $\mathcal{S}$ of oriented (n-1)-dimensional spheres of S^n and the quadric Λ.

2.2 Pencils (of circles, of spheres).

Linear family of circles of the plane, that is the equation of which are the linear combination of the equations of two different circles form a *pencil*. Among such a linear family, one circle is a line, two or exceptionally one are points. If the circles of a pencil have two common points we say it is *with base points* if the circles are all disjoints we say it has *limit* or *Poncelet* points. The last case is the tangential pencil, where all the circles are tangent at the same point.

Figure 1: Pencils of circles

Naturally, the notion extends to circles of the sphere S^2. One can then observe that the circles are intersection of the sphere with affine planes. The pencils are then the circles intersection of the sphere with a *pencil of planes* family of planes the equations of which are the linear combinations of the equations of two different planes. When the planes of the pencil are not parallels, they all intersect in a line Δ, the *axis* of the pencil. When the planes are parallel, ore when the axis Δ does not intersect the sphere, the pencil of circles has limit points. When the axis intersect the sphere it has base points: the points of the intersection $\Delta \cap S^2$; the left case is when Δ is tangent to S^2, the circles of the pencil are tangent at $\Delta \cap S^2$.

Figure 2: Pencils in the sphere

We can proceed in higher dimensions, and in particular consider pencils of spheres in $\mathbb{R}^3$ or S^3. Let us see how pencils can be described using the quadric Λ.

Dimension 2 planes through the origin intersect the quadric $\Lambda \subset \mathbb{R}^5$ in curves; those curves are the "geodesics" of Λ, in the sense that the energy $\int_a^b \mathcal{L}(c'(t))$ of any arc $c : [a,b] \to$ *geodesic* of the curve is extremal (but not minimal). Notice that the energy of a path can be negative.

The energy of any segment of light-ray is zero. Using segments of light-rays one can approximate any path by a path of zero energy. This explains why there is no hope to find a path of minimal positive length joining two points of Λ.

When all the vectors of the plane P are space-like the intersection Γ with the quadric Λ is connected and compact. The tangent vectors to this intersection are space-like. The corresponding set of spheres is a *pencil* of spheres with a common circle $\Gamma = P^\perp \cap S_\infty$. A segment of Γ is of positive energy. One can therefore define its length: $\int_a^b \sqrt{L}(\gamma'(t))dt$, where γ is a parametrisation of a segment of Γ. We will show later that the length of the segment of Γ joining $\gamma(t)$ and $\gamma(t_0)$ is just the angle between the corresponding spheres Σ_t and Σ_{t_0}.

When the plane is time-like the intersection is formed of two non-compact curves. The pencil of spheres has limit (or Poncelet) points which are $P \cap S_\infty$.

When the plane is tangent to the light cone, the intersection is made of two parallel lines, and the pencil is made of spheres all tangent at the point $P \cap S_\infty$.

2.3 The Lorentz group $\mathcal{G}$

Let $\mathcal{G}$ be the group of linear isomorphisms of R^{n+2} leaving L invariant. It acts transitively on the lines of the light-cone, on the space-like lines (lines conaining a pace-like vector) and on the time-like lines (lines containing a time-like vector). It acts therefore transitively on Λ (the quadrci of equation $L = +1$) and on each sheet of the quadric of equation $L = -1$.

2.4 Metrics and pseudo-metrics

As the tangent space at a point x to an euclidean sphere centered at the origin is orthogonal the vector x, the tangent space at a point x to a level of the quadratic form L is L-orthogonal to the vector x. The equation of the tangent space at x to the quadric $L = c$ is given by the linear part in u of the right hand of the equation:

$$L(x+u, x+u) = c$$

providing the equation $L(x, u) = 0$ of the tangent space.

The quadric $\mathbb{H} = \{L = -1\}$, endowed with the restriction to each tangent space of the quadratic form L is a riemannian manifold.

The quadric Λ of equation $L = +1$ does not inherit a riemannian metric: the restriction of L to a tangent space $T_\sigma\Lambda$ is a quadratic form of type (3,1). This tangent space contains therefore a light-cone of dimension 3. Let us prove that this light-cone is in fact the intersection $\Lambda \cap T_{sigma}\Lambda$.

We have to solve the system:

$L(\sigma + v, \sigma + v) = +1$

$L(\sigma, v) = 0$

As $L(\sigma + v, \sigma + v) = L(\sigma, \sigma) + 2L(\sigma, v) + L(v, v)$, we deduce from the two equation that for $v \in T_\sigma\Lambda$, $L(v, v) = 0$, which is the equation of the light-cone in $T_\sigma\Lambda$.

Let us now prove that the upper sheet of the quadric $L = -1$ is a model of $\mathbb{H}$, the hyperbolic space. We have observed that the conformal group $\mathcal{G}$ acts transitively on the time-like rays. Let us prove it acts transitively on the pairs: $(x, U); x \in \mathbb{H}; U an\, othonormal\, frame\, in\, T_x\mathbb{H}$. It is enough to prove it for two different frames at the same point say $0, 0, ...0, 1$. The tangent space to $\mathbb{H}$ is then horizontal (parallel to the (x_1, x_2, x_3, x_4)-space. The linear tranformations leaving the x_5-axis fixed, and the restriction of which to the horizontal space are isometries belong to the conformal group $\mathcal{G}$. They act transitively on unitary frames in the origin proving the transitivity assertion.

The restriction of $\mathcal{G}$ to $\mathbb{H}$ is the group of isometries of the hyperbolic space $\mathbb{H}$.

As it acts transitively on pairs (x, U), the curvature of $\mathbb{H}$ has to be constant. It cannot be positive, otherwise $\mathbb{H}$ should be compact (by Myers' theorem, see [dCa2]).

To chose a point z in $\mathbb{H}$ determines a metric on the sphere S^n. This metric is the projection on S^n, sphere at infinity of $\mathbb{H}$, of the metric on $T_z(\mathbb{H})$ using the geodesic rays of origin z. It can be seen also slicing the light-cone by the affine 4-space of equation $L(x, z) = -1$ and taking the metric on the sphere we get induced by the euclidean metric on the 4-dimensional affine space restriction of L to this 4-dimensional affine space.

Different choices of the point z determine conformally equivalent metrics on the sphere S^n.

So no metric on the sphere is invariant by $\mathcal{G}$. In fact the sphere does not even admit a measure invariant by the conformal group $\mathcal{G}$.

2.5 Pathes in the space of spheres

Proposition 2.5.1 *Let c be a path in Λ. If, in any point c(t) of the path, the tangent vector v(t) satisfies:*

$$L(v) > 0$$

(space-like curve) the corresponding family of spheres Σ_t admits an envelope; if , on the contrary , in any point of the path,

$$L(v) < 0$$

(time-like curve) The spheres Σ_t are nested.

Proof: As c(t) belongs to Λ, that is satisfies $L(c(t)) = 1$, one has $L(c(t), v) = 0$. The condition for a 1-parameter family of spheres to admit an envelope is that the L-orthogonal space P to the plane generated by $c(t)$ and $v(t)$ intersects the light cone. The intersection of P with the sphere at infinity is the a circle $\Gamma \in \Sigma_t$ that is called *characteristic circle* of the envelope. As $c(t)$ and $v(t)$ are L-orthogonal, it is equivalent to $L(v(t)) > 0$, or in other terms if the curve is space-like, the corresponding family of sphere admits an envelope.

On the contrary, when P does not intersect the light-cone, nearby spheres are disjoint, and therefore nested. □

The length of a space-like curve is then the total rotation of the family of spheres along its axis.

3 Measure on the set of spheres

Fortunately the sets of spheres of S^n do. In particular, Λ is endowed with a measure ω invariant by $\mathcal{G}$. The Lebesgue measure $|dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5|$ is invariant by the action of $\mathcal{G}$. The quadric Λ is a subset of $\mathbb{R}^5$. It inherits from $\mathbb{R}^5$ a measure invariant by the action of the group $\mathcal{G}$, defined on each tangent space at a point $x \in \Lambda$ by: $\omega(x) = |\iota_x(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5)|$.
Remark: Let $(v_1, v_2, ..., v_{n+1})$ be $n+1$ vectors of $T_x\Lambda$. The volume of the parallelepiped constructed on these vectors is

$$|det(x, v_1, v_2, ..., v_{n+1})| = \sqrt{(} - det(\mathcal{L}(v_i, v_j))$$

Not only Λ is unbounded, but its total measure of is infinite. One can explicitely compute, given a metric of volume 4π conformally equivalent to the standard one on S^3, the volume V_r of the set of spheres of geodesic radius larger than r:

$$V_r = 2Sh(ArcTh(1 - cos(r)))$$

(no garantee)

Worse, the measure of the set of spheres intersecting a given arc, whatever its size, is infinite.

Remark: This measure can also be seen as a measure on the set of totally geodesic hyperplanes of the hyperbolic space $\mathbb{H}$ which is invariant by the action of the hyperbolic isometries [**?**].

Let us now project the sphere S^n stereographically on an affine space $\mathbb{R}^n$ with euclidean coordinates $(x_1, x_2, ..., x_n$. There, a sphere Σ is given by its center $x_1, x_2, ..., x_n$ and its radius r. The measure ω is expressed by:

$$\omega = |[1/(r^{n+1})]dx_1 \wedge dx_2 \wedge ... \wedge dx_n \wedge dr|$$

3.1 Spheres of dimension 0

We will start with spheres of dimension 0 in S^1, and study their positions with respect to a "torus" T made of 4 distinct points.

Notice that, if the conformal image of four points is $(-R, -1, 1, R)$, the cross ratio is R. We will use this remark later, to give an interpretation of the modulus of a ring or of the zone between two non-intersecting spheres.

An oriented sphere σ disjoint from T bounds an interval I. We will say that σ is non-trivial if I contains two points of T. Informally we may say that the small enough spheres will all be trivial.

Figure 3: trivial (a) and (b), and a non-trivial (c) 0-sphere

Proposition 3.1.1 *The torus T which minimises the measure of the set of non trivial spheres is the torus made of the four vertices of a square (or its image by the conformal group of the circle).*

The domain Z of $\mathcal{S}$ formed by the non trivial spheres is bounded by segments of light rays formed by the spheres containing one of the four points of T.brace

Figure 4: the set of non-trivial 0-spheres

As the only conformal invariant of a set of four points is their cross-ratio, The measure $m(Z)$ is a function of this cross-ratio.

Proof: The proof of the proposition is a computation. Using the stereographic projection of S^1 on $\mathbb{R}$ the measure on $\mathcal{S} = \{pairs\ of\ points\ of\ S^1\}$ is $\frac{2}{(y-x)^2}|dx \wedge dy|$. Without loss of generality, we can suppose that the four points of the "torus" T are $\{\infty\,,\,0\,,\,1\,,\,z\}$. We will make the computation of the measure of "half" of the points of Z, that is $\{\infty < x < 0; 1 < y < z\}$, supposing $z > 1$. The other cases are analogous. One has $m(Z_z) = m(\{\infty < x < 0; 1 < y < z\}) + m(\{0 < x < 1; z < y < \infty\})$. One has :

$$m(\{\infty < x < 0; 1 < y < z\}) = \int_{1<y<z} \int_{-\infty<x<0} \frac{2}{(y-x)^2}|dx \wedge dy| = 2log(z).$$

In the same way we compute:

$$m(\{0 < x < 1; z < y < \infty\}) = 2log(z) - 2log(z-1)$$

The minimum of $m(Z_z)$ is achieved for $z = 2$, $m(Z_2) = 4log(2)$. This correspond to the "square" torus $T = \{e^{ik\pi/2}\}$.

□

3.2 Non-trivial spheres in $\mathcal{S}$, tangent spheres and twice tangent spheres.

Observe first that a circle is intersected by a sphere in at most two points if the sphere does not contain the circle. Generically, a sphere can intersect a knot K in 0, 2 or any even number of points.

We remark that the volume of the set of spheres that intersect a given knot K is infinite for any knot K. In order to have finite valued functional, we should get rid of the contribution of the set of "small spheres", which intersect K in at most 2 points.

Definition 3.2.1 An oriented 2-sphere Σ is called a *non-trivial sphere* **(in the wider sense)?** for a knot K if Σ intersects K in at least 4 points, where the number of the intersection points in $K \cap \Sigma$ is counted with multiplicity. An oriented 2-sphere Σ with $\Sigma = \partial D^3$ is called a *non-trivial sphere in the strict sense* for a knot K if $\mathrm{Int} D^3 \cap K$ contains at least 2 connected components of K.

Let $\overline{NT}(K)$ (or respectively, $NT(K)$) denote the set of the non-trivial spheres (or respectively, the non-trivial spheres in the strict sense).

If we fix a metric on the ambient sphere S^3, then for a constant α depending on the curve K, spheres of **geodesical (?)** radius smaller than α can intersect K in at most two points. This implies that the set $\overline{NT}(K)$ of non-trivial spheres is bounded in $\mathcal{S}$. It has therefore a finite volume.

Definition 3.2.2 Let $mnts_1(K)$ denote the volume of the set $\overline{NT}(K)$ of non-trivial spheres for a knot K.

The subset $\overline{NT}(K) \subset \mathcal{S}$ is a 4-dimensional region whose boundary consists of spheres which are tangent to the knot K. This boundary is not smooth. In particular, it has "corners" which are the intersections of two folds of the set of the tangent spheres. Therefore the corner of $\partial \overline{NT}(K)$ consists of spheres which are tangent to the knot in two distinct points, that is, the twice tangent spheres. The boundary $\partial \overline{NT}(K)$ may have other singularities.

Definition 3.2.3 Let $TWT(K)$ denote the set of the twice tangent spheres and $atwt(K)$ its area.

Remark: (1) Then the set $TWT(K)$ of the twice tangent spheres forms a surface of space type. Namely, the restriction of the Lorentz form to the tangent space at each regular point of this surface is positive definite. Therefore its area $atwt(K)$ is a positive number, which is a conformal invariant of the knot K.

(2) Unlike $E^{(2)}$ or $E_{|\sin\theta|}$, $atwt(K)$ can not be expressed as an integral over T^2 of a funcition of the infinitesimal cross-ratio Ω_{CR} which does not contain its derivatives. This is because the local contribution of a neighborhood of a pair of points on K to $atwt(K)$ depends on the curvature, whereas Ω_{CR} itself is independent of f''.

Question : Can $atwt(K)$ be expressed in terms of the coefficient of the infinitesimal cross-ratio:

$$\frac{e^{i\theta}}{|f(s)-f(t)|^2}, \ \frac{\partial}{\partial s}\left(\frac{e^{i\theta}}{|f(s)-f(t)|^2}\right), etc.?$$

*** This is not the 'Question' in the paper.**

(3) The area $atwt(K)$ is not an energy functional for knots.

Let K_n $(n > 3)$ be a knot with length 1 lying on a fixed sphere Σ_0 which has a pair of points $p_n, q_n \in K_n$ with $\delta_{K_n}(p_n, q_n) = 1/2$ and $|p_n - q_n| = 1/n$. Then $atwt(K_n)$ does not blow up as n goes to ∞.

3.3 The volume of the set of the non-trivial spheres.

The measure $mnts_1(K)$ of the set of the non-trivial spheres is not expressed as a "bilocal integral" like $E^{(2)}$ or $E_{|\sin\theta|}$ since it does not takes into account the number of intersection points of K with spheres. In order to get such an integral, we have to count each sphere Σ with multiplicity, which is a number of pairs of points in $\Sigma \cap K$ where the algebraic intersection number has the same sign.

Definition 3.3.1 The **measure of acyclicity** $mnts(K)$ is defined by

$$mnts(K) = \int_{NT(K)} C_2^n \, d\Sigma, \text{ where } n = n_K(\Sigma) = \frac{\sharp(\Sigma \cap K)}{2},$$

where the number of the intersection points $\sharp(\Sigma \cap K)$ is counted with multiplicity, and $d\Sigma$ denotes the $\mathcal{G}$-invariant measure of Λ.

Another related functional of interest is the *measure of the image of the 4-tuple map for a knot with multiplicity*, which is given by

$$mnts^{(4)}(K) = \int_{\mathrm{Conf}_4(S^1)\setminus\mathcal{C}c(K)} {\psi_K}^* \omega_\Lambda = \int_{NT(K)} C_4^{2n} \, d\Sigma,$$

where ω_Λ is $\mathcal{G}_+$-invariant volume 4-form of Λ. This definition of $mnts^{(4)}$ is based on the suggestion by J. Cantanella to the second author when he gave a talk on $mnts(K)$. It can not be expressed as a "bilocal integral".

Claim 3.3.2 *These three functionals,* $mnts_1$*,* $mnts$*, and* $mnts^{(4)}$ *are conformally invariant.*

Proof: Let $g \in \mathcal{G}$ be a conformal transformation. Then g maps a non-trivial sphere Σ of K to a non-trivial sphere $g \cdot \Sigma$ of $g \cdot K$ preserving the number of the intersection points. Therefore $g(NT(K)) = NT(g \cdot K)$ and $n_K(\Sigma) = n_{g \cdot K}(g \cdot \Sigma)$. Since the measure $d\Sigma$ is $\mathcal{G}$-invariant, i.e. $d\Sigma = d(g \cdot \Sigma)$, we have $mnts(g \cdot K) = mnts(K)$ etc. □

As we mentioned before, the measure $mnts_1(K)$ of the set $\overline{NT}(K)$ of the non-trivial spheres is finite for any smooth knot K since $NT(K)$ is bounded in Λ. On the other hand, it is not easy to control the number of the intersection points of K and a sphere Σ. We will show later at the end of the next subsection that the measure of acyclicity is finite for a smooth knot using its formula.

Question 3.3.3 *Is* $mnts^{(4)}$ *finite for any smooth knot?*

We conjecture that for any smooth knot K there exists a natural number n such that the measure of the set of spheres that intersect K in at least n points is 0, and hence the answer to the above Question is affirmative.

Remark: These three functionals, $mnts_1$, $mnts$, and $mnts^{(4)}$ are different. There holds $mnts_1(K) \leq mnts(K) \leq mnts^{(4)}(K)$. In fact, if an oriented 2-sphere Σ has exactly 6 transversal intersection points with K, then Σ is counted 1, $3 = C^3_2$, and $15 = C^6_4$ times respectively in these functionals.

3.4 The measure of acyclicity in terms of the infinitesimal cross-ratio.

Assume that K is oriented. Let $p \in K \cap \Sigma$ be a transversal intersection point. We say that Σ *intersects K at p positively* if the orientation of K is "*outward*" at p with respect to the orientation of Σ, namely, if the algebraic intersection number $K \cdot \Sigma$ is equal to $+1$ at p. We call p a *positive intersection point.* Then the measure of acyclicity satisfies

$$mnts(K) = \int_{NT(K)} C^{n'}_2 \, \omega,$$

where n' is the number of positive intersection points of K and Σ. We remark that the above definition does not depend on the choice of the orientation of a knot K.

Let $|NT^{(4)}_K(dx, dy)|$ denote the volume of the set of the non-trivial spheres for K which intersect both dx and dy positively. Then the measure of acyclicity of K is then equal to

$$mnts(K) = \lim_{dx,dy \to 0} \sum |NT^{(4)}_K(dx, dy)| = \int_{K\times K\setminus\triangle} |NT^{(4)}_K(dx, dy)| dx dy.$$

Proposition 3.4.1 *The measure of acyclicity is expressed as*

$$mnts(K) = \frac{\pi}{2} \int_{K\times K} (\sin\theta - \theta\cos\theta) \frac{dx dy}{|x-y|^2},$$

where θ is the argument of cross-ratio.

The proof decomposes into the following two lemmas.

Lemma 3.4.2 *There holds the dimension reduction formula:*

$$|NT^{(4)}_K(dx, dy)| = \frac{\pi}{2} |NT^{(3)}_K(dx, dy)|,$$

where $|NT^{(3)}_K(dx, dy)|$ denotes the volume of the set of the non-trivial circles which intersect both dx and dy positively.

Proof: Since $|NT^{(4)}_K(dx, dy)|$ is conformally invariant, we may assume that $x, x + dx, y$ and $y + dy$ are in $\mathbb{R}^2 = \{(X, Y, 0)\} \subset \mathbb{R}^3$ by a stereographic projection from the twice tangent sphere. Suppose the set of the non-trivial spheres or circles is parametrized by

$$CTR^{(3)}(dx, dy) \quad = \quad \left\{ (X, Y, r) \in \mathbb{R}^3_+ \;\middle|\; \begin{array}{l} \text{a circle with center } (X, Y) \text{ and radius } r \\ \text{is a non-trivial circle for } K \\ \text{which intersects both } dx \text{ and } dy \text{ positively.} \end{array} \right\}$$

$$
\begin{aligned}
CTR^{(4)}(dx,dy) &= \left\{ (X,Y,Z,r) \in \mathbb{R}^4_+ \;\middle|\; \begin{array}{l} \text{a sphere with center } (X,Y,Z) \text{ and radius } r \\ \text{is a non-trivial sphere for } K \\ \text{which intersects both } dx \text{ and } dy \text{ positively.} \end{array} \right\} \\
&\cong \{(X,Y,Z,\sqrt{r^2+Z^2}) \,|\, (X,Y,r) \in CTR^{(3)}(dx,dy), \in \mathbb{R}\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|NT_K^{(4)}(dx,dy)| &= \int_{CTR^{(4)}(dx,dy)} \frac{1}{r^4} dX\,dY\,dZ\,dr \\
&= \int_{CTR^{(3)}(dx,dy)} \left(\int_{-\infty}^{\infty} \frac{1}{(r^2+Z^2)^2} dZ \right) dX\,dY\,dr \\
&= \frac{\pi}{2} \int_{CTR^{(3)}(dx,dy)} \frac{1}{r^3} dX\,dY\,dr = \frac{\pi}{2} |NT_K^{(3)}(dx,dy)|.
\end{aligned}
$$

□

In what follows we calculate $|NT_K^{(3)}(dx,dy)|$ in the case when θ satisfies $0 \le \theta \le \frac{\pi}{2}$. Similar calculation works for $\frac{\pi}{2} \le \theta \le \pi$.

May assume that

$$
x = \infty,\ x + dx = (0,0),\ y = (1,0), \text{ and } y + dy = (1+a, b) \quad (b \ge 0)
$$

by a suitable orientation preserving Möbius transformation. Then, as the cross-ratio is preserved,

$$
a = \cos\theta \frac{|dx|\,|dy|}{|x-y|^2},\ b = \sin\theta \frac{|dx|\,|dy|}{|x-y|^2}.
$$

Then $CTR^{(3)}(dx,dy)$ is given by

$$
CTR^{(3)}(dx,dy) = \left\{ (X,Y,r) \in \mathbb{R}^3_+ \;\middle|\; \begin{array}{l} X^2 + Y^2 < r^2 \\ (X-1)^2 + Y^2 > r^2 \\ (X-(1+a))^2 + (Y-b)^2 < r^2 \end{array} \right\}.
$$

Let $\mathcal{N}(r_0)$ be the intersection of $CTR^{(3)}(dx,dy)$ and the level plane defined by $\{r = r_0\}$. Then

$$
|NT_K^{(3)}(dx,dy)| = \int_0^\infty \frac{1}{r^3} \operatorname{area}(\mathcal{N}(r)) dr.
$$

Let $C_i(r)$ be circles defined by

$$
\begin{aligned}
C_1(r) &= \{(X,Y) \,|\, X^2 + Y^2 = r^2\}, \\
C_2(r) &= \{(X,Y) \,|\, (X-1)^2 + Y^2 = r^2\}, \\
C_3(r) &= \{(X,Y) \,|\, (X-(1+a))^2 + (Y-b)^2 = r^2\}.
\end{aligned}
$$

Let

$$
P_{ij}(r) = (X_{ij}(r),\, Y_{ij}(r)) = C_i(r) \cap C_j(r) \quad (i \ne j).
$$

Then

$$
\begin{aligned}
P_{12}(r) &= \left(\frac{1}{2}, \sqrt{r^2-\frac{1}{4}}\right), \\
P_{13}(r) &= \left(\frac{1+a}{2} - b\sqrt{\frac{r^2}{(1+a)^2+b^2}-\frac{1}{4}}, \frac{b}{2}+(1+a)\sqrt{\frac{r^2}{(1+a)^2+b^2}-\frac{1}{4}}\right), \\
P_{23}(r) &= \left(1+\frac{a}{2} - \frac{b}{\sqrt{a^2+b^2}}\sqrt{r^2-\frac{a^2+b^2}{4}}, \frac{b}{2}+\frac{a}{\sqrt{a^2+b^2}}\sqrt{r^2-\frac{a^2+b^2}{4}}\right).
\end{aligned}
$$

Let $(\xi_{ij}(r), \eta_{ij}(r))$ be the coordinate of $P_{ij}(r)$ with respect to the frame (O, e'_1, e'_2) where $e'_1 = (\cos\theta, \sin\theta)$ and $e'_2 = (-\sin\theta, \cos\theta)$.

Lemma 3.4.3 *There holds*

$$
\begin{aligned}
|NT^{(3)}_K(dx,dy)| &= \int_{\frac{1}{2\sin\theta}}^{\infty} \frac{1}{r^3}\cdot\sqrt{a^2+b^2}\,(\eta_{23}(r)-\eta_{12}(r))dr + o(\sqrt{a^2+b^2}) \\
&= \left(\int_{\frac{1}{2\sin\theta}}^{\infty} \frac{\eta_{23}(r)-\eta_{12}(r)}{r^3}dr\right)\frac{|dx|\,|dy|}{|x-y|^2} + o\left(\frac{|dx|\,|dy|}{|x-y|^2}\right).
\end{aligned}
$$

Proof: We first remark that $\mathcal{N}(r)$ is not empty if and only if $X_{12}(r) \geq X_{23}(r)$, which is equivalent to $1 - r\sin\theta \leq \frac{1}{2}$ when $a, b \ll 1$.

The second equality is a consequence of

$$\sqrt{a^2+b^2} = \frac{|dx|\,|dy|}{|x-y|^2}.$$

Let $M_1(r)$ and $M_2(r)$ be as in Figure 5.

Figure 5:

We have only to show that

$$\int_{\frac{1}{2\sin\theta}}^{\infty} \frac{1}{r^3}\cdot \text{area}(M_i(r))\,dr = o(\sqrt{a^2+b^2}) \quad (i=1,2). \tag{1}$$

Since $\text{area}(M_1(r)) \leq a^2+b^2$, (1) holds for $i=1$. On the other hand

$$
\begin{aligned}
&\text{area}(M_2(r)) = \frac{1}{2}\sqrt{a^2+b^2}\,(\eta_{13}(r)-\eta_{12}(r)) + o(\sqrt{a^2+b^2}) \\
&= \frac{1}{2}\sqrt{a^2+b^2}\,\{\sin\theta\cdot(X_{12}(r)-X_{13}(r)) - \cos\theta\cdot(Y_{12}(r)-Y_{13}(r))\} + o(\sqrt{a^2+b^2}).
\end{aligned}
$$

Since

$$X_{12}(r) - X_{13}(r) = \sqrt{a^2+b^2}\left(-\frac{\cos\theta}{2} + \sin\theta\sqrt{\frac{r^2}{(1+a)^2+b^2}-\frac{1}{4}}\right),$$

we have

$$\int_{\frac{1}{2\sin\theta}}^{\infty} \frac{1}{r^3}\cdot\frac{1}{2}\sqrt{a^2+b^2}\cdot\sin\theta\cdot(X_{12}(r)-X_{13}(r))\,dr$$
$$\leq \frac{\sin\theta}{2}(a^2+b^2)\int_{\frac{1}{2\sin\theta}}^{\infty}\left(-\frac{\cos\theta}{2r^3}+\frac{\sin\theta}{r^2}\right)dr = O(a^2+b^2).$$

On the other hand since

$$\begin{aligned} Y_{13}(r)-Y_{12}(r) &= \frac{b}{2}+a\sqrt{\frac{r^2}{(1+a)^2+b^2}-\frac{1}{4}}+\sqrt{\frac{r^2}{(1+a)^2+b^2}-\frac{1}{4}}-\sqrt{r^2-\frac{1}{4}} \\ &= \frac{b}{2}+a\sqrt{\frac{r^2}{(1+a)^2+b^2}-\frac{1}{4}}-\frac{\frac{2a+a^2+b^2}{(1+a)^2+b^2}r^2}{\sqrt{r^2-\frac{1}{4}}+\sqrt{\frac{r^2}{(1+a)^2+b^2}-\frac{1}{4}}} \end{aligned}$$

we have

$$\int_{\frac{1}{2\sin\theta}}^{\infty}\frac{1}{r^3}\cdot\frac{1}{2}\sqrt{a^2+b^2}\cdot\cos\theta\cdot(Y_{13}(r)-Y_{12}(r))\,dr$$
$$\leq \frac{\cos\theta}{2}(a^2+b^2)\int_{\frac{1}{2\sin\theta}}^{\infty}\left(\frac{\sin\theta}{2r^3}+\frac{\cos\theta}{r^2}+\frac{\cos\theta}{r\sqrt{r^2-\frac{1}{4}}}\right)dr+o(\sqrt{a^2+b^2}),$$

as $\sqrt{r^2-\frac{1}{4}}\geq r\cos\theta$ when $r\geq\frac{1}{2\sin\theta}$, the right hand side satisfies

$$\leq\frac{\cos\theta}{2}(a^2+b^2)\int_{\frac{1}{2\sin\theta}}^{\infty}\left(\frac{\sin\theta}{2r^3}+\frac{1+\cos\theta}{r^2}\right)dr+o(\sqrt{a^2+b^2})=O(a^2+b^2).$$

This completes the proof of Lemma 3.4.3. □

Proof of Proposition 3.4.1: Since

$$\begin{aligned} \eta_{23}(r)-\eta_{12}(r) &= \sin\theta(X_{12}(r)-X_{23}(r))-\cos\theta(Y_{12}(r)-Y_{23}(r)) \\ &= \sqrt{r^2-\frac{a^2+b^2}{4}}-\frac{\sin\theta}{2}-\cos\theta\sqrt{r^2-\frac{1}{4}} \end{aligned}$$

we have

$$\begin{aligned} \int_{\frac{1}{2\sin\theta}}^{\infty}\frac{\eta_{23}(r)-\eta_{12}(r)}{r^3}dr &= \int_{\frac{1}{2\sin\theta}}^{\infty}\frac{1}{r^3}\left(r-\frac{\sin\theta}{2}-\cos\theta\sqrt{r^2-\frac{1}{4}}\right)dr+o(\sqrt{a^2+b^2}) \\ &= 2\int_0^{\sin\theta}(1-u\sin\theta-\cos\theta\sqrt{1-u^2})du+o(\sqrt{a^2+b^2}) \qquad \left(u=\frac{1}{2r}\right) \\ &= \sin\theta-\theta\cos\theta+o(\sqrt{a^2+b^2}), \end{aligned}$$

which, combined with Lemma 3.4.3, completes the proof of Proposition 3.4.1. □

Lemma **??** implies that the integrand $(\sin\theta-\theta\cos\theta)/|x-y|^2$ of the measure of acyclicity is 0 at the diagonal set. Therefore, the measure of acyclicity is finite for any knot of class C^4. Lemma **??** also implies that $E^{(2)}$, $E_{|\sin\theta|}$, and *mnts* are independent.

We remark that Claim **??** implies that the measure of acyclicity of K, and hence $mnts^{(4)}(K)$, is equal to 0 if and only if K is the standard circle.

4 Non-trivial zones.

4.1 Strongly non-trivial zones.

A circle is generically intersected by a sphere in 0 or two point. It is a necessary and sufficient condition for a knot to be the circle. A curve which is not a circle should therefore admit spheres intersecting it transversally in at least 4 points. Thickening such a non-trivial sphere, we get a region bounded by two disjoint spheres which is crossed by at least 4 strands.

Definition 4.1.1 (1) A *zone* Z is a region diffeomorphic to $S^2 \times [0,1]$ bounded by two disjoint spheres.

(2) Let Z be a zone with $\partial Z = S_1 \cup S_2$ and $T : \mathbb{R}^3$ (or S^3) $\to \mathbb{R}^3$ (or $\mathbb{R}^3 \cup \{\infty\}$ respectively) be a conformal transformation which maps the two boundary spheres of Z into a concentric position. Then the *modulus* $\rho(Z) > 1$ of the zone Z is the ratio of the radii of the two concentric boundary spheres of $T(Z)$.

(3) Instead of using particular conformal transformation T, one can use the *Lorentzian modulus* $\lambda(Z) = |L(S_1, S_2)|$ of a zone Z with $\partial Z = S_1 \cup S_2$, where S_1 and S_2 are considered [1] as points in Λ.

Remark: (1) A zone Z is uniquely determined by its modulus up to a motion by a Möbius transformation.

(2) The modulus $\rho(Z)$ is an increasing function of the Lorentzian modulus $\lambda(z)$ with $\rho = \lambda + \sqrt{\lambda^2 - 1}$ and $\lambda = \frac{1}{2}(\rho + 1/\rho)$.

In fact, one can assume that S_1 is an equator 2-sphere of S^3 and S_2 is a parallel 2-sphere to S^1 by a conformal transformation. Then the Poncelet pencil in Λ generated by S_1 and S_2, which is a hyperbola in the 2-plane in $\mathbb{R}^5$ generated by S_1 and S_2, can be parametrized by (Cht, Sht), where $t = 0$ corresponds to S_1. The restriction of L to this Poncelet pencil is a quadratic form of type $(1,1)$. Let t_0 be the parameter for S_2. Then $\rho(Z) = Cht_0 + Sht_0$ whereas $\lambda(Z) = Cht_0$.

Figure 6: A zone and its privileged representative

It can also be computed from the cross ratio of the four points intersections of the two spheres with a circle containing the two base points of the pencil generated by the two spheres.

Remark: If the zones Z_1 and Z_2 satisfy $Z_1 \subset Z_2$, one has $\rho(Z_2) \geq \rho(Z_1)$.

*** I do not find a word "large cross-strand in what follows (it might be called in other way).**

Definition 4.1.2 Let K be a knot and Z be a zone with $\partial Z = S_1 \cup S_2$.

(1) A *small strand (of K in Z)* is a closure of a connected component of the intersection of the interior of Z with K whose two end points lie on the same boundary sphere of Z. A *degenerate small strand* is a closed connected component of the intersection of K with

[1] S_1 and $-S_2$ are endowed with the orientation as the boundary of Z

∂Z. A *strand* is a closed connected component of K that is contained in the zone Z whose end points are on ∂Z.

(2) A *cross-strand* is a strand of K with one end point contained in S_1 and another in S_2. A **minimal** *cross-strand* is a cross-strand whose intersection with the interior of Z is connected. If a cross-strand has a common end point with a (degenerate) small strand, one can connect them to obtain a longer cross-strand. We call a cross-strand *large* if it is maximal for this connecting procedure starting from a *minimal* cross-strand.

(3) A zone will be called *strongly non-trivial* for a knot K if it has at least 4 closed cross-strands of the knot K.

I added "minimal cross-strand".
Remark: A large cross-strand is maximal for the inclusion in the set of cross-strands that do not contain more than or equal to two minimal cross-strands.

The existence of spheres intersecting a curve in at least four points transversally implies the existence of strongly non-trivial zones. In particular such non-trivial zones exist for a non-trivial knot K.

Figure 7: Non-trivial zone

Definition 4.1.3 A strongly non-trivial zone for a knot K is called *maximal* if it has the maximum modulus among the set of strongly non-trivial zones for K.

The above Remark implies that a maximal strongly non-trivial zone is also maximal for the inclusion in the set of strongly non-trivial zones.
Remark: There exists a maximal strongly non-trivial zone for any non-trivial knot K since the set of pairs of boundary spheres of strongly non-trivial zones is a non-empty compact subset of $\Lambda \times \Lambda$.
Let us now look for properties of maximal strongly non-trivial zones.

Proposition 4.1.4 ***Let Z be a maximal strongly non-trivial zone for K with $\partial Z = S_1 \cup s_2$. Then it satisfies the following.***

(1) K is contained in Z.

(2) $K \cap S_1$ (or $K \cap S_2$) consists of two points where K is tangent to S_1 (or S_2, respectively).

(3) The two tangent points of $K \cap S_1$ (or $K \cap S_2$) are antipodal.

The proof follows from the proofs of the following two lemmas.

Lemma 4.1.5 *If a strongly non-trivial zone Z with $\partial Z = S_1 \cup S_2$ is maximal, then there are two cross-strands which are tangent to S_1 and there are two cross-strands which are tangent to S_2.*

Proof: To fix ideas and allow pictures, let us choose a metric such that the boundary spheres of the zone Z are symmetric with respect to the equator of the sphere S^3. The pencil generated by S_1 and S_2 contains two points that we call *poles*. We call N the pole closer to S_1 than to S_2. If four cross-strands intersect transversally one of the boundary sphere, say S_1, the zone can be enlarged replacing S_1 by another sphere closer to the pole N (*move 1*).

If only three cross-strands intersect transversally the sphere S_1, a fourth is tangent to it at a point m. We can replace S_1 by a sphere of the pencil tangent to S_1 at m closer to P (*move(2)*).

Figure 8: Increasing the zone Z (1)

Figure 9: Increasing the zone Z (2)

□

In order to study the necessary condition for the relative positions of the tangency points on the boundary spheres of a maximal strongly non-trivial zone, we fix the tangency points and forget the rest of the knot.

Definition 4.1.6 Let x_1, x_2, x_3, and x_4 be ordered four points in $\mathbb{R}^3$ (or S^3).

(1) A zone Z with $\partial Z = S_1 \cup S_2$ is called to be *separating* if x_1 and x_3 are contained in one of the connected component of $\mathbb{R}^3 \setminus \mathrm{Int}Z$ (or $S^3 \setminus \mathrm{Int}Z$ and x_2 and x_4 in another component.

(2) The *maximal (Lorentz) modulus* $\mu(x_1, x_2, x_3, x_4)$ (or $\mu_L(x_1, x_2, x_3, x_4)$ respectively) *of the ordered four points* (x_1, x_2, x_3, x_4) is the supremum of the (Lorentz) moduli $\rho(Z)$ (or respectively, $L(S_1, S_2)$) of separating zones Z with $\partial Z = S_1 \cup S_2$. We put $\mu = \mu_L = 1$ when there are not such separating zones.

(3) A separating zone that attains the maximal modulus will be called the *maximal separating zone of the four point.*

We remark that if Z is a maximal separating zone of the four points, then one of its boundary sphere should contain both x_1 and x_3, and another x_2 and x_4.

Lemma 4.1.7 *(1) There are no separating zone, i.e. $\mu = \mu_L = 1$ if and only if x_i's are concircular in such a way that x_1 and x_3 are not adjacent, i.e. when the the cross-ratio $(x_1, x_2; x_3, x_4) = \frac{x_1 - x_3}{x_1 - x_4} : \frac{x_2 - x_3}{x_2 - x_4}$ is a positive real number.*

(2) Otherwise the maximal separating zone Z is unique.

(3) Suppose Z with $\partial Z = S_1 \cup S_2$ is a maximal separating zone of (x_1, x_2, x_3, x_4) such that S_1 and S_2 are concentric. Then x_1 and x_3 are antipodal in one of the boundary spheres of Z, and x_2 and x_4 are antipodal in another boundary sphere.

Remark: The condition of being antipodal does not seem to be conformally invariant but dependent on the metric. Nevertheless, as it depends on a choice of a

Figure 10: Maximal modulus for four points

representative of the zone Z it has a conformal meaning. The condition can be rephrased as follows: the two tangencies have to belong to a circle containing the two limit points of the pencil generated by the two boundary spheres of the zone.

The proof is the consequence of the following lemma: [2]

Lemma 4.1.8 *Let $\mathcal{P}$ be a Poncelet pencil of spheres; it is the intersection of Λ with a plane P. Let $\mathcal{B}$ be the two dimensional linear family of spheres $\mathcal{B} = B \cap \Lambda$ where $B = P^{\perp}$. At a point $\sigma \in \mathcal{P}$ the 3-plane orthogonal to $\mathcal{P}$ is generated by the vectors tangent at σ to the pencils of spheres with base circle a circle of the form $\sigma \cap b\,(b \in \mathcal{B})$.*

Proof of Lemma 4.1.7 **:** The last lemma implies the previous one as, if two ends contained in S_1 are not opposite, there exists a non geodesic circle γ containing them. Let us call $\mathcal{B}_\gamma$ the pencil with base circle γ. Consider the Poncelet pencil $\mathcal{P}$ generated by S_1 and S_2. The modulus of the region bounded by the disjoint spheres S_1 and $\sigma \in \mathcal{P}$ is just $L(S_2, \sigma)$. Its level through S_1 is therefore tangent to the hypersurface of equation $L(S_2, \tau) = L(S_2, S_1)$. Its tangent plane at S_1 has the equation: $L(S_2, v - S_1) = 0$. Therefore, for w a vector tangent to $\mathcal{B}_\gamma$ at S_1 we have $L(S_2, w) \neq 0$. This implies that the derivative of the modulus of the region between S_2 and a sphere $b \in \mathcal{B}$ is not zero in S_1 allowing to increase the modulus keeping four cross-strands. The extremities may change as first the two tangencies become a priori only intersection points of the knot with the sphere $b \in \mathcal{B}$ close to S_1, second some tangency end may disappear but is replaced by a point close to one of the ends belonging to the rotation circle. □

Proof of the Lemma 4.1.8**:** The intersection $\mathcal{B} = B \cap \Lambda$ is the set of spheres containing the two limit points of the Poncelet pencil $\mathcal{P}$. In the model we chose, all the spheres $\tau \in \mathcal{E}$ intersect S_1 and S_2 in a totally geodesic circle in S_1 or S_2. Let $F = B \oplus \mathbb{R} \cdot S_1$. The tangent vector at S_1 is orthogonal to S_1 and to B, therefore it is also orthogonal to the tangent space to $F \cap \Lambda$ at S_1 which contains all the tangent vectors at S_1 to the pencils through geodesic circles in S_1 (in our model where S_1 and S_2 belong to a Poncelet pencil with limit points two antipodal points (the poles) of S^3. □

Definition 4.1.9 The *strong modulus ρ_1 of a knot K* is the supremum of the modulus of a strongly non-trivial zone for K.

We need to relax our definition of strongly non-trivial zone. Let Z be a zone with $\partial Z = S_1 \cup S_2$.

Definition 4.1.10 A strand is called *trivial* if it is a small strand with end points in S_1 (or S_2) and it can be homotoped with its end points fixed to an arc in S_1 (or respectively, S_2) (which is not necessarily embedded), in the complement in Z of the set of small strands with end points in S_2 (or respedtively, S_1).

We call end points of non-trivial strands *non-trivial ends.*

The number of the non-trivial ends will be counted with multiplicity.

[2] We state other proofs in the subsection 5.1.

Definition 4.1.11 A zone Z is a *non-trivial zone* if it contains four non-trivial strands.

Lemma 4.1.12 *Any sphere contained in a non-trivial zone Z intersects the knot K in at least four points.*

Proof: We have only to check the case when two of the non-trivial strands are given by a pair of "hooked small strands".

Suppose γ_1 (or γ_2) is a small strands with end points in S_1 (or respectively, S_2) such that γ_1 can not be homotoped to an arc in S_1 with its end points fixed in $Z \setminus \gamma_2$, and vice versa. Then each end point of γ_1 must be joined to one of the end points of γ_2. Hence the existence of a pair of "hooked small strands" implies the existence of two cross-strands γ_3 and γ_4. Then any sphere contained in Z intersects $\gamma_1 \cup \gamma_2$ in at least two points and $\gamma_3 \cup \gamma_4$ in at least two points. $\square$

Figure 11: non-trivial zone

The definition and the existence of the *maximal non-trivial zone* is parallel to those of maximal strongly non-trivial zone. They satisfy the same properties as the maximal strongly non-trivial zones. The proofs are similar.

Proposition 4.1.13 *Let Z be a maximal non-trivial zone for K with $\partial Z = S_1 \cup S_2$. Then it satisfies the following.*

(1) K is contained in Z.

(2) $K \cap S_1$ (or $K \cap S_2$) consists of two points where K is tangent to S_1 (or S_2, respectively).

(3) The two tangent points of $K \cap S_1$ (or $K \cap S_2$) are antipodal.

Definition 4.1.14 The *modulus $\rho(K)$ of a knot* K is the supremum of the modulus of a non-trivial zone of K.

Theorem 4.1.15 *There exists a constant $a > 0$ such that if a knot K is a non-trivial knot, then its modulus is larger than a.*

Let us now start with a curve K of modulus ρ. Let us consider a stereographic projection to $\mathbb{R}^3$ such that the two boundary spheres of a maximal zone Z are concentric, with center the origin. We may imagine a very thin knitted curve contained in a zone of small modulus. In order to prove such a zone is not a maximal zone for the knot we need to use the condition that the maximal modulus is very small to control the knot at all scales.

Figure 12: Knitted curve

We remark that the modulus $\rho(K)$ of a knot K is larger than the maximal modulus $\mu(x_1, x_2, x_3, x_4)$ for any four points on the knot chosen in that cyclic order, since any separating zone of (x_1, x_2, x_3, x_4) is a non-trivial zone for K.

Fixing three points x_1, x_2, x_3 defines a function μ_{x_1,x_2,x_3} by $\mu_{x_1,x_2,x_3}(x) = \mu(x_1, x_2, x_3, x)$. It is a continuous function, the zero level of which is the arc γ of the circle trough (x_1, x_2, x_3) which does not contain the point x_2. Moreover, given a small value ϵ of μ_{x_1,x_2,x_3}, if the sphere S^2 is endowed with a metric such that the length of the circle through x_1, x_2, x_3 is of the order of 1, then the distance from the level $\mu_{x_1,x_2,x_3} = \epsilon$ to the circle γ satisfies:

$$\forall x \in \{\mu_{x_1,x_2,x_3} = \epsilon)\}\,;\; a \cdot \mathrm{length}(\gamma) \;<\; d(x\,,\,\gamma) \;<\; A \cdot \mathrm{length}(\gamma).$$

Let us now suppose that the curve K has a small modulus, say less than $\frac{1}{10000}$, and chose a stereographic projection of S^3 on $\mathbb{R}^3$ such that the two boundary spheres S_1 and S_2 of a maximal zone Z for K are concentric with center the origin (the north pole of the projection is one of the limit points of the pencil generated by S_1 and S_2, the tangency point of S^3 and $\mathbb{R}^3$,which we chose as origin of $\mathbb{R}^3$, is the other limit point). Finally, compose the stereographic projection with an homothety to transform the inner sphere into the unit sphere centered at the origin.

Lemma 4.1.16 *The knot K in contained is a thin tubular neighborhood of a geodesic circle of the middle sphere S_m (defined by the intersection of the ray containing $\frac{1}{2}(\sigma_1+\sigma_2)$) and Λ, where σ_1 and σ_2 are the points in Λ corresponding to the spheres S_1 and S_2) of the zone Z.*

Proof: We know that the knot K is tangent to S_1 in two antipodal points, they will be x_1 and x_3. We chose x_2 on the intersection of the knot with the "equatorial" plane associated to x_1 and x_3. This defines a strand γ_1, of extremities x_1 and x_3, containing x_2, of K. The strand of K joining x_3 to x_1 which does not contain x_2, that we will call γ_2, have to stay in a neighborhood of the arc of extremities x_1 and x_3, of the circle defined by the three points x_1, x_2, x_3 which does not contain x_2. Let x_4 be a point of the intersection of γ_2 with the equatorial plane previously defined. Using now the circle through x_1, x_2, x_4 we deduce that the strand γ_1 is contained in a neighborhood of the arc of extremities x_1 and x_2 of the circle determined by x_1, x_2, x_4 which does not contain x_4.

Figure 13: The knot is trapped in a neighborhood of a circle

As the two circles containing x_1, x_2, x_3 and containing x_1, x_2, x_4 are close to each other, and close to a great circle G_1 of the sphere S_m, which completes the proof. $\square$

Consider now the normal bundle to G_1, and discs $D_1^1, \ldots D_{n_1}^1$ of this bundle through equidistant point of G_1, say, such that the length of those arc is roughly 100 times the diameter of the tubular neighborhood of G_1 containing the knot K. Chose a point x_i^2 of K in each disc D_i^1. Joining consecutive (for the cyclic order) points x_i^2 and x_{i+1}^2 by a small arc of the circle defined by the two consecutive points x_i^2 and x_{i+1}^2 and one of the almost antipodal points to x_i^2: x_I^2, that we will call γ_i^2, we get an unknotted polygon G_2 with all the angles almost flat (the tangent of those angle is at most, say, $\varphi_2 \;<\; 1,5 \cdot Arctg(\frac{1}{100}) \;<\; \frac{3}{100}$).

Figure 14: The first polygonal approximation

Using the the same idea as for the proof of the corollary, we deduce that the small strand of the knot joining x_i^2 to x_{i+1}^2 has to stay in a neighborhood of the small arc of the circle γ_i^2 of "diameter of the order of $\mu \cdot$ (length ofγ_i^2).

The knot is now confined in a one order of magnitude thinner neighborhood of G_2; let us call δ_2 the diameter of this tube. Consider now the normal disc bundle to G_2, which is defined with no ambiguity, except in a neighborhood of the vertices of G_2, union of ball of diameter, say, $2 \cdot \delta_2$. Let us now chose a sequence of consecutive discs of this normal bundle, $D^2_1 ... D^2_{n_2}$ distant of roughly $100 \cdot \delta_2$. The choice of the discs near the vertices of Γ_1 has to be made with more care: they should be roughly at a distance of $50\delta_2$ form the vertices of Γ_1. Again we chose a point x^3_i in each normal disc D^2_i. We get that way a new unknotted polygon Γ_3. The not K in now contained in a tubular neighborhood of Γ_3 of one order of magnitude thinner.

The only obstruction left to proceed with inductive construction of unknotted polygons inscribed in K is the angles of the polygons. We need to guarantee they stay flat enough. The angles θ^2_i of Γ_2 are bounded by φ_2 see next picture.

Figure 15: The first polygon Γ_2 inscribed in K

That is also true for the angles at all the vertices of the successive polygons except for the ones which are adjacent to some vertex of the previous polygon. A priori the angle cannot grow more than linearly with the index of the polygone. In fact $8 \cdot \varphi_2$ bounds all the angles of all the Γ_k. If the angle θ at a vertex of Γ_{k-1} is larger than $5 \cdot \varphi_2$, then our construction will replace it by two angles of Γ_k smaller than $\frac{\theta}{2} + 3\varphi_2$, therefore never reaching $8 \cdot \varphi_2$

Figure 16: Construction of Γ_k

Getting to a polygon Γ_k such that the diameter δ_k of its tubular neighborhood containing the knot K is smaller than, $\frac{1}{3} \cdot \Delta(K)$ where $\Delta(K)$ is the global radius of curvature defined by Gonzales and Maddocks [?] (see section **??**), we prove that the knot K is trivial. This ends the proof of the theorem 4.1.15.

4.2 Jump of the measure of acyclicity for non-trivial knots.

Given the curve K, the spheres contained in a non-trivial zone A intersect K in at least four points. The region R_A in Λ corresponding to the spheres contained in the annulus A is bounded by two light cones (half light cones to be precise) with vertex the boundary spheres S_1 and S_2 of A. The volume of R_A is a monotonous function of the modulus of A. Therefore the existence of a strictly larger than 0 lower bound for the moduli of maximal non-trivial zones associated to non-trivial knots implies the existence of a strictly positive constant $c > 0$ such that for any non-trivial knot, the measure $mnts_1(C)$ of the set of non-trivial spheres is larger or equal to c.

The modulus and the measure of acyclicity probably do not behave the same way for connected sums of knots. Let $\rho([K])$ (or $mnts([K])$) denote the infimum of the modulus (or the measure of acyclicity respectively) of a knot that belongs to an isotopy class $[K]$.

Question 4.2.1 *Is it true that:*

$$\rho([K_1 \sharp K_2]) = \max\{\rho([K_1])\,,\,\rho([K_2])\}\,?$$

Question 4.2.2 *Is it true that:*

$$mnts([K_1 \sharp K_2]) = (or \leq \; or \geq) \; mnts([K_1]) + mnts([K_2]) \,?$$

Question 4.2.3 *Do the modulus and the measure of acyclicity take their minimum values for non-trivial knots at a (2,3)-torus knot on a torus of revolution?*

4.3 Explosion of the measure of acyclicity for singular knots.

Theorem 4.3.1 *The measure of acyclicity is an energy functional for knots.*

Proof: Fix δ $(0 < \delta \leq \frac{1}{2})$. For $0 < d \leq \delta$ put

$$\mathcal{K}(d) = \mathcal{K}_\delta(d) = \left\{ \begin{array}{c|l} K : \text{a knot with} & \exists x, y \in K \text{ such that} \\ \quad\text{length } l(K) & \text{i) the shorter arc-length} \\ & \quad \text{between } x \text{ and } y \text{ is } \delta l(K). \\ & \text{ii) } |x-y| \leq d l(K). \end{array} \right\},$$

$$\rho(d) = \inf_{K \in \mathcal{K}(d)} \{mnts(K)\}$$

We remark that ρ is a monotonely decreasing function of d because $d_1 < d_2$ implies $\mathcal{K}(d_1) \subset \mathcal{K}(d_2)$, and that $\rho(d) = 0$ for $d \geq (\sin \pi\delta)/\pi$ because $\mathcal{K}(d)$ contains the standard circle.

Assume that there is a positive constant M' such that

$$\sup_{d>0} \rho(d) = \lim_{d \to +0} \rho(d) = M'.$$

There is a positive constant $M = M(M')$ such that if the measure of acyclicity of K is smaller than $1.1M'$ then the ratio of the radii of the two spheres of any concentric non-trivial zone for K is smaller than M.

Let K_d be a knot with length 1 in $\mathcal{K}(d)$ whose measure of acyclicity is smaller than $1.1M'$. Let x, y be a pair of points on K_d such that the shorter arc-length between them is δ and that $d_0 = |x-y| \leq d$. [3] The knot K is divided into two arcs, A_1 and A_2, by x and y. Put

$$a = \min\left\{ \sup_{z \in A_1} \left| z - \frac{x+y}{2} \right|, \sup_{w \in A_2} \left| w - \frac{x+y}{2} \right| \right\}.$$

Let Σ_r denote the 2-sphere with center $(x+y)/2$ and radius r. Since $(\Sigma_a, \Sigma_{d_0/2})$ forms a non-trivial zone, $a \leq M d_0/2$ and hence $a \leq Md/2$. Let $\bar{K}_d$ be the part of K_d which is contained in $\Sigma_{Md/2}$. The length $l(\bar{K}_d)$ of $\bar{K}_d$ is greater than or equal to δ since $\Sigma_{Md/2}$ contains at least one of the arcs A_1 and A_2. Then by Poincaré's formula ([Sa] page 259, see also pages 111, 277) there holds

$$\int_G \#(g(\Sigma_r) \cap \bar{K}_d) dg = \frac{O_3 O_2 O_1 \cdot O_0}{O_2 O_1} \cdot \text{area}(\Sigma_r) \cdot l(\bar{K}_d),$$

[3] We use the axiom of choice here.

where O_n denotes the volume of the n dimensional unit sphere, G denotes the group of the orientation preserving motions of $\mathbb{R}^3$, which is isomorphic to the semidirect product of $\mathbb{R}^3$ and $SO(3)$, and dg denotes the kinematic density ([Sa] page 256). We have $dg = dP \wedge dK_{[P]}$, where $P \in \mathbb{R}^3$, dP is the volume element of $\mathbb{R}^3$, and $dK_{[P]}$ is the kinematic density of the group of special rotations around P, which is isomorphic to $SO(3)$. Since

$$\int dK_{[P]} = O_2 O_1$$

the Poincaré's formula implies

$$\int_{\Sigma_r(X,Y,Z)\cap K_d \neq \phi} \#(\Sigma_r(X,Y,Z)\cap K_d)dXdYdZ = 2\pi r^2 l(\bar{K}_d),$$

where $\Sigma_r(X,Y,Z)$ denotes the sphere with radius r and center (X,Y,Z). Suppose $d/2 \leq r \leq Md/2$. Then the measure of acyclicity of K_d satisfies

$$\begin{aligned} mnts(K_d) &= \int_{NT(K_d)} C_2^{\#(\Sigma_r(X,Y,Z)\cap K_d)} \cdot \frac{1}{r^4} dXdYdZdr \\ &\geq \int_{d/2}^{Md/2} \frac{1}{r^4}\left\{ \int_{D^3_{Md/2+r}((x+y)/2)} \frac{\#(\Sigma_r(X,Y,Z)\cap \bar{K}_d) - 1}{2} dXdYdZ \right\} dr, \end{aligned}$$

where $D^3_c(R)$ denotes the 3-ball with center c and radius R. Thus

$$\begin{aligned} mnts(K_d) &\geq \frac{1}{2}\int_{d/2}^{Md/2} \frac{1}{r^4}\left\{ 2\pi r^2 l(\bar{K}_d) - \frac{4}{3}\pi\left(\frac{Md}{2}+r\right)^3 \right\} dr \\ &\geq \frac{1}{2}\left\{ 2\pi\delta\left[-\frac{1}{r}\right]_{d/2}^{Md/2} - \frac{4}{3}\pi(Md)^3\left[-\frac{1}{3r^3}\right]_{d/2}^{Md/2} \right\}, \end{aligned}$$

which blows up as d goes down to 0, which contradicts the assumption that $\lim_{d\to+0}\rho(d) \leq M < \infty$. □

5 Appendix.

5.1 The maximal modulus and cross-ratio of four points.

In this subsection we give a formula to express the maximal Lorentz modulus $\mu_L(x^1,x^2,x^3,x^4)$ of the four points in terms of the cross-ratio $(x^2,x^3;x^1,x^4)$. It implies that the integrand of $E^{(2)}$ is equal to the *infinitesimal maximal modulus* $\mu_L(x, x+dx, y, y+dy)$.

But we start with an alternative proof of Lemma 4.1.7 by showing it in lower dimension by one. The terms in the following lemma can be defined in a parallel way.

Lemma 5.1.1 *Let x_1, x_2, x_3, and x_4 be ordered four points on S^2. Suppose the cross-ratio $(x_1, x_2; x_3, x_4)$ is not a positive real number.*

Then the maximal modulus of a separating annulus is attained by the unique separating annulus, which will be called the maximal separating annulus. Then the pair x_1, x_3 (or x_2, x_4) is on a circle of the pencil whose base points are the limit points of the pencil defined by the boundary circles of the maximal separating annulus.

Proof: First observe that the two boundary circles C_1 and C_2 of a separating annulus with the maximal modulus should contain, one x_1 and x_3, the other x_2 and x_4. The idea of the proof is the same as the idea of the proof of Lemma 4.1.7. If these point where not on a circle Γ through the limit point generated by the two boundary circles, say for $x_1 \in C_1$ and $x_3 \in C_1$, rotating from C_1 in the pencil of base points x_1 and x_3 will define a curve in the quadric Λ of circles in S^2 transverse at C_1 to the level $L(C_2, C) = L(C_2, C_1)$ of L, therefore to the level $m(C_2, C) = m(C_2, C_1)$ contradicting the fact that a maximum is a critical point. $\square$

Lemma 5.1.2 *Let C_1 and C_2 be a pair of concentric circles on $\mathbb{R}^2 \subset \mathbb{R}^3$. Then among the zones Z with $\partial Z = S_1 \cup S_2$ satisfying $(S_1 \cup S_2) \cap \mathbb{R}^2 = C_1 \cup C_2$, the maximal modulus is attained by the unique zone which has concentric boundary spheres whose center is in $\mathbb{R}^2$.*

The proof will be omitted.

Corollary 5.1.3 *Let x_1, x_2, x_3, and x_4 be ordered four points in $\mathbb{R}^3$ or S^3, and let Σ be a 2-sphere passing through them. Then the maximal modulus $\mu(x_1, x_2, x_3, x_4)$ of the four points is equal to the maximal modulus of a separating annulus in Σ.*

Claim 5.1.4 *The maximal Lorentz modulus $\mu_L(x^1, x^2, x^3, x^4)$ is expressed in terms of the cross-ratio $cr = (x^2, x^3; x^1, x^4)$ as*

$$\mu_L(x^1, x^2, x^3, x^4) = \sqrt{1 + 2|1 - cr|^2 \left\{ \left| \frac{cr}{1 - cr} \right| - \Re\left(\frac{cr}{1 - cr} \right) \right\}}.$$

Remark: Lemma 4.1.7 implies the above Claim as follows. The boundary spheres of the maximal separating zone can be mapped into a concentric position by a Möbius transformation. Then Lemma 4.1.7 asserts that x_1 and x_3 are antipodal in one of the boundary spheres, and x_2 and x_4 are antipodal in another. We can assume without loss of generality that $x_1, x_3 = \pm 1$ and $x_2, x_4 = \pm z$ in $\mathbb{C} \subset \mathbb{R}^3$, and the boundary spheres of the maximal separating zone are concentric spheres with center the origin and radii 1 and $|z|$. Then the maximal Lorentz modulus $\mu_L(x^1, x^2, x^3, x^4)$ is given by $(|z| + 1/|z|)/2$ and the cross-ratio $cr = (x^2, x^3; x^1, x^4)$ is given by $-(z-1)^2/(4z)$, and therefore $cr/(1-cr) = (z-1)^2/(z+1)^2$. One can verify the equality by the direct calculation.

Lemma 5.1.5 *Let Π be a plane and Σ be a 2-sphere in $\mathbb{R}^3$ such that the minimum and maximum distance between Π and Σ are a and b $(0 < a < b)$ respectively. Then the modulus $\rho(\Sigma, \Pi)$ of the zone bounded by Σ and Π is given by*

$$\rho(\Sigma, \Pi) = \frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}}.$$

Proof: May assume that Π is the y-z plane and Σ has $(\frac{a+b}{2}, 0, 0)$ as its center. Then any inversion with respect to a 2-sphere with center $(\pm\sqrt{ab}, 0, 0)$ maps Σ and Π to a concentric position. $\square$

This Lemma immediately implies the following Lemma.

Lemma 5.1.6 *Let Σ be a 2-sphere in $I\!R^3$ with center O, and $A \in I\!R^3$ be a point outside Σ. Then among planes P passing through A, the plane P_0 perpendicular to the line OA gives the maximal modulus $\rho(\Sigma, P)$.*

Let T_0 be an inversion with respect to a 2-sphere whose center lies on the line OA that maps Σ and P_0 to a concentric position as is given in the proof of the previous Lemma. Then $T_0(A)$ and $T_0(\infty)$ are antipodal in $T_0(P_0)$. Therefore this lemma gives an alternative computational proof of Lemma 4.1.7.

Lemma 5.1.7 *Let C be a circle with center O in a plane Π, and $A \in \Pi$ be a point outside C. Then among pairs of a sphere Σ passing through C and a plane P passing through A, the pair of the sphere Σ_0 whose center belongs to Π and the plane P_0 perpendicular to the line OA gives the maximal modulus.*

Proof: Let r be the radius of C and $a = |O - A|$. Let $\Sigma(h)$ denote the 2-sphere whose center is apart from Π by h ($h \geq 0$). Then the preceding two Lemmas imply that the maximum of the modulus $\rho(\Sigma(h), P)$ with $P \ni A$ is given by

$$\sqrt{1 + \frac{a^2 - r^2}{r^2 + h^2}} + \sqrt{\frac{a^2 - r^2}{r^2 + h^2}},$$

which is a decreasing function of h. $\square$

Proof of Claim 5.1.4: Without loss of generality we can assume that $x^1 = \infty$, $x^2 = 0$, $x^3 = 1$, and $x^4 = z = u + iv$ in $\mathbb{C} \cup \{\infty\} \cong \mathbb{R}^2 \cup \{\infty\} \subset \mathbb{R}^3 \cup \{\infty\}$. Then the cross-ratio $cr = (0, 1; \infty, z) = 1 - \frac{1}{z}$ and hence $\frac{cr}{1-cr} = z - 1$. We assume that $z \notin \{z \in \mathbb{R} \,|\, z > 1\}$, otherwise the both sides of the formula coincide with 1. The previous lemma implies that the maximal modulus $\mu_L(\infty, 0, 1, z)$ is attained by a zone $Z(\xi)$ bounded by $S_1(\xi)$ and $S_2(\xi)$ for some $\xi \in \mathbb{R}$, where $S_2(\xi)$ has center

$$O(\xi) = \left(\frac{u}{2} - \frac{v}{\sqrt{u^2+v^2}}\xi, \frac{v}{2} + \frac{u}{\sqrt{u^2+v^2}}\xi\right) \in \mathbb{R}^2,$$

and $S_1(\xi)$ is a plane perpendicular to the line joining $O(\xi)$ and 1. Then $\rho(S_1(\xi), S_2(\xi))$ is given by

$$\rho(Z(\xi)) = \frac{\sqrt{1 - u + \frac{u^2+v^2}{4} + \frac{2v}{\sqrt{u^2+v^2}}\xi + \xi^2} + \sqrt{1 - u + \frac{2v}{\sqrt{u^2+v^2}}\xi}}{\sqrt{\frac{u^2+v^2}{4} + \xi^2}},$$

which takes the maximum value

$$\sqrt{1 + \frac{2\left\{\sqrt{(u-1)^2 + v^2} - (u-1)\right\}}{u^2 + v^2}}.$$

$\square$

As a corollary of Claim 5.1.4 there holds:

Corollary 5.1.8 *The integrand $|\Omega_{CR}| - \Re\Omega_{CR}$ of $E^{(2)}(K)$ can be interpreted as the maximal Lorentz modulus $\mu_L(x, x + dx, y, y + dy)$ of a pair of infinitesimal segments.*

6 Möbius energy

The author thanks D.Rolfsen for pointing out the reference [F-H-W] to him. Recently M.F.Freedman, Z-X.He and Z.Wang using an previous work of O'Hara, [**?**] defined the Möbius energy of a rectifiable curve embedded in $\mathbb{R}^3$ by:

$$E(\gamma) = \int\int_{\gamma\times\gamma\setminus\Delta} \frac{1}{|\gamma(v)-\gamma(u)|^2} - \frac{1}{[dist_{\mathbb{R}^3}(\gamma(v),\gamma(u))]^2}$$

where Δ is the diagonal of the product $\gamma \times \gamma$. Separately the integral of the two fractions would diverge, but the sum converges. They prove that this function is invariant by the Möbius group.

Let $c([\gamma])$ be the crossing number of the knot type $[\gamma]$ (the infimum of the number of crossings of the projection of the knot $\gamma \in [\gamma]$ on a plane, when γ describes the isotopy class.

Theorem 6.0.9 *[F-H-W] The energy $E(\gamma)$ of a simple closed curve $\gamma \subset \mathbb{R}^3$ satisfies the inequality:*

$$E(\gamma) \geq 2\pi c([\gamma]) + 4$$

The equality $E(\gamma) = 4$ is achieved only when the curve is a (plane, round) circle.

Conjecture The Möbius energy and the measure of the set of spheres intersecting the curve γ in at least four points are linked by inequalities. One may have to take a multiplicity involving the number of intersection points into account.

7 Taut embeddings

7.1 The circle two piece property

Definition 7.1.1 *A top circle $C \subset S^2$ for a closed subset $A \subset S^2$ is a circle intersecting A, bounding two discs D_1 and D_2 such that the closure of one , say $\overline{D}_1$ contains A whilst the interior of D_2 is disjoint from A.*

We will call the intersection $C \cap A$ of a top circle with A a *topset* of A.

Definition 7.1.2 *A simple closed curve of S^2 has the circle two piece property , if it is divided by any circle in at most two connected components.*

Proposition 7.1.3 *A simple closed curve satisfying the circle two piece property is a (round) circle.*

The result is clear as, for any other simple closed curve, rotating through the tangency point an osculating circle with generic contact with the curve will give circles which contradict the circle two piece property. The circle two piece property is also meaningful for 2-dimensional submanifolds with smooth boundary of S^2.

remark

Notice that if a curve C does not satisfy the circle two piece property, one can find a top circle intersecting it in at least two connected components. Suppose that a disc

D_1 intersects C in two components. Let us call D_2 the interior of the complement of D_1. Chose a point a in $D_2 \setminus C$. The circles of the pencil generated by a and ∂D_1 , ordered from ∂D_1 by the inclusion of the discs they bound, starting with D_1, have a first tangency with C; that circle C_t is tangent to C in at least a point b, and the intersection $C_t \cap C$ still has at leaszt two components; it is the boundary of a disc D_t containing D_1. Consider now the pencil of the circles tangent to C_t at b ordered from C_t using as before the inclusion of discs that they bound. One of them is a top circle, and the corresponding topset has two connected components at least.

Proposition 7.1.4 *The only 2-dimensional manifolds W with smooth boundary of S^2 having the circle two piece property are obtained by removing from S^2 a finite number of disjoint closed discs D_i with boundary (round) circles.*

Lemma 7.1.5 *The top sets of a closed set A satisfying the circle two piece property are connected , that is are either a point or an arc of the corresponding top circle. Reciprocally if all the topsets of a closed set are connected, it satisfies the circle two piece property.*

Proof: Consider a top disc D_1. As we suppose that $A \cap \partial D_1$ is not connected we can take a point $x \in \partial D_1 \, x \notin A \cap \partial D_1$. Consider now the pencil of circles tangent at x to ∂D_1. By the stereographic projection of S^2 to $\mathbb{R}^2$ we send the previous pencil to a family of parallel lines that, as the circles of the pencil, cut the image of A into two pieces. We can use now the argument we used for usual topsets in the previous paragraph. □

Proof: of the proposition If the boundary of W is not a union of circles, then consider one component of ∂W which is not a circle. Performing a suitable inversion, this component can be seen as the outer boundary component of $W \in \mathbb{R}^2$. If this component is not a circle, we proved in a previous remark that some circle C_1 bounding a disc $D_1 \in \mathbb{R}^2$ which contains W will touch the outer boundary component of W in two connected components. The previous lemma provides a contradiction. The condition of the proposition are sufficient because any circle C tranverse to ∂W intersects each circle ∂D_i in zero or two points. Then C and the discs D_i it crosses form a necklace the complement of which has two open connected components which are the components of $W \setminus C \cap W$. □

7.2 The spherical two-piece property

Definition 7.2.1 *A closed surface $M \subset S^3$ satisfies the sperical two piece property, S.T.P.P. if for any sphere Σ the difference $M \setminus (M \cap \Sigma)$ has at most two connected components. Such a surface is called taut*

In 1970 T.Banchoff proved the following theorem:

Theorem 7.2.2 *[Ban2] A surface embedded in S^3 satisfying the sherical two piece property is either a round sphere embedded or a Dupin cyclide, that is the conformal image of a torus of revolution of (complex) equation*

$$|z_1| = a, |z_2| = b; a^2 + b^2 = 1; (z_1, z_2) \in \mathbb{C}^2$$

Remark: The Dupin cyclides are in two different ways the envelopes of one dimensional families of spheres tangent to three spheres bounding three disjoint balls.

The proof of this theorem is analogous to the proof of Kuiper's result about tight immersions. One need to consider *spherical topsets* and *top spheres.*

Definition 7.2.3 *A sphere Σ is a top sphere if it bounds two balls B_1 and B_2 such that:*
-the interior of say, B_2 does not meet M
-both $\overline{B}_2$ and $\overline{B}_1$ do meet M.

We can weaken that definition:

Definition 7.2.4 *A sphere Σ is said to be a local topsphere of M at $m \in M$ if m belongs to $\Sigma \cap M$ and if m has a neighbourhood $U \subset M$ which is contained in one, say B_1 of the balls B_1 and B_2 of boundary Σ. If the neighbourhood $U \subset M$ can be chosen to intersect Σ only in m then we say that the sphere Σ is a strict local topsphere.*

Proposition 7.2.5 *A surface $M \subset S^3$ has the spherical two piece property if and only if every local topsphere is a topsphere.*

Proof: Suppose it is not the case, then there exists a point $q \in int(B_2)$. For a sphere Σ' tangent to M at m, but bounding a closed ball B'_1 which strictly contains B_1. It is a strict local topsphere of M at m, and the intersection $\overline{B}'_2 \cap M$ has at least two connected components, one reduced to m, and one containing q. A third sphere Σ" tangent in p to Σ', very close to Σ'_1 and contained in B'_1 contradicts the spherical two piece property. □

At a point m, we can consider the pencil of spheres tangent to M at m which, with the point m is a circle $\mathcal{P}(m)$. The support spheres of M form, if M is not a (round) sphere an interval of this pencil. Let us call $\Sigma^+(M,m)$ and $\Sigma^-(M,m)$ the boundary spheres of this interval. Applying this construction to nested neighbourhood $U_i \subset M; i \in \mathbb{N}$ such that $\cap_{i\in\mathbb{N}} U_i = m$ we get spheres $\Sigma_i^+(M,m)$ and $\Sigma_i^-(M,m)$ which converge to the two osculating spheres of M at m $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$. We can also define them using a stereographic projection of center different from m and the principal curvatures of $stereo(M)$ at $stereo(m)$. That last observation implies that, when $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$ are different, the intersection $\Sigma_1(M,m) \cap M$ is tangent to a line $L_1(m) \subset T_m M$ and the intersection $\Sigma_2(M,m) \cap M$ is tangent to a line $L_2(m) \subset T_m M$. We call those direction *principal directions*. A point where $\Sigma_1(M,m) = \Sigma_2(M,m)$ is called an *umbilic*.

Lemma 7.2.6 *If M is a taut smooth surface of S^3 then $\Sigma^+(M,m)$ and $\Sigma^-(M,m)$ coincide with $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$.*

Proof: The interval of $\mathcal{P}(m)$ containing the point sphere m and bounded by $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$ is in that case equal to the set of topspheres. □

We are ready to prove the:

Theorem 7.2.7 *A smooth taut surface embedded in S^3 is either a (round) sphere or a smooth torus*

Proof: **Remark:** A (round) circle of S^3 has the spherical two piece property.

If M has an umbilic m, then it "lies" between identical spheres $\Sigma_1(M,m) = \Sigma_2(M,m)$, and is therefore a sphere. If it does not have any umbilical point, then there exist to transverse line fields on M $L_1(m)$ and $L_2(m)$, as M is embedded in S^3 it is orientable, and therefore is a torus. □

Proposition 7.2.8 *A Dupin cyclide is taut.*

Proof: The enveloppe of a time like curve in Λ is a canal surface, union of the caracteristic circles of the family. The directions tangents to this family of circles are principal direction. A Dupin cyclide is in two different ways a canal surface, and therefore admits two transverse foliation by circles (tangent to the principal directions). the component of $M \setminus \Sigma$ are union of plaques of those two foliations. the circle two piece property applied to the leaves of the two foliation imply that they are cut in at most two intervals, and can match in at most two connected components. $\square$

Then an essential lemma is:

Lemma 7.2.9 *A spherical top set of a taut embedded torus M satisfies the circle two piece property.*

Proof: As before B_1 is the ball of boundary a topsphere Σ which contains M in its closure and B_2 the other ball of boundary Σ. If the topset does not satisfy the two piece property,in the topsphere Σ we can find a circle C which is a topcircle of $\Sigma \cap M$ such that the intersection $C \cap \partial\Sigma \cap M = C \cap M$ is not connected. As before the intersection $M \cap \Sigma$ is contained in $\overline{D}_1$ disc of boundary C, and the other disc D_2 of boundary C does not meet $M \cap \Sigma$. Chose a and c on different components of $C \cap M$ and b and d in different components of $C \setminus (C \cap M)$, in way those points are in cyclic order on C. Let γ be a geosdesic arc from b to d in D_2 and V a neighbourhood of γ in S^3 disjoint from M. Turning Σ around C we get a family Σ^t.We chose the rotation sign to leave γ out of the component, but small enough to garantee the existence of a continuous family of pathes γ^t joining a to c in $\Sigma^t \cap V$ (B_1^t obtained by continuity from B_1). Then the points a and c will be in different components of $B_2^t \cap M$,as there is no path connecting a and c in $M \cap C = M \cap \Sigma \cap \Sigma^t$, and as any path in the union of the hemispheres Σ^t containing the arcs γ^t joining a and b should cross V. Therefore, for t small enough, (with the right sign), Σ^t cuts M in at least three connected components. (This last argument is quite analogous to Kuiper's for tight surfaces). $\square$

Proposition 7.2.10 *If M is a taut torus in S^3 then for any topsphere Σ, $\Sigma \cap M$ is a point or a circle.*

Proof: We know by the previous proposition that the top set satisfies the circle two piece property. It cannot be Σ as M is a torus, nor contain interior points, which would be umbilical points of M, and imply again the equality $M = \Sigma$. The topset could apriori also be $\Sigma \setminus \{non\ finite\ family\ of\ round\ discs\}$. the boundary of those discs cannot bound a disc in M without contradicting tautness (consider a Poncelet pencil of spheres containing Σ), but then those boundary curve would be disjoint simple closed curves on M ; three disjoint simple curves on a torus always disconnect it into more than two pieces, so $M \cap int(B_1)$ would have at least two components. Moving Σ slightly into B_1 provides a sphere Σ' bounding a ball B_1' such that $M \cap \overline{B}_1'$ has at least two connected components. The only left possibilities are a point and a circle. $\square$

The interval of topspheres tangent at $m \in M$ to the taut torus M is bounded by the two osculating spheres at m, Σ_1 and Σ_2. Let us consider a sphere Σ tangent at m to M close to Σ_1 which is not a topsphere. It intersects M in the neighbourhood of m into two transverse arcs crossing at m the tangent of which are close to the principal direction $L_1 \subset T_m M$. Suppose that the intersection $\Sigma_1 \cap M$ is the point m. Chose a neighbourhood

$U \subset S^3$ of m Such that the intersection $M \cap U$ is a small disc. For non topsphere Σ^t tangent to M at m close enough to Σ_1 the intersection $\Sigma^t \cap M$ is contained in U. As, at m start four arcs of $\Sigma^t \cap M$ with distinct tangents, we can find two points p and q in $\Sigma^t \cap M$ such that any path from p to q in D_1^t passes through m. Chose in $U \cap \Sigma_1$ a very small circle σ centered at m, such that that the small disc $\delta_\sigma$it bounds does not contain any of the points p and q. In the pencil of spheres containing Σ_1 ,and following by continuity the ball B_1, some interval of spheres Σ^τ starting at $\Sigma_1$will be such that $\overline{B_1^\tau}$ contains p and q but does not contain m . For τ small enough and with the right sign, Σ^τ does not satisfy the two piece property. Then we can conclude that the osculting spheres intersect a taut torus M in circles. Those circles are necessarily lines of curvature, so M is a Dupin cyclide [Dar] . this ends the proof of the theorem listing taut surfaces in S^3.

8 Willmore functional, conformal length, conformal area

Let us now show that we can associate to a closed surface of S^3 a subset of Λ, the measure of which is a conformal invariant of the surface.

First we should mentionned the so-called *Willmore conjecture* (also the question was already asked by Thomsen around 1930). The following 2-form on a surface M embedded or immersed in S^3 is invariant by the action of the conformal group on S^3:

$$dw = (k_1 - k_2)^2 \cdot dv$$

where k_1 and k_2 are the principal curvatures and dv the area form of M.

The integral on M of this form:

$$W(M) = \int_M dw$$

is then a conformal invariant of the immersed surface. Looking first at revolution tori of equation

$$|z_1| = a, |z_2| = b; a^2 + b^2 = 1; (z_1, z_2) \in \mathbb{C}^2$$

Conjecture [Wil1] [Wil2] When M is a torus:

$$W(M) \geq 2\pi^2$$

The value of $W(M)$ can be interpreted as an area in the quadric Λ [Bry] . View M as embeddedd in $\mathbb{R}^3$. Consider at each point of $m \in M$ the sphere $\Sigma_b(m)$, tangent at m to M, and with mean curvature the mean curvature of M in M.

Remark: First observe that this is a conformal property, equivalent to impose that the intersection curves of M and $\Sigma_b(m)$ intersect at right angles in m. There may be an inequality linking $W(M)$ and the measure of the spheres with non trivial intersection with M.

Proof: We can write local equations of M and $\Sigma_b(m)$ in the neighbourhood of m, using axis tangent to the principal directions of M at m and to the normal to M at m (k_1 and k_2 are the principal curvatures of M at m) :

$$z = \frac{1}{2}[k_1 x^2 + k_2 y^2] + \textit{higher order}$$

$$z=\frac{1}{2}[\frac{k_1+k_2}{2}x^2+\frac{k_1+k_2}{2}y^2]+\textit{higher order}$$

The equation of the intersection is

$$\frac{k_2-k_1}{2}x^2+\frac{k_1-k_2}{2}y^2+\textit{higher order}\ =0$$

proving that the projection of the intersection $M\cap\Sigma_b(m)$ on the tangent plane is, in the neighbourhood of m, two curves intersecting at m with right angles.

□

This defines a map $\gamma_b : M \to \Lambda$. The image of this map is a surface which is space-like (that is the tangent plane to it, in every point, contains only space-like non-zero vectors).

The measure of the image of this map is then again $W(M)$.

This conjecture has proved to be particulary rich in connection with other problems see [Wil1] [Wil2] [Li-Ya].

Theorem 8.0.11 *Let M be a two dimensional flat torus in $\mathbb{R}^3$ with lattice generated by $\{(0,1),(x,y)\}$ where $0\leq x\leq \frac{1}{2}$ and $\sqrt{1-x^2}\leq y\leq 1$, then*

$$\int_M |H^2|\geq 2\pi^2$$

8.1 Conformal structures on tori.

First recall the possible conformal structures on a torus.[Bri-Kno] and [Jo-Si] .

Let ω be a complex number with positive imaginary part, and let Γ_ω be the lattice in $\mathbb{C}$ consisting of all complex numbers $n+m\omega$, where m and n are integers. Then $\mathbb{C}/\Gamma_\omega$ has the structure of a one-dimensional complex manifold. As a topological manifold, $\mathbb{C}/\Gamma_\omega$ is homeomorphic to $S^1\times S^1$. We are interested in the possible conformal structures. Then (z_1,z_2) and $(az_1,az_2,a\in\mathbb{C}^*)$ define the same conformal structure. Chosing the proper orientation we may also chose between $(1,\omega)$ and $(1,\overline{\omega})$. Brieskorn and Knörrer's book contains also an algebraic geometry interpretation; they prove that all complex tori can be interpretated as a cubics. One can prove (see [Jo-Si] p.273) that the space of moduli describing the conformal structures on a torus is a quotient of:

$$\mathcal{D}=\{\omega,\ |\omega|\geq 1,\ |Re\omega|\leq 1/2\}$$

identifying the corresponding sides by the maps $z\to z+1$, $z\to -1/z$.

Allowing the reversing orientation conformal map $z\to -1/\overline{z}$ one reduces the modulus to:

$$\{\omega,\ |\omega|\geq 1,\ 0\leq Re\omega\leq 1/2\}$$

8.2 Conformal area, conformal length.

Definition 8.2.1 *Let M be a surface or a curve embedded in S^3. Define*

$$Volconf(M)=sup_{g\in\mathcal{M}}vol(g(M))$$

where $\mathcal{M}$ is, as before the Möbius group of conformal diffeomorphisms of S^3.

Proposition 8.2.2

- *The conformal length of a curve in at least 2π, the length of a geodesic circle.*
- *The conformal area of a surface is at least 4π, the area of a geodesic sphere.*
- *The conformal area of an immersed surface admitting double points is at least 8π.*
- *The conformal area of an immersed surface admitting triple points is at least 12π.*

Let $proj$ be the steoreographic projection with "south pole" a point m of M. The maps $(proj)^{-1} \bullet \mathcal{H}_R \bullet proj$, when $R \to \infty$ expand, up to cover almost all the sphere, any small neigbourhood of m, and expand the small arc or disc of M contained in that neighbourhood up to a curve or surface very close to a geodesic circle or sphere.

Proposition 8.2.3 *If M is a minimal (zero mean curvature) tori embeddeded in S^3 then $vol(M) = Volconf(M)$*

We use again the relation between the extrinsinc Gaussian curvature K_{ext} and the intrinsic Gaussian curvature K_{int} of a surface in S^3: $K_{int} = 1 + K_{ext}$ and the fact that in R^3or S^3, the 2-form $(k_1 - k_2)^2 dv$ and its integral on M are conformally invariant. Denote also H_S the mean curvature of this surface $M \in S^3$.

Let $proj$ be again a stereographic projection. Denote by H and K the mean curvature and Gauss curvature of $proj(M) \in \mathbb{R}^3$, and by k_1 and k_2 its principal curvatures. Then the quantity

$$\int_{proj(M)} (k_1 - k_2)^2 dv = \int_{proj(M)} H^2 - K$$

is equal to:

$$\int_M (H_S)^2 - K_{ext} = \int_M (H_S)^2 - K_{int} + 1 = Vol(M)$$

as M is minimal and is a torus. Using again the fact M is a torus and Gauss Bonnet theorem, one gets that $\int_{proj(M)} H^2$ is a conformal invariant, and therefore that

$$\int_{proj(M)} H^2 = \int_{g(M)} H_S^2 + vol(g(M), \forall g \in \mathcal{M}$$

Then

$$Volconf(M) \leq vol(M)$$

The other inequality is immediate.

The study of the first eigenvalue of the Laplacian on M [Li-Ya], implies the inequality:

Proposition 8.2.4 *A flat torus with lattice generated by $\{(1,0),(x,y)\}$ where $0 \leq x \leq \frac{1}{2}$ and $\sqrt{1-x^2} \leq y \leq 1$ satisfies:*

$$Volconf(M) \geq 2\pi^2$$

The theorem now follows directly from the inequality: $\int_M H_S^2 + 1 \geq \int_M 1 = vol(M)$ applied to a sequence of embedding the volume of which approximates the conformal volume.

References

[Ban2] Thomas F. Banchoff. *The spherical two-piece property and tight surfaces in spheres,* Journal of differential geometry 4 (1970) p 193-205.

[Bo-Fe] T.Bonnesen und W.Fenchel. *Theorie de konvexen Körper.* Springer (1934).

[Bri-Kno] E. Brieskorn and H. Knörrer. Plane algebraic curves. Birkhäuser (1986).

[Bry] R.Bryant. A duality theorem for Willmore surfaces, Journal of differential geometry 20 (1984) p 23-53.

[Bu] Georges Louis Leclerc, comte de Buffon. *Essai d'arithmétique morale.* (1777) 4^o volume des suppléments de l'édition *in quarto* de l'imprimerie royale (France).

[doCa] M.do Carmo.*Differential geometry of curves and surfaces.* Prentice Hall (1976).

[Cau] A.L. Cauchy. *Mémoire sur la rectification des courbes et la quadratures des surfaces courbes.* Mémoires des l'académie des sciences Paris 22 (1850)p 3-15.

[Ch-La] S.S.Chern and R.K.Lashof. *On the total curvature of immersed manifolds I and II.* American journal of mathematics 79 (1957) and Michigan mathematical journal 5 (1958).

[Cro] W. Crofton.*On the theory of local probability.* Phil. Trans. of the Royal soc. London 158 (1868)p 181-199.

[Dar] G. Darboux.*Leçons sur la théorie générale des surfaces.* Gauthier-Villars Paris (1887).

[dCa] M. do Carmo. *Differential geometry of curves and surfaces.* Prentice Hall (1976).

[dCa2] M. do Carmo. *Riemannian geometry.*

[Far] I. Fary. Sur la courbure totale d'une courbe gauche faisant un noeud, Bulletin de la société mathématique de France 77 (1949)p.128-138.

[Fe1] W. Fenchel. Uber Krümmung und Windung geschlössener Raum Kurven. Math. Ann. 101 (1929).

[Fe2] W. Fenchel. On total curvature of riemannan manifolds I; Journal of London mathematical society N^o15 (1940).

[F-H-W] M.H.Freedman, Z-X.He and Z.Wang. Möbius energy of knots and unknots, Annals of mathematics 139 (1994) p 1-50.

[Ho] E. Hopf. *Differential geometry in the large.* Springer lecture notes N^o 1000, second edition (1989).

[Jo-Si] G.Jones and D. Singerman. Complex fonctions (an algebraic and geometric viewpoint). Cambridge University Press (1987).

[La] R. Langevin.*Introduction to integral geometry*, notes prepared for the 21th brazilian coloquim of mathematics at IMPA , july 1997, publication of IMPA Rio de Janeiro, Brazil.

[La-O'H] R. Langevin and J. O'Hara. *Conformally invariant energies of knots*, manuscript.

[Li-Ya] P.Li and S.T.Yau. *A new conformal invariant and its application to the willmore conjecture and the first eigenvalue of compact surfaces*, Inventiones math. 69 (1982) p 269-291.

[Mil1] J.Milnor.*On total curvature of knots.* Annals of mathematics 52 (1950) p.248-260.

[Mil4] J.Milnor.*On total curvature of closed space curves.* Math. Scand 1, p.248-260 (1953).

[New2] Isaac Newton. **trouver la reference du polygone de Newton**

[Ru] Walter Rudin. *Real and complex analysis.* Mac Graw Hill (1970).

[Sa1] L.A. Santalò. *Introduction à la géométrie intégrale.* Paris (1951).

[Sa2] L.A. Santalò. *Integral geometry and geometric probability.* Encyclopedia of mathematics and its applications. Addison Wesley (1976).

[Wil1] T.J.Willmore. Total curvature in riemannian geometry, (1982) Ellis Horwood Series John Wiley and Sons.

[Wil2] T.J.Willmore. Riemannian geometry, Clarendon press Oxford (1993)