A Nonuniform Helical Equilibria of Nonisotropic Rods

We assume a uniform, hyperelastic rod, i.e., there exists a convex, coercive strain-energy density function $W(\mathbf{w})$ with $W_{\mathbf{w}}(0) = 0$ that generates the constitutive relations

$$\mathbf{m} = \partial_{\mathbf{u}} W(\mathbf{u} - \hat{\mathbf{u}}). \qquad \textbf{[A.1]}$$

Here the constants $\hat{\mathbf{u}}$ are the strains in the uniform, unstressed reference configuration at which $\mathbf{m} = \mathbf{0}$. As W is convex and coercive, it has a unique convex, coercive Legendre transform or conjugate function

$$W^*(\mathbf{x}) = \max_{\mathbf{w}} \{\mathbf{x} \cdot \mathbf{w} - W(\mathbf{w})\}. \qquad \textbf{[A.2]}$$

In terms of this conjugate function, the constitutive relations given by Eq. **A.1** can be inverted to yield

$$\mathbf{u} = \hat{\mathbf{u}} + \partial_{\mathbf{m}} W^*(\mathbf{m}). \qquad \textbf{[A.3]}$$

We now make extensive use of the symmetries and associated integrals of the equilibrium equations in the absence of body loads. When the rod is uniform the Hamiltonian function

$$W^*(\mathbf{m}) + \mathbf{m} \cdot \hat{\mathbf{u}} + \mathbf{n} \cdot \mathbf{d}_3 \qquad \textbf{[A.4]}$$

is constant along equilibria (see ref. 1). When the rod is isotropic the tangential twist $m_3 = \boldsymbol{m} \cdot \boldsymbol{d}_3$ is another integral, which is associated through Noether's Theorem with invariance of the Hamiltonian **A.4** to the angle ψ in moments of the form $\mathbf{m} = (M \sin\psi, M \cos\psi, m_3)$ for arbitrary values of M and m_3, i.e., at all points in $\mathbf{m}$-space. In the absence of distributed body loadings, $\boldsymbol{f} \equiv \mathbf{0} \equiv \boldsymbol{l}$, the equilibrium equations (given in the main text by Eq. **6**) have the integrals

$$\boldsymbol{n} = \boldsymbol{c}_1, \qquad \boldsymbol{m} + \boldsymbol{r} \times \boldsymbol{n} = \boldsymbol{c}_2, \qquad \textbf{[A.5]}$$

where $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are constant vectors. And if $\boldsymbol{c}_1 \neq \mathbf{0}$, there is a shift of origin such that the second integral in Eq. **A.5** can be rewritten

$$\boldsymbol{m} + (\boldsymbol{r} - \boldsymbol{r}_0) \times \boldsymbol{n} = \alpha \boldsymbol{n} \qquad \textbf{[A.6]}$$

for some scalar α.

If the resultant forces $\boldsymbol{n}$ vanish identically, the Hamiltonian system becomes completely integrable, and the existence of nonuniform helical solutions can be analyzed directly. Thus, the only nontrivial case is that of nonzero force. We assume that an equilibrium with $\boldsymbol{n} \neq \mathbf{0}$ has a helical centerline that is either of infinite extent or could be continued to be of infinite extent. Then, the force $\boldsymbol{n}$ must be parallel to the axis of the helix. This conclusion follows from the following two observations (1,2): First, as the rod is uniform, the function **A.4** is constant in s, which implies that $|\boldsymbol{m}|$ is bounded, because W^* is coercive (2). Second, as $|\boldsymbol{m}|$ is bounded, the integral **A.5** implies that $|\boldsymbol{r} \times \boldsymbol{n}|$ is bounded, so that for an infinite helix $\boldsymbol{n}$ must be parallel to the axis $\boldsymbol{e}_3$. Once the direction of $\boldsymbol{n}$ is known there is a simple expression for the moment $\boldsymbol{m}$. We consider helical centerlines of the form

$$\boldsymbol{r}(s) = r\cos(\eta_1\, s)\, \boldsymbol{e}_1 + r\sin(\eta_1\, s)\boldsymbol{e}_2 + p\, \eta_1\, s\, \boldsymbol{e}_3, \qquad \textbf{[A.7]}$$

where $\{\boldsymbol{e}_i\}$ is a fixed basis, r is the *radius* of the helix, $2\pi p$ is the *pitch*, and $\eta_1 = 1/\sqrt{r^2+p^2}$. On these helices, the integral **A.6** takes the particular form $\boldsymbol{m} - r\boldsymbol{\nu}\times\boldsymbol{n} = \alpha\boldsymbol{n}$. Taking into account the fact that the principal normal $\boldsymbol{\nu}$ is perpendicular to $\boldsymbol{e}_3$, this integral may be rewritten as

$$\boldsymbol{m} = r\,N\,\boldsymbol{\nu}\times\boldsymbol{e}_3 + N\,\alpha\,\boldsymbol{e}_3, \qquad \textbf{[A.8]}$$

or

$$\boldsymbol{m}(s) = m_\beta\boldsymbol{\beta}(s) + m_3\boldsymbol{\tau}(s), \qquad \textbf{[A.9]}$$

where the components m_β and $m_3 \equiv m_\tau$ are constant

$$m_\tau = \boldsymbol{m}\cdot\boldsymbol{\tau} = \frac{N}{\sqrt{\kappa^2+\tau^2}}\left(\frac{\kappa^2}{\kappa^2+\tau^2} + \alpha\tau\right), \qquad \textbf{[A.10a]}$$

$$m_\beta = \boldsymbol{m}\cdot\boldsymbol{\beta} = \frac{N}{\sqrt{\kappa^2+\tau^2}}\left(\frac{\kappa\tau}{\kappa^2+\tau^2} + \alpha\kappa\right), \qquad \textbf{[A.10b]}$$

$$m_\nu = \boldsymbol{m}\cdot\boldsymbol{\nu} = 0. \qquad \textbf{[A.10c]}$$

In particular as $\boldsymbol{m}\cdot\boldsymbol{\nu} = 0$ the moment $\boldsymbol{m}$ lies in the $(\boldsymbol{\beta},\boldsymbol{\tau})$ or rectifying plane to the helix. Now the components in the director frame can be computed

$$m_1 = \boldsymbol{m}\cdot\boldsymbol{d}_1 = m_\beta\sin\varphi, \qquad \textbf{[A.11a]}$$

$$m_2 = \boldsymbol{m}\cdot\boldsymbol{d}_2 = m_\beta\cos\varphi, \qquad \textbf{[A.11b]}$$

$$m_3 = \boldsymbol{m}\cdot\boldsymbol{d}_3 = m_\tau. \qquad \textbf{[A.11c]}$$

Consequently, constitutive relations **A.3** can be re-written

$$0 = -\kappa\sin\varphi + \hat{u}_1 + \partial_{m_1}\bar{W}^*, \qquad \textbf{[A.12a]}$$

$$0 = -\kappa\cos\varphi + \hat{u}_2 + \partial_{m_2}\bar{W}^*, \qquad \textbf{[A.12b]}$$

$$\varphi' = -\tau \qquad + \hat{u}_3 + \partial_{m_3}\bar{W}^*, \qquad \textbf{[A.12c]}$$

where $\bar{W}^* := W^*(m_\beta\sin\varphi, m_\beta\cos\varphi, m_3)$. For given κ and τ and for given constitutive parameters $\hat{\mathbf{u}}$, and function W^*, the system **A.12** is to be solved for constants m_3 and m_β and register function $\varphi(s)$. From the existence of helical equilibria, we know that there are at least two solutions with $\varphi(s)$ constant, and these are the uniform helical solutions. Nonuniform helical equilibria exist when there are solutions with $\varphi(s)$ non-constant, and such solutions exist whenever Eqs. **A.12 a** and **b** have a family of non-isolated solutions of the form $(\bar{\varphi}\pm\delta, m_\beta, m_3)$. Solutions of **A.12c** with $\varphi'(s) \neq 0$ require such a family, and if there is such a family, then **A.12c** is an ordinary differential equation to be integrated for $\varphi(s)$ for any value of τ and with initial data $\varphi(0) = \bar{\varphi}$, and the solution will give a non-constant register except for the one critical value of the torsion that makes the right-hand side vanish.

Thus the existence of non-uniform helical equilibria is reduced to the study of the two algebraic Eqs. **A.12 a** and **b** . But a solvability condition for Eqs. **A.12 a** and **b** is

$$0 = -\cos\varphi\,\hat{u}_1 - \cos\varphi\,\partial_{m_1}\bar{W}^* + \sin\varphi\,\hat{u}_2 + \sin\varphi\,\partial_{m_2}\bar{W}^*, \qquad \textbf{[A.13]}$$

(for $m_\beta \neq 0$) and **A.13** is the statement that the Hamiltonian **A.4** is invariant under changes in φ for arguments of the form $\mathbf{m} = (m_\beta\sin\varphi, m_\beta\cos\varphi, m_3)$. Contrariwise if the solvability condition **A.13** is satisfied on an interval $\bar{\varphi}\pm\delta$ then a modified version of Eqs **A.12 a** and **b** have a solution on the same interval with κ replaced by a scalar function $f(\varphi)$ satisfying

$$f(\varphi) = \sin\varphi\,\hat{u}_1 + \sin\varphi\,\partial_{m_1}\bar{W}^* + \cos\varphi\,\hat{u}_2 + \cos\varphi\,\partial_{m_2}\bar{W}^*, \qquad \textbf{[A.14]}$$

and to recover a solution to the original system we need that $f(\varphi)$ is in fact a positive constant that can be taken as κ. Differentiation of Eq. **A.14**, and use of Eq. **A.13** reveals that f is independent of φ precisely if a mixed second partial derivative vanishes

$$\partial_{m_\beta}\partial_\varphi \bar{W}^* = 0, \qquad \textbf{[A.15]}$$

which is another symmetry condition on the Hamiltonian given by Eq. **A.4**. And f can be shown to be positive as a consequence of the vanishing of the second derivative of the Hamiltonian with respect to φ taken with Eq. **A.13** and convexity of W^*, specifically

$$\kappa = f = m_\beta \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix} \cdot \begin{pmatrix} \partial^2_{m_1 m_1} W^* & \partial^2_{m_1 m_2} W^* \\ \partial^2_{m_2 m_1} W^* & \partial^2_{m_2 m_2} W^* \end{pmatrix} \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix}. \qquad \textbf{[A.16]}$$

Thus, we conclude that a nonuniform helical equilibrium (with force $\boldsymbol{n}$ parallel to the axis of the helix) exists if and only if there is an arc $\mathbf{m} = (m_\beta \sin\varphi, m_\beta \cos\varphi, m_3)$ in moment or $\mathbf{m}$-space with $\varphi \in \bar{\varphi} \pm \delta$, along which the Hamiltonian in Eq. **A.4** is constant, and the second derivative in Eq. **A.15** also vanishes. When both of these conditions are met, there is a nonuniform helical equilibrium with curvature given by Eq. **A.16**, and for any torsion except that for which the right-hand side of Eq. **A.12c** vanishes, which would be one of a family of non-isolated uniform helical equilibria.

In the case of isotropic rods, moment space is completely filled by arcs of the appropriate type, so the negation of the existence of any such arc is a strong condition forbidding even a local form of isotropy anywhere in $\mathbf{m}$-space. In the particular case of a quadratic strain-energy density function with coefficient matrix of the general form

$$\boldsymbol{K} = \begin{pmatrix} K_1 & 0 & K_{13} \\ 0 & K_2 & K_{23} \\ K_{13} & K_{23} & K_3 \end{pmatrix}, \qquad K_1 \le K_2, \qquad \textbf{[A.17]}$$

no appropriate arc exists unless

$$K_1 < K_2, \quad K_{13} = 0, \quad K_{23} = \sqrt{K_3(K_2 - K_1)}, \quad \hat{u}_1 = 0. \qquad \textbf{[A.18]}$$

In this case

$$\boldsymbol{K}^{-1} = \begin{pmatrix} \frac{1}{K_1} & 0 & 0 \\ 0 & \frac{1}{K_1} & -\frac{\sqrt{K_3(K_2-K_1)}}{K_1 K_3} \\ 0 & -\frac{\sqrt{K_3(K_2-K_1)}}{K_1 K_3} & \frac{K_2}{K_1 K_3} \end{pmatrix}, \qquad \textbf{[A.19]}$$

and circles giving nonuniform helical equilibria fill the

$$m_3 = \frac{K_1 K_3 \hat{u}_2}{\sqrt{K_3(K_2 - K_1)}} \qquad \textbf{[A.20]}$$

plane.

As an example, we consider the case where the strain energy takes the particular form

$$W(u_1, u_2, u_3) = {u_1}^2 + 26\,(u_2 - 1)^2 + 10\,(u_2 - 1)\,u_3 + {u_3}^2. \qquad \textbf{[A.21]}$$

Then it can be verified that there is the nonuniform helical equilibrium, see Fig. 5, with $\kappa = \sqrt{3}/2, \tau = 1/2$ and register

$$\varphi(s) = 2\ \arctan\left(\frac{\sqrt{334}\tan\left(\frac{s\sqrt{334}}{20}\right)}{47 + 25\sqrt{3}} \right). \qquad \textbf{[A.22]}$$

[1] Kehrbaum, S. & Maddocks J. H. (1997) *Philos. Trans. R. Soc. London* **355**, 2117–2136.
[2] Kehrbaum, S. (1997) Ph. D. thesis, (University of Maryland, College Park).