MATHEMATICS

Around the Möbius band

A newly derived set of differential equations provides a numerical solution to the classic question of predicting the shape of a Möbius strip.

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One of the classic problems in mechanics is to find the shape assumed by a Möbius strip — the famous band that is closed with a half-twist and which has the intriguing topological property that it only has one side (Fig. 1). This abstract mathematical question, dating back to at least 1930 (refs 1,2), is also of practical scientific interest as single crystals in the form of a Möbius band have now been grown[3,4]. On page 563 of this issue, Starostin and van der Heijden derive a new set of differential equations governing the elastic equilibrium shapes of developable strips, which allow numerical computation of the geometry of Möbius bands[5].

To explain the notion of a developable strip it will be useful to arm oneself with a few sheets of plastic overhead transparencies (A4 size is ideal), a ruler, an appropriate pen, a pair of scissors and Sellotape. A narrow rectangular strip of about 3 cm width should be cut from one of the long edges of the transparency, a straight line drawn down the centre of the strip, and a closed band formed by bending (with, for the moment, no twisting) and the ends overlapped and sealed with the Sellotape. When released, the band should adopt a shape in which the centre line is almost exactly circular, with the band itself forming part of the walls of a circular cylinder.

This circular shape is not pre-ordained; the centre line can be deformed substantially away from a circle to form planar or non-planar closed curves. There is clearly a strong coupling between the shape of the centre line and the configuration of the entire strip. It is the elastic properties of the band that select the circular centre line as the equilibrium configuration.

There is a two-dimensional elasticity theory that can describe deformations of stretch and shear of, for example, a tablecloth that always remains in the two- dimensional plane of the tabletop.

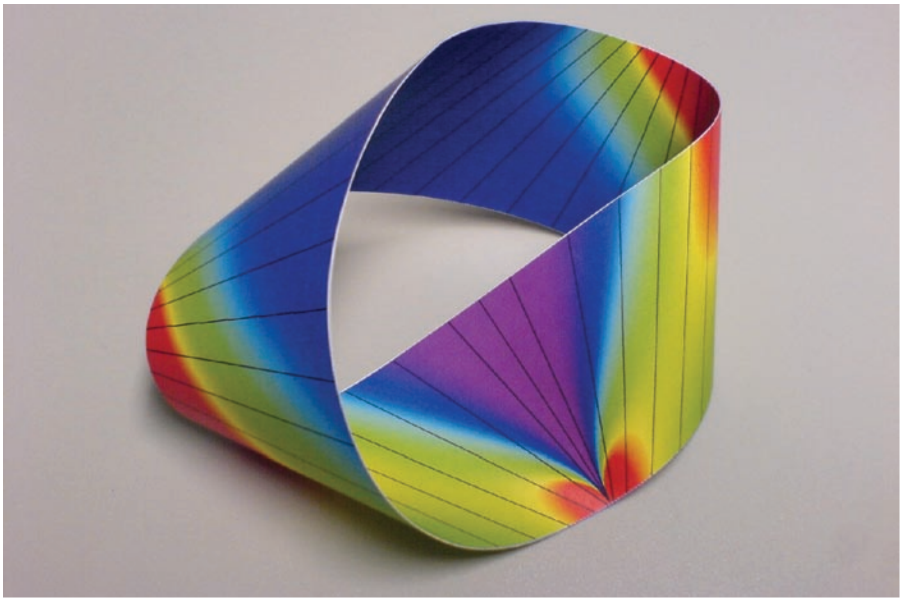

Figure 1 Photograph of a Möbius strip[5].

There are also elasticity theories that describe two-dimensional objects embedded in three-dimensional space, for example the shape of the surface of an inflated balloon.

It seems that the problem of finding the shape of a strip is of this last form. But there is another crucial feature: the overhead transparency cannot be significantly deformed while keeping it flat on a tabletop. Unlike a piece of cloth or a rubber balloon, stretch and shear of the transparency are negligible, and significant deformation can only be obtained by bending the sheet into three-dimensions to form what is, by definition, a developable surface.

It can be shown (see, for example, ref. 6) that any smooth developable surface is generated by a one-parameter family of straight lines, and further that if the three-dimensional image of a reference curve on the surface — for example the centre line of our rectangular strip — is known, then the entire configuration of the surface is known. This fact has been used to write the elastic energy of the entire developable surface in terms of the curvature and torsion of the reference curve[7]. Starostin and van der Heijden then use mathematical machinery developed in ref. 8 to derive the equilibrium equations that express the stationarity of this elastic energy. In our experiment with the untwisted strip, the special property of the circular centre-line is that it is a — presumably minimum energy — solution of these equilibrium equations. In this case the straight lines generating the cylindrical surface are all parallel and perpendicular to the circular centre line.

If the above experiment is repeated, but with the imposition of a half-twist before closing the ribbon, the image of the centre line becomes a non-planar curve. There is no explicit expression known for this shape, but Starostin and van der Heijden have computed a numerical solution of their equilibrium conditions for which the associated developable surface is a Möbius band. Now the family of straight lines generating the surface are not all parallel, and have varying angles with the centre line.

There remains one free parameter of significance, namely the width of the rectangular strip compared with its length. The Starostin–van der Heijden computations demonstrate that the shape of the centre line depends quite sensitively on this aspect ratio, and the transparency strip can easily be used to lend physical credence to these computations.

How large can the width be? For the untwisted loop there is no restriction. However, an entire A4 sheet cannot be smoothly closed into a Möbius band. In addition to this self-avoidance condition, there is a second, more subtle, width restriction. Unless the family of generators of the developable surface are all parallel, as in the cylindrical case of our first experiment, it can be shown that they generate a singularity, usually a curve called the edge of regression. The image of the centre line and the width must be such that the edge of regression lies outside the physical strip. The Starostin–van der Heijden computations reveal that as the maximal width is approached there are localized regions of high deformation. A fuller understanding of both types of width restriction presents a number of interesting, and largely open, mathematical questions related to understanding the shapes of developable strips. Indeed the study of the analogous, but simpler, self-avoidance conditions for tubes with circular cross-sections is a subject of contemporary interest both in modelling biological macromolecules[9,10] and in explaining why a telephone hand-set cord frequently exhibits regions of both left- and right- handed helices[11].

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ARTIFICIAL TONGUES

Tastes good to me

The mammalian sense of taste has an exquisite ability to differentiate subtle variations in flavour. An artificial tongue has now been developed with the ability to amplify and sense analytes that before may have gone unnoticed.

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Sweet, salty, sour and bitter — some also include umami (savoury) — are the basic tastes that the mammalian tongue can classify. For the past couple of decades researchers have attempted to mimic this process to develop what many refer to as an 'electronic tongue.' These multichannel taste sensors most often derive from arrays of small molecules or polymers that respond in some way to the desired targets. Such sensor arrays are commonly impregnated in thin films, attached to polymeric beads, or simply used in solution. Artificial tongues have been designed to identify specific analytes in complex mixtures and also to define general taste quality. Moreover, they can be trained to test for non-food tastes such as finding differences in blood or saliva samples, and for substances such as toxins, chemical warfare agents and bacteria[1–8]. Even with the advances to date, there are still problems in identifying certain elusive analytes while maintaining a breadth of substances that can be identified with efficient selectivity and sensitivity.

On page 576 of this issue, Matile and co-workers describe a 'tongue' platform consisting of a synthetic multifunctional pore system that is capable of identifying many of the traditional flavour components in complex mixtures[9]. Their analysis uses a process that is analogous to the mammalian sense of taste, in that both involve transport through transmembrane pores. The researchers develop a means to amplify the response for analytes that are difficult to detect, with the response modifying the release of a dye that allows for the detection of the targeted compounds by the naked eye.

In earlier work[10], Matile's group created a synthetic pore for broad-based sensing. The stimuli-responsive pores were built from synthetic rigid aromatic supports, held together by short cationic peptides, to provide a pore interior to which anionic, aromatic analytes can bind. The basic method hinges on incorporating these pores into the walls (membranes) of dye-containing vesicles. The sensing chemistry is on the outside of the vesicle, separated from the signalling dye, the release of which relies on the ability of the pore to transduce the signal across the membrane. The researchers used the pore with an enzyme that defined selectivity for specific analytes, and developed a system that was able to sense saccharides in soft drinks by monitoring the consumption of ATP — the energy source used in such enzymatic reactions — making the approach adaptable for monitoring other such routes. If the desired target is present in the sample a series of enzymatic and chemical reactions take place and the pore transmits this information to the interior of the vesicle, modulating the release of the dye, thus allowing the analyte to be detected by eye.

The current work by Matile and co-workers[9] extends this chemistry by incorporating 'reactive amplification' to detect analytes that would normally be missed by the universal pore (Fig. 1). Unlike most efforts to enhance detection that typically focus on improving the host, in this elegant approach the binding ability of the analyte is boosted instead. Reactive amplification introduces secondary signalling molecules (Fig. 1b) that react with functional groups created on the analytes during the reactions with enzymes (Fig. 1a), covalently modifying these elusive targets (Fig. 1b), and producing compounds that can react with the pores more efficiently (Fig. 1c) than the original analyte, thereby amplifying the pore response. It is possible to carry out these functionalization reactions *in situ*, producing more potent signal generators from these otherwise undetected analytes. Matile *et al.*