Mathematical Models and Methods in Applied Sciences
Vol. 19, No. 6 (2009) 911–937

Doi: 10.1142/S0218202509003668

A 1D MODEL FOR A CLOSED RIGID FILAMENT IN STOKES FLOW

PH. CAUSSIGNAC

LCVMM IMB SB EPFL, Station 8,
CH-1015 Lausanne, Switzerland

Received 3 June 2008
Revised 3 July 2008
Communicated by N. Bellomo

We adapt an existing asymptotic method to set up a one-dimensional model for the fall of a closed filament in an infinite fluid in the Stokes regime. Starting from the single-layer integral representation of the fluid velocity around the filament, we get, for a very slender filament, a Fredholm integral equation on the filament centerline. From this equation, we can compute the drag and force acting on the filament and consequently the resistance matrix. The integral equation is discretized with a collocation method. The study of a scalar model problem yields existence and uniqueness results together with an error estimate for the discretization scheme. Then, we compare the resistance matrix of thin ideal knots obtained from the discretization of the present model to a boundary element method; numerical convergence results and a good agreement of both methods validate our model.

Keywords: Knots sedimentation; 1D-model; Fredholm integral equation.

AMS Subject Classification: 74K10, 74F10, 65R20

1. Introduction

The free fall of a rigid body under gravity in a Stokes fluid can be described by the so-called resistance matrix,[7] the entries of which are coefficients in the equation of motion of the solid. This can be done for the steady state and also for the quasi-static approximation of the fluid.[3] The resistance matrix is obtained from the force and torque of six elementary Stokes problems in the (infinite) domain exterior to the solid; these problems are completely decoupled from the equations for the motion of the solid. Hence, a boundary element method for solving these Stokes problems numerically is very natural; but, if the solid is very slender, it can become expensive and inaccurate. In this case, it is preferable to set up a model in which the solid is one-dimensional (a curve). The popular Rotne–Prager process[20,10] belongs to this kind of models: the solid is represented by a sequence of overlapping beads. For slender bodies, such as tubes with cylindrical cross-section, one can postulate a density distribution of Stokeslets and Rotlets on the centerline for representing the fluid velocity; then, a process of matched asymptotic expansion gives, via an integral equation, an approximation of the force and torque. This kind of models has first been developed by Keller and Rubinow,[14] and then improved by Johnson[13]; a good description can be found in Ref. 8, where such a model has been applied to the growth of fibers. Although these models have been designed for non-closed fibers, they can be used in the case of closed filaments; but, the computations leading to integral equations are quite complicated. Other singularity methods have been used to simulate long filaments or fibers and their interactions, see for example Ref. 16; the linear relationship between the fluid velocity and the force obtained in matched asymptotic expansion models has been used, by postulating the form of the force, for simulating flexible fibers in Ref. 22 and interaction between them in Ref. 24.

Another class of models is obtained by a direct asymptotic expansion on the boundary integral representation of the fluid velocity[11,21]; they are used in general to compute analytically some first few terms representing the velocity of the fluid around straight or slightly curved slender bodies. But, at least to our knowledge, they were not used in the case of closed curves. The purpose of this paper is to set up an asymptotic expansion of the latter kind for closed filaments and to develop a numerical method for computing the density in the one-dimensional integral representation for the fluid velocity. Numerical examples are given for the resistance matrix of ideal knots.

The paper is organized as follows. In Sec. 2, we present the geometry and some hypotheses leading to the one-dimensional problem. Section 3 is devoted to the asymptotic expansion of integrals and to the final integral equation formulation; the Hadamard finite part integral is briefly described in Appendix A and some detailed computations of the asymptotic expansion are done in Appendix B. The discretization of the one-dimensional equation is presented in Sec. 4. In Sec. 5, we study the model problem of the circular torus; the proofs of the theorems are given in Appendix C. Finally, numerical results for knotted tubes are given in Sec. 6 and compared to results obtained with a boundary element method.

2. Notations and Basic Assumptions

The filament is a closed rigid slender body defined as a curved cylinder with circular cross-section of constant radius ε; its centerline Γ is defined by a smooth closed curve, parametrized by its normalized arc length s:

$$s \in [0,1] \mapsto \mathbf{r}(s) \in \mathbb{R}^3, \quad \mathbf{r}(0) = \mathbf{r}(1); \tag{2.1}$$

when necessary, we also denote by $\mathbf{r}$ the 1-periodic extension of this function to $\mathbb{R}$. The usual Frénet frame is defined by the unit vectors:

$$\mathbf{t} = \mathbf{r}', \quad \mathbf{n} = \frac{\mathbf{r}''}{|\mathbf{r}''|}, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}. \tag{2.2}$$

The surface of the filament is parametrized by:

$$\Sigma_\varepsilon = \{\mathbf{y}(s,\varphi) = \mathbf{r}(s) + \varepsilon\cos\varphi\mathbf{n}(s) + \varepsilon\sin\varphi\mathbf{b}(s) \in \mathbb{R}^3; 0 \le \varphi \le 2\pi, s \in [0,1]\}. \quad (2.3)$$

The slenderness property consists in assuming that the cross-section radius is much smaller than the curve length, that is, $\varepsilon \ll 1$.

The starting point for establishing our model is the single-layer representation of the displacement velocity $\mathbf{v}$ of the fluid surrounding the filament in the Stokes regime[15] (we use the summation convention on repeated indices):

$$\mathbf{v}(\mathbf{x}) = -\int_{\Sigma_\varepsilon} \mathbf{U}^i(\mathbf{x}-\mathbf{y})\psi_i(\mathbf{y})dS(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3, \quad (2.4)$$

with the Stokeslets $\mathbf{U}^i$ given, in cartesian coordinates, by:

$$U_j^i(\mathbf{x}) = \frac{1}{8\pi\nu}G_j^i(\mathbf{x}), \quad G_j^i(\mathbf{x}) = \frac{1}{|\mathbf{x}|}\left(\delta_{ij} + \frac{x_i x_j}{|\mathbf{x}|^2}\right). \quad (2.5)$$

The unknown density $\boldsymbol{\psi} : \Sigma_\varepsilon \to \mathbb{R}^3$ is determined by solving the following Fredholm integral equation:

$$\mathbf{v}(\mathbf{x}) = -\int_{\Sigma_\varepsilon} \mathbf{U}^i(\mathbf{x}-\mathbf{y})\psi_i(\mathbf{y})dS(\mathbf{y}), \quad \mathbf{x} \in \Sigma_\varepsilon, \quad (2.6)$$

together with a compatibility condition to ensure the uniqueness.[6]

First, we assume that, on the surface Σ_ε, the density does not depend on the angle[a]:

$$\boldsymbol{\psi}(\mathbf{y}(s',\varphi')) = \boldsymbol{\alpha}(s'), \quad (2.7)$$

and, for a specific point

$$\mathbf{x}(s,\varphi) = \mathbf{r}(s) + \varepsilon\cos\varphi\mathbf{n}(s) + \varepsilon\sin\varphi\mathbf{b}(s) \in \Sigma_\varepsilon. \quad (2.8)$$

In a way consistent with our asymptotic method (see below), we keep only the terms of lowest order in ε for the surface measure dS (assuming that the curvature and torsion of the centerline are of order 1 in ε) and replace (2.6) by:

$$\mathbf{v}(\mathbf{x}(s,\varphi)) = -\frac{\varepsilon}{8\pi\nu}\int_0^{2\pi}\int_0^1 \mathbf{G}^i(\mathbf{x}(s,\varphi) - \mathbf{y}(s',\varphi'))\alpha_i(s')ds'd\varphi'. \quad (2.9)$$

For obtaining a one-dimensional equation, we eliminate the φ-dependence by taking the average of (2.9) over the angle:

$$\overline{\mathbf{v}}(s) = -\frac{\varepsilon}{16\pi^2\nu}\int_0^{2\pi}\int_0^{2\pi}\int_0^1 \mathbf{G}^i(\mathbf{x}(s,\varphi) - \mathbf{y}(s',\varphi'))\alpha_i(s')\,ds'd\varphi'd\varphi, \quad s \in [0,1], \quad (2.10)$$

with the velocity average:

$$\overline{\mathbf{v}}(s) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{v}(\mathbf{x}(s,\varphi))d\varphi. \quad (2.11)$$

[a]For example, the density is represented by the average $\boldsymbol{\alpha}(s') = \frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{\psi}(s',\varphi')d\varphi'$.

The integrals over the arclength in (2.10) depend on ε, φ and φ', but we do not indicate the angle dependence and write them as:

$$\mathbf{I}(s,\varepsilon)=\int_0^1 \mathbf{G}^i(\mathbf{x}(s,\varphi)-\mathbf{y}(s',\varphi'))\alpha_i(s')ds'. \tag{2.12}$$

The integrals (2.12) can be hypersingular when $\mathbf{x}=\mathbf{y}$; thus, for small ε, we shall find an expansion of them in terms of ε, involving Hadamard finite part integrals like in Refs. 11 and 21. The definition and some properties of the Hadamard finite part integral are given in Appendix A. Since the curve Γ is closed, the functions $\boldsymbol{\alpha}$ shall be 1-periodic.

In the general case, we do not know explicitly the arclength parametrization of the curve Γ, but only its asymptotic behavior for s' close to s:

$$\mathbf{r}(s')=\mathbf{r}(s)+(s'-s)\mathbf{t}(s)+O(\delta^2), \quad |s'-s|\le\delta. \tag{2.13}$$

We split the integral (2.12) into two parts:

$$\begin{aligned}\mathbf{I}(\overline{s},\varepsilon) &= \left(\int_{s+\delta}^{s+1-\delta}+\int_{s-\delta}^{s+\delta}\right)\mathbf{G}^i(\mathbf{x}(s,\varphi)-\mathbf{y}(s',\varphi'))\alpha_i(s')ds' \\ &\doteqdot \mathbf{I}_1(s,\varepsilon)+\mathbf{I}_2(s,\varepsilon),\end{aligned} \tag{2.14}$$

for some δ such that $\varepsilon\le\delta\ll 1$.

Since $\mathbf{x}(s,\varphi)-\mathbf{y}(s',\varphi')=\mathbf{r}(s)-\mathbf{r}(s')+O(\varepsilon)$, we get, using (2.5):

$$\mathbf{I}_1(s,\varepsilon)=\int_{s+\delta}^{s+1-\delta}\mathbf{G}^i(\mathbf{r}(s)-\mathbf{r}(s'))\alpha_i(s')ds'+O(\varepsilon). \tag{2.15}$$

For the domain $|s'-s|<\delta$, we make use of (2.13), which implies that

$$G^i_j(\mathbf{x}(s,\varphi)-\mathbf{y}(s,\varphi'))=\frac{1}{q(s'-s,\varepsilon;s')}\left(\delta_{ij}+\frac{D_{ij}(s'-s,\varepsilon;s')}{q^2(s'-s,\varepsilon;s')}\right)+O(\delta^2), \tag{2.16}$$

where again we did not indicate the angle dependence on the right-hand side and also the one in the variable s alone. After some easy computations, one finds:

$$\begin{aligned}q^2(x,\varepsilon;y)&=|x|^2+2\varepsilon x\mathbf{t}(s)\cdot[\cos\varphi'\mathbf{n}(y)+\sin\varphi'\mathbf{b}(y)]\\ &\quad+2\varepsilon^2(1-[\cos\varphi\mathbf{n}(s)+\sin\varphi\mathbf{b}(s)]\cdot[\cos\varphi'\mathbf{n}(y)+\sin\varphi'\mathbf{b}(y)]),\end{aligned} \tag{2.17}$$

$$\begin{aligned}D_{ij}(x,\varepsilon;y)&=|x|^2t_i(s)t_j(s)\\ &\quad-\varepsilon xt_i(s)[\cos\varphi n_j(s)+\sin\varphi b_j(s)-\cos\varphi' n_j(y)-\sin\varphi' b_j(y)]\\ &\quad-\varepsilon xt_j(s)[\cos\varphi n_i(s)+\sin\varphi b_i(s)-\cos\varphi' n_i(y)-\sin\varphi' b_i(y)]\\ &\quad+\varepsilon^2[\cos\varphi n_i(s)+\sin\varphi b_i(s)][\cos\varphi n_j(s)+\sin\varphi b_j(s)]\\ &\quad+\varepsilon^2[\cos\varphi' n_i(y)+\sin\varphi' b_i(y)][\cos\varphi' n_j(y)+\sin\varphi' b_j(y)]\\ &\quad-\varepsilon^2[\cos\varphi n_i(s)+\sin\varphi b_i(s)][\cos\varphi' n_j(y)+\sin\varphi' b_j(y)]\\ &\quad-\varepsilon^2[\cos\varphi n_j(s)+\sin\varphi b_j(s)][\cos\varphi' n_i(y)+\sin\varphi' b_i(y)].\end{aligned} \tag{2.18}$$

By replacing the variable s' by $t = s' - s$, we can write the jth component of the second integral in (2.14) as:

$$(\mathbf{I}_2)_j(s,\varepsilon) = \int_{-\delta}^{\delta} \frac{1}{q(t,\varepsilon;t+s)} \left(\delta_{ij} + \frac{D_{ij}(t,\varepsilon;t+s)}{q^2(t,\varepsilon;t+s)}\right) \beta_i(t)dt + O(\delta^2), \tag{2.19}$$

where

$$\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(t+s). \tag{2.20}$$

In the next section, we will need the generic type of each scalar integral appearing in (2.19), i.e.

$$\mathcal{I}(s,\varepsilon) = \int_{-\delta}^{\delta} h(t,\varepsilon;t+s)f(t)dt, \tag{2.21}$$

with $f = \beta_i$ and where the function h is of the form:

$$h(t,\varepsilon;x) = \frac{A}{(t^2 + 2\varepsilon t q_2(x) + 2\varepsilon^2 q_3(x))^{1/2}} + \frac{d_1 t^2 + 2\varepsilon t d_2(x) + 2\varepsilon^2 d_3(x)}{(t^2 + 2\varepsilon t q_2(x) + 2\varepsilon^2 q_3(x))^{3/2}}, \tag{2.22}$$

with $A = 0$ or 1.

3. Asymptotic Expansion

In this section, we make a formal asymptotic expansion of the integral (2.21) in the variable ε, up to order $O(\varepsilon)$, and then we write the integral equation resulting from applying this expansion to the right-hand side of the integral equation (2.10). We essentially proceed like in Ref. 11, assuming that all the necessary regularity conditions are satisfied.

The basic property we shall make use of is:

$$\frac{\partial^l}{\partial \varepsilon^l} h(\lambda t, \lambda\varepsilon; x) = \frac{\mathrm{sgn}(\lambda)}{\lambda^{1+l}} \frac{\partial^l}{\partial \varepsilon^l} h(t,\varepsilon;x). \tag{3.1}$$

We will also need the following Taylor expansions:

$$f(t) = f(0) + tf'(t_*), \quad t_* \in (0,t), \tag{3.2}$$

$$f(t) = f(0) + tf'(0) + R_f(t), \tag{3.3}$$

$$R_f(t) = \frac{t(t-t_*)}{2} f''(t_*), \quad t_* \in (0,t), \tag{3.4}$$

$$h(t,\varepsilon;x) = h(t,0;x) + \varepsilon h_\varepsilon(t,0;x) + R_h(t,\varepsilon;x), \tag{3.5}$$

$$R_h(t,\varepsilon;x) = \int_0^{\varepsilon} (\varepsilon - \xi) \frac{\partial^2}{\partial \xi^2} h(t,\xi;x) d\xi, \tag{3.6}$$

$$h(t,1;x+s) = h(t,1;s) + x h_x(t,1;s) + O(x^2), \tag{3.7}$$

$$h_\varepsilon(1,0;x+s) = h_\varepsilon(1,0;s) + x h_{\varepsilon x}(1,0;s) + O(x^2), \tag{3.8}$$

where the x- or ε-indices denote the corresponding partial derivatives.

The first step consists of using (3.5) for decomposing (2.21):

$$\begin{aligned}\mathcal{I}(s,\varepsilon) &= \int_{-\delta}^{\delta} f(t)(h(t,0;t+s)+\varepsilon h_\varepsilon(t,0;t+s))dt\\ &\quad -\left(\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}\right) f(t)R_h(t,\varepsilon;t+s)dt + \int_{-\varepsilon}^{\varepsilon} f(t)R_h(t,\varepsilon,t+s)dt\\ &\quad +\left(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\right) f(t)R_h(t,\varepsilon,t+s)dt\\ &\doteq \mathcal{I}_1(s,\varepsilon)+\mathcal{I}_2(s,\varepsilon)+\mathcal{I}_3(s,\varepsilon)+\mathcal{I}_4(s,\varepsilon). \end{aligned} \tag{3.9}$$

The detailed computations are presented in Appendix B and yield (see (B.9)):

$$\begin{aligned}\mathcal{I}(s,\varepsilon) &= h(1,0;0)Pf\int_{-\delta}^{\delta}\frac{f(t)}{|t|}dt + f(0)Pf\int_{-\infty}^{\infty} h(t,1;s)dt - 2f(0)h(1,0;0)\ln\varepsilon\\ &\quad -2f'(0)h_\varepsilon(1,0;s)\varepsilon\ln\varepsilon + O(\varepsilon). \end{aligned}\tag{3.10}$$

Let us come back to the original variables. First, we notice that the term $h_\varepsilon(1,0;s)$ corresponds in (2.12) to

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\frac{1}{q}\left(\delta_{ij}+\frac{D_{ij}}{q^2}\right)(1,0;s) &= -t_i(s)[n_j(s)(\cos\varphi-\cos\varphi')+b_j(s)(\sin\varphi-\sin\varphi')]\\ &\quad - t_j(s)[n_i(s)(\cos\varphi-\cos\varphi')+b_i(s)(\sin\varphi-\sin\varphi')];\end{aligned}$$

hence, while integrating over the angles, the contribution of the term of order ε ln ε disappears. Consequently, we will neglect it in the sequel. Using (3.10), (2.19), (2.20), (2.14) and (2.15), we get:

$$\begin{aligned}\mathbf{I}(s,\varepsilon) &= \mathbf{Q}^i(s)\alpha_i(s) - 2\ln\varepsilon\mathbf{R}^i(s)\alpha_i(s) + \int_{s+\delta}^{1+s-\delta}\mathbf{G}^i(\mathbf{r}(s)-\mathbf{r}(s'))\alpha_i(s')ds'\\ &\quad +\mathbf{R}^i(s)Pf\int_{s-\delta}^{s+\delta}\frac{\alpha_i(s')}{|s'-s|}ds' + O(\varepsilon)+O(\delta^2).\end{aligned}\tag{3.11}$$

The matrices **R** and **Q** are given by:

$$R^i_j(s) = \delta_{ij}+t_i(s)t_j(s), \tag{3.12}$$

$$\begin{aligned}Q^i_j(s) &= Pf\int_{-\infty}^{\infty}\left(\frac{1}{(t^2+2(1-\cos(\varphi-\varphi'))^{1/2}}\left(\delta_{ij}+\frac{D_{ij}(t,1;s)}{t^2+2(1-\cos(\varphi-\varphi')}\right)\right)dt\\ &= \int_{-1}^{1}\left(\frac{1}{(t^2+2(1-\cos(\varphi-\varphi'))^{1/2}}\left(\delta_{ij}+\frac{D_{ij}(t,1;s)}{t^2+2(1-\cos(\varphi-\varphi')}\right)\right)dt\\ &\quad + Pf\int_{-1}^{1}\left(\frac{1}{(1+2t^2(1-\cos(\varphi-\varphi'))^{1/2}}\left(\delta_{ij}+\frac{D_{ij}(1,t;s)}{1+2t^2(1-\cos(\varphi-\varphi')}\right)\right)\frac{dt}{|t|}.\end{aligned}\tag{3.13}$$

The integral equation corresponding to (2.10) with the terms of order ε and δ^2 neglected reads:

$$\frac{4\nu}{\varepsilon}\overline{\mathbf{v}}(s) = -\overline{\mathbf{Q}}^i(s)\alpha_i(s) + 2\ln\varepsilon\mathbf{R}^i(s))\alpha_i(s) - \int_{s+\delta}^{1+s-\delta}\mathbf{G}^i(\mathbf{r}(s)-\mathbf{r}(s'))\alpha_i(s')ds'$$
$$-\mathbf{R}^i(s)Pf\int_{s-\delta}^{s+\delta}\frac{\alpha_i(s')}{|s'-s|}ds', \quad s\in[0,1], \tag{3.14}$$

where

$$\overline{Q}_j^i(s) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}Q_j^i(s)d\varphi' d\varphi. \tag{3.15}$$

4. Discretization

We want to find an approximate solution $\boldsymbol{\alpha}_h$ of (3.14) using a collocation method with the help of the following partition of [0, 1]:

$$s_i = ih, \quad i = 0,1,\ldots,N, \quad h = \frac{1}{N}; \tag{4.1}$$

furthermore, we set

$$m_j = s_{j-1} + \frac{h}{2}, \quad j = 1,2,\ldots,N. \tag{4.2}$$

Assuming that $\boldsymbol{\alpha}_h$ is piecewise constant, we use the notation $\alpha_i^j = (\boldsymbol{\alpha}_h)_i(m_j)$ and replace (3.14) by the following discrete scheme:

$$\frac{4\nu}{\varepsilon}\overline{\mathbf{v}}(m_i) = -\overline{\mathbf{Q}}_h^k(m_i)\alpha_k^i + 2\ln\varepsilon\mathbf{R}^k(m_i)\alpha_k^i - \sum_{\substack{j=1\\|j-i|>p}}^{N} h\mathbf{G}^k(\mathbf{r}_h(m_i)-\mathbf{r}_h(m_j))\alpha_k^j$$
$$-\mathbf{R}^k(m_i)\sum_{|j-i|\le p}\alpha_k^j\int_{s_{j-1}}^{s_j}\frac{dt}{|m_i-t|}. \tag{4.3}$$

We build the mesh $\mathbf{r}_h$ of the knot centerline with the method described in Ref. 2, using biarcs, based on ideal knot shapes of E. Rawdon, available from his website and computed by the method presented in Ref. 17; the points $\mathbf{r}_h(m_i)$ coincide with biarcs ends.

If we compare (4.3) to (3.14), we notice that $\delta = (p+1/2)h$ in the discrete case and that, for $i = j$ the last integral is a finite part one, so that one has:

$$\int_{s_{j-1}}^{s_j}\frac{dt}{|m_i-t|} = \operatorname{sgn}(s_j-m_i)\ln(|s_j-m_i|)$$
$$-\operatorname{sgn}(s_{j-1}-m_i)\ln(|s_{j-1}-m_i|), \quad |j-i|\le p. \tag{4.4}$$

The approximation $\overline{Q}_h$ of the matrix $\overline{Q}$ defined by (3.13) and (3.15) is computed in the following way. Given some positive integer M, we set $\eta_i = 2\pi/M$, $\sigma_i = 2/M$,

$i = 0, 1, 2, \ldots, M$, and we partition the integration domain $\Pi = [0, 2\pi]^2 \times [-1, 1]$ into the M^3 subdomains $\Pi_{ijk} = [\eta_{i-1}, \eta_i] \times [\eta_{j-1}, \eta_j] \times [\sigma_{k-1}, \sigma_k]$, $1 \leq i, j, k \leq M$. The integrals over Π_{ijk} are approximated by the barycenter quadrature rule and the derivative required for the second integral in (3.13) is computed by the two-sided finite difference formula involving the points $\sigma_k \pm 1/M$. Finally, the linear system (4.3) with dense matrix is solved with the help of a LU factorization.

5. A Model Problem: The Circular Torus

5.1. *Continuous problems*

We choose as the curve Γ the circle of unit length in the $x_1 - x_2$ plane parametrized by

$$\mathbf{r}(s) = \frac{1}{2\pi}(\cos(2\pi s)\mathbf{e}_1 + \sin(2\pi s)\mathbf{e}_2), \quad s \in [0, 1), \tag{5.1}$$

which yields

$$\mathbf{t}(s) = -\sin(2\pi s)\mathbf{e}_1 + \cos(2\pi s)\mathbf{e}_2, \quad \mathbf{n}(s) = -2\pi\mathbf{r}(s), \quad \mathbf{b} = \mathbf{e}_3. \tag{5.2}$$

We want to study the scalar case, the starting point of which is given by the integral representation of the solution of the Laplace problem instead of the Stokes problem:

$$v(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Sigma_\varepsilon} \frac{\psi(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|} dS(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{5.3}$$

After some easy computations, we obtain by expanding $\mathbf{r}(s')$ and $\mathbf{n}(s')$ as well:

$$\begin{aligned} |\mathbf{y} - \mathbf{x}|^2 = {} & |s' - s|^2 + 2\varepsilon^2(1 - \cos(\varphi' - \varphi) + O(\varepsilon|s' - s|^2) \\ & + O(|s' - s|^3) + O(\varepsilon^2|s' - s|), \end{aligned} \tag{5.4}$$

and the function q^2 corresponding to (2.17) is given here by:

$$q^2(x, \varepsilon) = |x|^2 + 2\varepsilon^2(1 - \cos(\varphi' - \varphi)). \tag{5.5}$$

Using the same asymptotics as for the Stokes case, we get the integral equation:

$$\frac{2}{\varepsilon}\overline{v}(s) = (2\ln\varepsilon - Q)\alpha(s) - \int_{s+\delta}^{1+s-\delta} \frac{\alpha(s')}{|\mathbf{r}(s') - \mathbf{r}(s)|} ds' - Pf\int_{s-\delta}^{s+\delta} \frac{\alpha(s')}{|s' - s|} ds', \tag{5.6}$$

with

$$\begin{aligned} Q &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} Pf\int_{-\infty}^{+\infty} \frac{dt}{\sqrt{t^2 + 2(1 - \cos(\varphi' - \varphi))}} d\varphi' d\varphi \\ &= \frac{1}{2\pi}\int_{-1}^{1}\int_0^{2\pi} \frac{d\varphi dt}{\sqrt{t^2 + 2(1 - \cos(\varphi))}} + \frac{1}{2\pi} Pf\int_{-1}^{1}\int_0^{2\pi} \frac{d\varphi dt}{|t|\sqrt{1 + 2t^2(1 - \cos(\varphi))}} \\ &\simeq 1.386294. \end{aligned}$$

If we set $\mathcal{C} = Q - 2\ln\varepsilon$, $f(s) = -2\overline{v}(s/2\pi)/\varepsilon$, $u(s) = \alpha(s/2\pi)$, $t = 2\pi s'$ and replace δ by $2\pi\delta$, we get the Fredholm integral equation:

$$\mathcal{C}u(s) + \int_{s+\delta}^{2\pi+s-\delta} \frac{u(t)}{2|\sin(\frac{t-s}{2})|}dt + Pf\int_{s-\delta}^{s+\delta} \frac{u(t)}{|t-s|}dt = f(s), \quad s \in [0,1). \tag{5.7}$$

Since

$$Pf\int_0^{2\pi} \frac{u(t)}{2|\sin(\frac{t-s}{2})|}dt = \int_{s+\delta}^{2\pi+s-\delta} \frac{u(t)}{2|\sin(\frac{t-s}{2})|}dt + Pf\int_{s-\delta}^{s+\delta} \frac{u(t)}{2|\sin(\frac{t-s}{2})|}dt, \tag{5.8}$$

we can rewrite the integrals in (5.7) as

$$\begin{aligned} &\int_{s+\delta}^{2\pi+s-\delta} \frac{u(t)}{2|\sin(\frac{t-s}{2})|}dt + Pf\int_{s-\delta}^{s+\delta} \frac{u(t)}{|t-s|}dt \\ &\quad = Pf\int_0^{2\pi} \frac{u(t)}{2|\sin(\frac{t-s}{2})|}dt + \int_{s-\delta}^{s+\delta}\left(\frac{1}{|t-s|} - \frac{1}{2|\sin(\frac{t-s}{2})|}\right)u(t)dt \\ &\quad = Pf\int_0^{2\pi} \frac{u(t)}{2|\sin(\frac{t-s}{2})|}dt + O(\delta^2). \end{aligned} \tag{5.9}$$

Thus, we neglect the term of order δ^2 and consider the following model problem; we look for a 2π-periodic function u such that:

$$\mathcal{C}u(s) + Pf\int_0^{2\pi} \frac{u(t)}{2|\sin(\frac{t-s}{2})|}dt = f(s), \tag{5.10}$$

the given function f being also 2π-periodic. We define the operator T formally by:

$$Tu(s) = \int_0^{2\pi} \frac{u(t)-u(s)}{2|\sin(\frac{t-s}{2})|}dt, \tag{5.11}$$

and, thanks to (A.7), we can write (5.10) in the following form:

$$cu + Tu = f, \tag{5.12}$$

with

$$c = \mathcal{C} + 2\ln 4. \tag{5.13}$$

The problem (5.12) is close to the one studied in Refs. 8 and 9, where (5.11) is replaced by

$$Ru(s) = \int_{-1}^{1} \frac{u(t)-u(s)}{|t-s|}dt, \tag{5.14}$$

without a periodicity condition. It corresponds, when we impose 2-periodicity, to a straight "periodic" rod; we can show that, in this case, (3.14) reduces to three identical problems of this type. The operators (5.11) and (5.14) being similar, most of the results of Refs. 8 and 9 will also apply to the present case and we shall mainly adapt the proofs of these references.

Eigenvalues and eigenvectors of T are given in the next lemma.

Lemma 5.1. *The trigonometric polynomials* $p_0 = 1$, $p_n(x) = \cos nx$, $p_{-n}(x) = \sin nx$, $n = 1, 2, \dots$, *are eigenvectors of* T, *and correspond to the eigenvalues* (*geometrically double for* $n > 0$)

$$\alpha_0 = 0, \quad \alpha_n = -4\sum_{k=1}^{n} \frac{1}{2k-1} = -2\left(\psi\left(n + \frac{1}{2}\right) + \gamma + 2\ln 2\right), \tag{5.15}$$

where ψ *is the digamma function and* γ *the Euler constant.*

We denote by C^λ the vector space of 2π-periodic functions which are Hölder continuous of index λ, $0 < \lambda < 1$, that is the functions such that

$$|u(s) - u(s')| \le K|s - s'|^\lambda; \tag{5.16}$$

we recall that C^λ is a Banach space with the norm:

$$\|u\|_{C^\lambda} = \max\left\{\|u\|_\infty, \sup_{s\neq t} \frac{|u(t) - u(s)|}{|t - s|^\lambda}\right\}. \tag{5.17}$$

Considering T as an operator from Hölder spaces, yields the following results, which are analogous to those for the operator R.[9]

Lemma 5.2. (1) *The operator* T *is a linear bounded operator from* C^λ *into* C^μ *for all positive* $\mu < \lambda$.
(2) *One has* $TC^\lambda \subseteq C^\mu$ *for all* $\lambda \in (0, 1]$ *and for all* $\mu \in [0, \lambda)$.

As a consequence of this lemma, we see that the operator T does not map C^λ into itself, but that it maps the set

$$X = \cup_{\mu\in(0,1]} C^\mu, \tag{5.18}$$

into itself.

For obtaining existence and uniqueness results, we have first to extend T from the space spanned by the trigonometric polynomials:

$$\Pi = \mathrm{span}[(p_k)_{k\in\mathbb{Z}}], \tag{5.19}$$

which is dense in L^2 (the space of functions in $L^2(0, 2\pi)$ extended by 2π-periodicity); for a function $f \in L^2$, let f_k, $k \in \mathbb{Z}$ be its Fourier coefficients, i.e.

$$f_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt, \quad f_k = \frac{1}{\pi}\int_0^{2\pi} f(t)p_k(t)dt, \quad k \in \mathbb{Z} - \{0\}. \tag{5.20}$$

Then, we have the following result.

Proposition 5.1. *If* $c + \alpha_n \neq 0$, $n \in \mathbb{N}_0$, *the operator* $T|_\Pi$ *can be extended by closure to a self-adjoint operator* $\overline{T} : L^2 \to L^2$, *whose domain is*

$$\mathrm{dom}(\overline{T}) = \left\{f \in L^2;\ \sum_{k=-\infty}^{+\infty} \alpha_k^2 f_k^2 < \infty\right\},$$

and which has a pure point spectrum, given by the eigenvalues α_n and eigenvectors of Lemma 5.1.

From this proposition, we conclude that, provided $c + \alpha_n \neq 0$, for all $n \in \mathbb{N}_0$, the equation

$$c\varphi + \overline{T}\varphi = f, \tag{5.21}$$

has, for $f \in L^2$, the unique solution

$$\varphi = \sum_{k=-\infty}^{+\infty} \frac{f_k}{c + \alpha_k} p_k. \tag{5.22}$$

The asymptotic behavior of c and α_n are given by

$$c \approx -2\ln\varepsilon, \quad \varepsilon \to 0, \qquad \alpha_n \approx -2\ln n, \quad n \to \infty,$$

and consequently for small ε of order of $1/n$, $(c + \alpha_n)$ can be approximately zero. A typical value for our ideal knots is $\varepsilon = 0.005$, which gives $c \simeq 14.75552$ and we have $\alpha_{224} \simeq -14.75031$, $\alpha_{225} = -14.75922$; thus $\min_n |(c + \alpha_n)| \simeq 4 \cdot 10^{-3}$.

We obtained a result about the solvability of (5.21), which of course has a relation to the true problem of solving Eq. (5.12). For the latter one, there are several possible setups for getting a Hölder continuous function as a solution. We give the simplest one below.

Proposition 5.2. *Assume that $c + \alpha_n \neq 0, n \in \mathbb{N}_0$ and that the Fourier coefficients of f have the asymptotic behavior*

$$f_n = O\left(\frac{\ln|n|}{|n|^{1+\lambda}}\right), \quad |n| \to \infty, \quad 0 < \mu < \lambda < 1. \tag{5.23}$$

Then, the unique solution φ of (5.21) belongs to C^λ and is a solution of (5.12).

5.2. *Numerical scheme and error estimate*

As in the true case of the vector problem we define the numerical scheme based on the discretization of (5.7) by

$$\mathcal{C}u^i + h\sum_{|j-i|>p} \frac{u^j}{2|\sin\frac{m_j - m_i}{2}|} + \sum_{|j-i|\le p} u^j \int_{s_{j-1}}^{s_j} \frac{dt}{|t - m_i|} = f(m_i), \quad i = 1, 2, \ldots, N, \tag{5.24}$$

with $u^i = u_h(m_i), u_h$ a piecewise constant approximation of u and $\delta = (p + 1/2)h$; the integral for $i = j$ is understood as a finite part integral. We consider the linear system (5.24) as an approximation of the model problem (5.10).

Lemma 5.3. *For h sufficiently small, the linear system (5.24) has a unique solution.*

From (5.10) and (5.24), we get the linear system

$$M\boldsymbol{\delta}\mathbf{u} = \mathbf{b}, \tag{5.25}$$

with M defined in (C.11) and

$$\delta u_i = u(m_i) - u^i,$$

$$b_i = h \sum_{|j-i|>p} \frac{u(m_j)}{2|\sin \frac{m_j - m_i}{2}|} - \frac{1}{2} \int_{m_i+\delta}^{2\pi+m_i-\delta} \frac{u(t)}{|\sin \frac{t-m_i}{2}|} dt$$

$$+ \sum_{|j-i|\leq p} u(m_j) \int_{s_{j-1}}^{s_j} \frac{dt}{|t-m_i|} - Pf \int_{m_i-\delta}^{m_i+\delta} \frac{u(t)}{2|\sin \frac{t-m_i}{2}|} dt. \tag{5.26}$$

Consequently, from (C.12), the solution of the linear system (5.25) satisfies:

$$\|\boldsymbol{\delta}\mathbf{u}\| = O\left(\frac{\|\mathbf{b}\|}{|\ln h|}\right), \quad h \to 0. \tag{5.27}$$

It remains to estimate the norm of the right-hand side $\mathbf{b}$ according to the following lemma.

Lemma 5.4. *Assume that the solution of* (5.10) *is* $u \in C^\lambda$; *then one has*:

$$|b_i| \leq K(\lambda)|\ln h| h^\lambda. \tag{5.28}$$

With the help of (5.27) and (5.28), we obtain the final estimate quoted in the next theorem.

Theorem 5.1. *Under the hypotheses of Proposition* 5.2, *we have for the discretized scheme* (5.24):

$$|u(m_i) - u^i| = O(h^\lambda), \quad h \to 0, \quad 1 \leq i \leq N. \tag{5.29}$$

6. Numerical Results

We want to compute the resistance matrix of a filament, which can be written blockwise:

$$R = \begin{pmatrix} K & C \\ S & \Theta \end{pmatrix}; \tag{6.1}$$

its entries are given by the total force and torque acting on the body surface resulting from six basic solutions $(\mathbf{h}^{(i)}, p^{(i)}), (\mathbf{H}^{(i)}, P^{(i)}), i = 1, 2, 3$, of the external Stokes flow[7] with the velocity boundary conditions:

$$\mathbf{h}^{(i)} = \mathbf{e}_i, \quad \mathbf{H}^{(i)} = \mathbf{e}_i \times \mathbf{y}, \quad \text{on } \Sigma_\varepsilon, \quad i = 1, 2, 3, \tag{6.2}$$

where $\mathbf{e}_i$ are the standard cartesian basis vectors. Denoting by $\boldsymbol{\psi}^{(i)}, \boldsymbol{\Psi}^{(i)}$ the corresponding density distributions on Σ_ε according to (2.4), the blocks of R are given by[4,3]:

$$K_{ji} = \int_{\Sigma_\varepsilon} \psi_j^{(i)}(\mathbf{y}) dS(\mathbf{y}), \quad C_{ji} = \int_{\Sigma_\varepsilon} \Psi_j^{(i)}(\mathbf{y}) dS(\mathbf{y}),$$

$$S_{ji} = \int_{\Sigma_\varepsilon} (\mathbf{y} \times \boldsymbol{\psi}^{(i)}(\mathbf{y}))_j dS(\mathbf{y}), \quad \Theta_{ji} = \int_{\Sigma_\varepsilon} (\mathbf{y} \times \boldsymbol{\Psi}^{(i)}(\mathbf{y}))_j dS(\mathbf{y}), \quad 1 \leq i, \ j \leq 3. \tag{6.3}$$

Recall that the resistance matrix is symmetric positive-definite; furthermore it is diagonal for bodies with a rotational symmetry, as for example a circular torus.

In our model, we replace the density distributions in the integrands appearing in (6.3) by their corresponding approximations which are solutions of (3.14). Keeping only the terms of order ε, we get, for the total force and torque, integrals of the type

$$\int_{\Sigma_\varepsilon} \boldsymbol{\alpha} dS = 2\pi\varepsilon \int_0^1 \boldsymbol{\alpha}(s) ds, \quad \int_{\Sigma_\varepsilon} \boldsymbol{\alpha}_h \times \mathbf{y}(s,\varphi) dS = \varepsilon \int_0^1 \boldsymbol{\alpha}(s) \times \mathbf{r}(s) ds. \tag{6.4}$$

Of course, in the numerical computations, we use in place of $\boldsymbol{\alpha}$ and $\mathbf{r}$, their approximation $\boldsymbol{\alpha}_h$ and $\mathbf{r}_h$; we compute the integrals on each interval $[s_{j-1}, s_j]$ using the quadrature formula

$$\int_{s_{j-1}}^{s_j} \boldsymbol{\alpha}_h(s) ds \simeq \frac{h}{2} \boldsymbol{\alpha}_h(m_j), \quad \int_{s_{j-1}}^{s_j} \boldsymbol{\alpha}_h(s) \times \mathbf{r}(s) ds \simeq \frac{h}{2} \boldsymbol{\alpha}_h(m_j) \times \mathbf{r}_h(m_j). \tag{6.5}$$

The filaments are the knots described in Ref. 3; the kinematic viscosity is $\nu = 1$ and we scale the knots if they are not of unit length. The dimensionless radius ε can take different values.

The first results concern the convergence of the diagonal elements of the matrix K in function of h and the number of integration points M for computing the matrices $\overline{\mathbf{Q}}_h^k$ in (4.3). Our example is the circular torus knot 0_1 shown in Fig. 1, with $\varepsilon = 0.005$; in this case, K is diagonal with two different entries ($K_{11} = K_{22} < K_{33}$).

In Fig. 2, we plot the first diagonal entry against $1/h$ for the three different values $\delta = (p + 1/2)h$, $p = 0, 1, 2$. We observe that the cases $p = 1, 2$ converge to the same value, while the result for $p = 0$ is slightly different. This was also confirmed by tests on a scalar model problem whose solution is known, showing that for $p = 0$, the approximation does not converge very well to the exact solution. Thus, we will usually choose $p = 2$.

The effect of the number M of integration points for computing the matrices $\overline{\mathbf{Q}}_h^k$ is shown in Fig. 3, for $h = 100$, $\delta = 5h/2$. At the light of these results, we decide to fix in the sequel $M = 50$.

Remark 6.1. Like for the model problem, the scheme (4.3) can be considered as an approximation of (3.14) without the last integral and where the preceding one is on $(0, 1)$. For the knot 0_1 the singularities in the integral are of the same type as those of the model problem and following the proof of Lemma 5.2 should yield that the

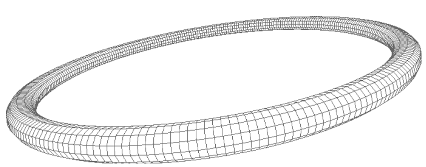

Fig. 1. The knot 0_1.

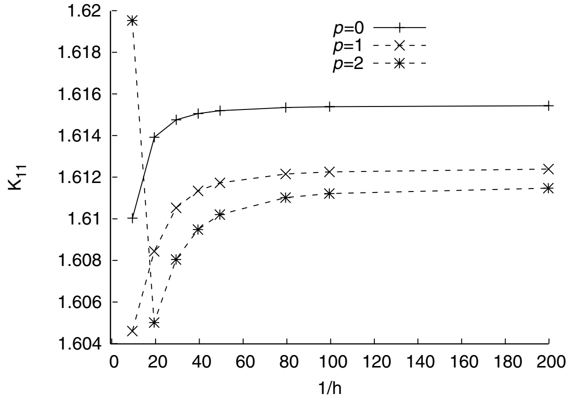

Fig. 2.

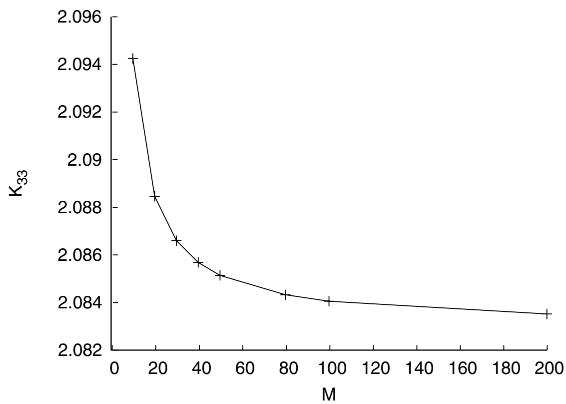

Fig. 3.

solution belongs to C^λ. Then, for constant boundary velocities (this is the case for computing two blocks of the resistance matrix), we would obtain a value of λ close to 1. In the case $\mathbf{v} = \mathbf{e}_1$, we computed the solution of (4.3) with the meshes defined by $h = 1/(2^k 20), k = 0, 1, \ldots, 6$. We considered the approximation for $k = 6$ as being exact for computing the error at the collocation nodes of the mesh with $k = 0$; then, for each node, we obtained an approximate value of λ in (5.29) with a least-square fit. The result, for $M = 50$, is $\lambda \geq 0.9$, which confirms our error estimate.

Next, we compare the two different diagonal entries of K (1D−, 1D+) obtained with our model to the boundary element method of Ref. 3 (bem−, bem+), still for the knot 0_1.

In Fig. 4 we plot, for the same parameter δ as in Fig. 3 and $M = 50$, the two different diagonal elements of K in function of $\log_2(1/h)$. The agreement is remarkable starting from $h = 1/20$.

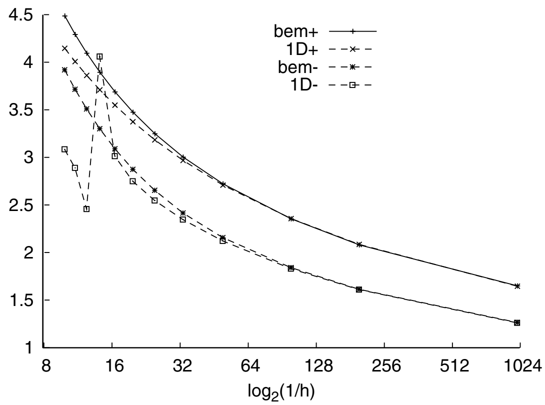

Fig. 4.

The eigenvalues of the matrix A appearing in the block decomposition of the inverse of the resistance matrix

$$R^{-1} = \begin{pmatrix} M & A' \\ A & N \end{pmatrix}, \tag{6.6}$$

determine the steady states of the free fall together with their stability.[7,10,3] We take the example of the knot 5_1 pictured on Fig. 5.

In Fig. 6 for $h = 200$, $\delta = 5h/2$, we plot the largest and smallest eigenvalues of the matrix A (which determines the stable steady states[10] for different values of ε. Again, the results for both methods are indentical. Notice that the boundary element method will not produce accurate results for $\varepsilon < 0.001$.

The asymptotic behavior of the force of the fluid on slender bodies is $O(C/\ln\varepsilon)$ when $\varepsilon \to 0$ (see for example Refs. 23 and 14); we check this by plotting in Fig. 7 the two different diagonal entries of the matrix K for the knot 0_1 against ε. The two curves are given by $-11.4/\ln\varepsilon$ and $-8.76/\ln\varepsilon$.

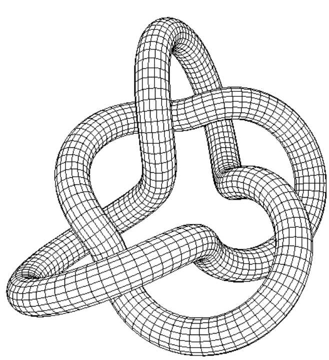

Fig. 5. The knot 5_1.

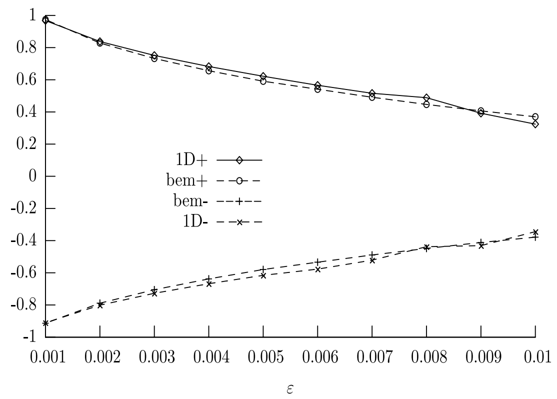

Fig. 6.

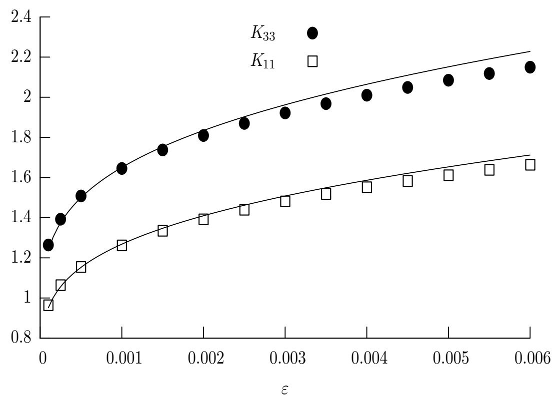

Fig. 7.

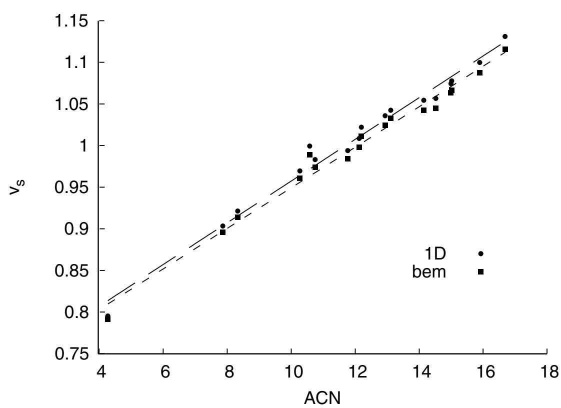

Fig. 8.

Finally, for the knots $3_1, 5_1, 5_2, 6_1, 8_{19}, 6_3, 7_1, 8_{20}, 7_2, 8_{21}, 7_7, 8_3, 8_1, 8_9, 8_5, 8_{17}$, and 9_2, we plot in Fig. 8 the mean sedimentation speed (see Refs. 10 and 3) versus the average crossing number ACN; the numerical parameters are $\varepsilon = 0.005$, $h = 100$, $\delta = 5h/2$. This is known to be a straight line. A least-square fit to a straight line gives $0.0244ACN + 0.7051$ for the BEM and $0.0251ACN + 0.7061$ for the present model.

In conclusion the numerical results, concerning the convergence and the comparison with the boundary element method, show the validity of our asymptotic model for computing the sedimentation of a closed solid filament in a Stokes flow. This is the first step for a study of the sedimentation of flexible filaments, for which we intend to set up a one-dimensional model with the same kind of asymptotic method.

Appendix A. The Hadamard Finite Part

We shall recall briefly some properties of the Hadamard finite part of an integral for the specific cases needed in this paper.

We start with a finite interval $I = (a, b)$ and proceed like in Ref. 5; we assume that we have a continuous function k: $(a - b, b - a) \to \mathbb{R}$ such that

$$\begin{aligned} &k(-x) = -k(x), \quad k(x) = k_1(x) + k_2(x), \quad x \to 0, \\ &k_1(x) \sim x^{-\lambda}, \quad \lambda \geq 1, \quad k_2(x) \to 0, \quad x \to 0, \\ &k_1 = K_1'. \end{aligned}$$

Given a continuous function u: $I \to \mathbb{R}$ we want to give a meaning to the integral:

$$I(s) = \int_I u(t)k(|t - s|)dt, \quad s \in (a, b), \tag{A.1}$$

which does not exist, even as a Cauchy principal value. The basic idea is to subtract the singularity at $t = s$ in the following way. Noticing that, for $\tau > 0$:

$$\left(\int_a^{s-\tau} + \int_{s+\tau}^b\right) k_1(|t - s|)dt = K_1(s - a) + K_1(b - s) - 2K_1(\tau),$$

we define the finite part of the integral by:

$$Pf\int_I u(t)k(|t - s|)dt \doteqdot \lim_{\tau \to 0^+} \left[\left(\int_a^{s-\tau} + \int_{s+\tau}^b\right) u(t)k(|t - s|)dt + 2u(s)K_1(\tau)\right]. \tag{A.2}$$

Our first kernel is defined by $k(x) = k_1(x) = 1/x$ and we have:

$$Pf\int_I \frac{u(t)}{|t - s|}dt \doteqdot \lim_{\tau \to 0^+} \left[\left(\int_a^{s-\tau} + \int_{s+\tau}^b\right) \frac{u(t)}{|t - s|}dt + 2u(s)\ln\tau\right]. \tag{A.3}$$

With an integration by parts of (A.2) for $u \in C^1(a, b)$, we get easily:

$$Pf\int_I \frac{u(t)}{|s - t|}dt = u(a)\ln(s - a) + u(b)\ln(b - s) - \int_I u'(t)\mathrm{sgn}(t - s)\ln|t - s|dt. \tag{A.4}$$

For example, for $\varepsilon > 0$:

$$Pf \int_{-\varepsilon}^{\varepsilon} \frac{1}{|t|} dt = 2 \ln \varepsilon, \tag{A.5}$$

which is used for example in (B.5).

Notice that an integral which is hypersingular at infinity can be transformed into an integral hypersingular at zero by the change of variable $t' = 1/t$.

Let us now come to the case of periodic functions, for which, we assume that k and u are 2π-periodic. We use again (A.3) with $a = 0$ and $b = 2\pi$; since

$$\frac{1}{2 \sin \frac{x}{2}} = \frac{1}{x} + O(x), \quad x \to 0,$$

we obtain for our specific case:

$$Pf \int_0^{2\pi} \frac{u(t)}{2|\sin \frac{t-s}{2}|} dt = \lim_{\tau \to 0^+} \left[\left(\int_0^{s-\tau} + \int_{s+\tau}^{2\pi} \right) \frac{u(t)}{2|\sin \frac{t-s}{2}|} dt + 2u(s) \ln \tau \right]. \tag{A.6}$$

Noticing that

$$-\frac{d}{dx} \arctan h\left(\cos \frac{x}{2}\right) = \frac{1}{2 \sin \frac{x}{2}},$$

we get

$$\int_0^{2\pi} \frac{u(t) - u(s)}{2|\sin \frac{t-s}{2}|} dt = \lim_{\tau \to 0^+} \left[\left(\int_0^{s-\tau} + \int_{s+\tau}^{2\pi} \right) \frac{u(t)}{2|\sin \frac{t-s}{2}|} dt - 2u(s) \arctan h\left(\cos \frac{\tau}{2}\right) \right],$$

and since

$$\arctan h\left(\cos \frac{x}{2}\right) = -\ln |x| + \ln 4 + O(x^2), \quad x \to 0,$$

we finally obtain, for any Hölder continuous function $u \in C^\lambda(0, 2\pi)$, $0 < \lambda < 1$:

$$Pf \int_0^{2\pi} \frac{u(t)}{2|\sin \frac{t-s}{2}|} dt = \int_0^{2\pi} \frac{u(t) - u(s)}{2|\sin \frac{t-s}{2}|} dt + 2 \ln 4 u(s). \tag{A.7}$$

Appendix B. Asymptotic Expansion of the Integrals in (3.9)

We present here the detailed expansion of the integrals defined in (3.9); for expanding them, we shall very often make use of (3.1), without mentioning it explicitly. Of course, some of these integrals can be hypersingular and shall be understood in the Hadamard finite part sense, but we will indicate this only in the final result.

The first integral is easy to expand:

$$\mathcal{I}_1(s) = \int_{-\delta}^{\delta} f(t) \frac{1}{|t|} h(1, 0; t + s) dt + \varepsilon \int_{-\delta}^{\delta} f(t) \frac{\operatorname{sgn}(t)}{t^2} h_\varepsilon(1, 0; t + s) dt,$$

and hence:

$$\mathcal{I}_1(s) = h(1,0;0) Pf \int_{-\delta}^{\delta} \frac{f(t)}{|t|} dt + O(\varepsilon). \tag{B.1}$$

We represent $\mathcal{I}_2$ as

$$\begin{aligned}
\mathcal{I}_2 &= -\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) f(t) \int_0^{\varepsilon} (\varepsilon - \xi) \frac{\partial^2}{\partial \xi^2} h(t, \xi; t+s) d\xi dt \\
&= -\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \frac{f(t)\text{sgn}(t)}{t^3} \int_0^{\varepsilon} (\varepsilon - \xi) h_{\varepsilon\varepsilon}\left(1, \frac{\xi}{t}; t+s\right) d\xi dt \\
&= -\varepsilon^2 \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \frac{f(t)\text{sgn}(t)}{t^3} \int_0^1 (1 - \xi) h_{\varepsilon\varepsilon}\left(1, \frac{\xi\varepsilon}{t}; t+s\right) d\xi dt.
\end{aligned}$$

A rough estimate shows that, since the vectors in (2.17) are of unit norm, $|q^2(1,\eta;y)| \geq 1 - 2|\eta|$; this implies that $h_{\varepsilon\varepsilon}(1, \frac{\xi\varepsilon}{t}; t+s)$ is bounded for $\delta > 2\varepsilon$. Hence:

$$\mathcal{I}_2(s,\varepsilon) = O(\varepsilon^2), \quad \delta > 2\varepsilon. \tag{B.2}$$

We have, from (3.5):

$$\begin{aligned}
\mathcal{I}_3 &= \int_{-\varepsilon}^{\varepsilon} f(t) \left[h(t, \varepsilon; t+s) - \frac{h(1,0;t+s)}{|t|} - \varepsilon \frac{\text{sgn}(t)}{t^2} h_{\varepsilon}(1,0;t+s) \right] dt \\
&\doteqdot \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned} \tag{B.3}$$

Using (3.2) and then (3.7), one finds:

$$\begin{aligned}
\mathcal{J}_1 &= \int_{-\varepsilon}^{\varepsilon} [f(0) + t f'(t_*)] h(t, \varepsilon; t+s) dt \\
&= f(0) \int_{-1}^{1} h(t, 1; \varepsilon t + s) dt + \varepsilon f'(t_*) \int_{-1}^{1} t h(t, 1; \varepsilon t + s) dt \\
&= f(0) \left\{ \int_{-1}^{1} h(t, 1; s) dt + \varepsilon \int_{-1}^{1} t h_x(t, 1; s) dt \right\} + O(\varepsilon);
\end{aligned}$$

hence, we get

$$\mathcal{J}_1 = f(0) \int_{-1}^{1} h(t, 1; s) dt + O(\varepsilon). \tag{B.4}$$

Since $h(1,0;x) = h(1,0;0)$, using again (3.2) yields for $\mathcal{J}_2$:

$$\begin{aligned}
\mathcal{J}_2 &= -f(0) h(1,0;0) Pf \int_{-\varepsilon}^{\varepsilon} \frac{dt}{|t|} - f'(t_*) h(1,0;0) \int_{-\varepsilon}^{\varepsilon} \text{sgn}(t) dt \\
&= -2 f(0) h(1,0;0) \ln \varepsilon.
\end{aligned} \tag{B.5}$$

From (3.3) and (3.8), we obtain:

$$\begin{aligned}\mathcal{J}_3 &= -\int_{-1}^{1} \frac{\operatorname{sgn}(t)}{t^2} h_\varepsilon(1,0;\varepsilon t+s)[f(0)+R_f(\varepsilon t)]dt - \varepsilon f'(0)\int_{-\varepsilon}^{\varepsilon} \frac{h_\varepsilon(1,0;\varepsilon t+s)}{|t|}dt \\ &= -f(0)h_\varepsilon(1,0;s)\int_{-1}^{1}\frac{\operatorname{sgn}(t)}{t^2}dt - \varepsilon f'(0)h_\varepsilon(1,0;s)Pf\int_{-\varepsilon}^{\varepsilon}\frac{dt}{|t|} + O(\varepsilon),\end{aligned}$$

which implies that

$$\mathcal{J}_3 = -2f'(0)h_\varepsilon(1,0;s)\varepsilon\ln\varepsilon + O(\varepsilon). \tag{B.6}$$

With the help of (B.3)–(B.6), one finds that the expansion of $\mathcal{I}_3$ is:

$$\begin{aligned}\mathcal{I}_3(s,\varepsilon) = f(0)Pf\int_{-1}^{1} h(t,1;s)dt - 2f(0)h(1,0;0)\ln\varepsilon \\ - 2f'(0)h_\varepsilon(1,0;s)\varepsilon\ln\varepsilon + O(\varepsilon).\end{aligned} \tag{B.7}$$

For the last integral, we write:

$$\begin{aligned}\mathcal{I}_4 &= \left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) f(\varepsilon t)\left[h(t,1;\varepsilon t+s) - \frac{h(1,0;0)}{|t|} - \frac{\operatorname{sgn}(t)}{t^2}h_\varepsilon(1,0;\varepsilon t+s)\right]dt \\ &= f(0)\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right)\left[h(t,1;s) - \frac{h(1,0;0)}{|t|} - h_\varepsilon(1,0;s)\frac{\operatorname{sgn}(t)}{t^2}\right]dt + O(\varepsilon);\end{aligned}$$

this implies, since the last two integrals are zero, that

$$\mathcal{I}_4(s,\varepsilon) = f(0)Pf\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right)h(t,1;s)dt + O(\varepsilon). \tag{B.8}$$

The final result for the asymptotic expansion of the integral (3.9) is:

$$\begin{aligned}\mathcal{I}(s,\varepsilon) &= h(1,0;0)Pf\int_{-\delta}^{\delta}\frac{f(t)}{|t|}dt + f(0)Pf\int_{-\infty}^{\infty} h(t,1;s)dt - 2f(0)h(1,0;0)\ln\varepsilon \\ &\quad - 2f'(0)h_\varepsilon(1,0;s)\varepsilon\ln\varepsilon + O(\varepsilon).\end{aligned} \tag{B.9}$$

Appendix C. Proofs

Proof of Lemma 5.1. By induction. First we check that $T1 = 0$, $T\sin s = -4\sin s$, $T\cos s = -4\cos s$, which is consistent with (5.15) for $n = 0, 1$. Then, assuming the affirmation true for n, $n-1$ and using $\cos((n+1)x) = 2\cos x\cos(nx) - \cos((n-1)x)$, one has

$$\begin{aligned} T\cos((n+1)s) &= \int_0^{2\pi} \frac{(\cos t - \cos s)\cos nt}{|\sin\frac{t-s}{2}|}dt \\ &\quad + \cos s \int_0^{2\pi} \frac{\cos nt - \cos ns}{|\sin\frac{t-s}{2}|}dt - \alpha_{n-1}\cos((n-1)s) \\ &= -2\int_0^{2\pi} \operatorname{sgn}(t-s)\sin\left(\frac{t+s}{2}\right)\cos(nt)dt \\ &\quad + 2\alpha_n \cos s\cos ns - \alpha_{n-1}\cos((n-1)s). \end{aligned}$$

The last integral can be computed with the help of

$$\begin{aligned} \sin\left(\frac{t+s}{2}\right)\cos(nt) &= \frac{\partial}{\partial t}\left(\frac{1}{2n-1}\cos\left(\frac{(2n-1)t-s}{2}\right)\right. \\ &\quad \left. -\frac{1}{2n+1}\cos\left(\frac{(2n+1)t+s}{2}\right)\right), \end{aligned}$$

and we obtain finally

$$\begin{aligned} T\cos((n+1)s) &= -\frac{4}{2n+1}\cos((n+1)s) + 2\alpha_n\cos s\cos ns \\ &\quad -\left(\alpha_{n-1} - \frac{4}{2n-1}\right)\cos((n-1)s) \\ &= \left(\alpha_n - \frac{4}{2n+1}\right)\cos((n+1)s). \end{aligned}$$

For $\sin ns$, we proceed in the same way using $\sin((n+1)x) = 2\cos x\sin(nx) - \sin((n-1)x)$ and

$$\begin{aligned} \sin\left(\frac{t+s}{2}\right)\sin(nt) &= \frac{\partial}{\partial t}\left(\frac{1}{2n-1}\sin\left(\frac{(2n-1)t-s}{2}\right)\right. \\ &\quad \left. -\frac{1}{2n+1}\sin\left(\frac{(2n+1)t+s}{2}\right)\right). \end{aligned}$$

□

Proof of Lemma 5.2. (1) We follow closely the proof of Theorem 7.2.5 (p. 234) in Ref. 12. Assuming $u \in C^\lambda$, we set $k_u = \inf\{K; K \text{ satisfies } (5.16)\}$ and

$$\varphi(t) = \frac{u(t) - u(s')}{2|\sin\frac{t-s'}{2}|} - \frac{u(t)-u(s)}{2|\sin\frac{t-s}{2}|},$$

which implies that

$$Tu(s') - Tu(s) = \int_0^{2\pi} \varphi(t)dt.$$

Then, we define $\varepsilon = 2|s' - s|$ (assuming it is small) and we split the integral:

$$\int_0^{2\pi} \varphi(t)dt = \int_{s-\varepsilon}^{s+\varepsilon} \varphi(t)dt + \int_{s+\varepsilon}^{2\pi+s-\varepsilon} \varphi(t)dt. \tag{C.1}$$

(i) $|t - s| < \varepsilon$

One has:

$$\begin{aligned} |\varphi(t)| &\leq k_u\left(\frac{|t-s|^\lambda}{2|\sin\frac{t-s}{2}|} + \frac{|t-s'|^\lambda}{2|\sin\frac{t-s'}{2}|}\right) \\ &= k_u\left(|t-s|^{\lambda-1}\frac{\frac{|t-s|}{2}}{|\sin\frac{t-s}{2}|} + |t-s'|^{\lambda-1}\frac{|\frac{t-s'}{2}|}{|\sin\frac{t-s'}{2}|}\right) \\ &\leq K_0 k_u(|t-s|^{\lambda-1} + |t-s'|^{\lambda-1}), \end{aligned} \tag{C.2}$$

the last inequality coming from

$$\frac{x}{\sin x} = 1 + O(x^2). \tag{C.3}$$

One computes:

$$\int_{s-\varepsilon}^{s+\varepsilon} |t-s|^{\lambda-1}dt = \frac{2}{\lambda}\varepsilon^\lambda = \frac{2^{\lambda+1}}{\lambda}|s-s'|^\lambda; \tag{C.4}$$

from $|t - s'| \leq |t - s| + |s' - s| \leq 3\varepsilon/2$, we get:

$$\int_{s-\varepsilon}^{s+\varepsilon} |t-s'|^{\lambda-1}dt \leq \int_{s'-3\varepsilon/2}^{s+3\varepsilon/2} |t-s'|^{\lambda-1}dt = \frac{2}{\lambda}\left(\frac{3\varepsilon}{2}\right)^\lambda = \frac{2\cdot 3^\lambda}{\lambda}|s-s'|^\lambda. \tag{C.5}$$

Using (C.2)–(C.5) yields:

$$\left|\int_{s-\varepsilon}^{s+\varepsilon} \varphi(t)dt\right| \leq K_0\frac{2(2^\lambda + 3^\lambda)}{\lambda}k_u|s-s'|^\lambda \doteqdot K_1(\lambda)k_u|s-s'|^\lambda. \tag{C.6}$$

(ii) $|t - s| \geq \varepsilon$

One sets

$$\begin{aligned} \varphi(t) &= \frac{u(s) - u(s')}{2|\sin\frac{t-s}{2}|} + (u(t) - u(s'))\left(\frac{1}{2|\sin\frac{t-s'}{2}|} - \frac{1}{2|\sin\frac{t-s}{2}|}\right) \\ &\doteqdot \varphi_1(t) + \varphi_2(t). \end{aligned} \tag{C.7}$$

Then, one has:

$$\begin{aligned} \left|\int_{s+\varepsilon}^{2\pi+s-\varepsilon} \varphi_1(t)dt\right| &\leq k_u|s-s'|^\lambda\int_{s+\varepsilon}^{2\pi+s-\varepsilon}\frac{dt}{2|\sin\frac{t-s}{2}|} \\ &= k_u|s-s'|^\lambda \ln\left(\frac{1+\cos\frac{\varepsilon}{2}}{1-\cos\frac{\varepsilon}{2}}\right) \\ &= k_u|s-s'|^\lambda((-2\ln\varepsilon) + O(1)) \\ &\leq K_2 k_u|s-s'|^\mu, \quad 0 < \mu < \lambda. \end{aligned} \tag{C.8}$$

From the bound:

$$\left| \left| \sin \frac{t-s}{2} \right| - \left| \sin \frac{t-s'}{2} \right| \right| \le \left| \sin \frac{t-s}{2} - \sin \frac{t-s'}{2} \right| \le 2 \left| \sin \frac{s'-s}{4} \right| \le \frac{|s-s'|}{2},$$

we get:

$$|\varphi_2(t)| \le \frac{|u(t)-u(s')|}{4|\sin\frac{t-s'}{2}||\sin\frac{t-s}{2}|}|s-s'|,$$

and hence, using again (C.3):

$$\left| \int_{s+\varepsilon}^{2\pi+s-\varepsilon} \varphi_2(t)dt \right| \le k_u|s-s'| \int_{s+\varepsilon}^{2\pi+s-\varepsilon} |t-s|^{-1}|t-s'|^{\lambda-1}(1+O(|t-s|^2+|t-s'|^2)dt;$$

finally, using $|t-s'| \ge ||t-s|-|s-s'|| \ge |t-s|-\varepsilon/2 \ge |t-s|/2$, we get, after some easy computations:

$$\left| \int_{s+\varepsilon}^{2\pi+s-\varepsilon} \varphi_2(t)dt \right| \le \frac{K}{|\lambda-1|}\left(\frac{3}{2}\right)^{\lambda-1} k_u|s-s'|^\lambda \doteqdot K_3(\lambda)k_u|s-s'|^\lambda. \tag{C.9}$$

Putting together (C.1), (C.6)–(C.9) gives the final estimate:

$$|Tu(s')-Tu(s)| \le K(\lambda)k_u|s-s'|^\mu \le K(\lambda)|s-s'|^\mu\|u\|_{C^\lambda}. \tag{C.10}$$

Consequently, the bound (C.10) implies that $Tu \in C^\mu$ and also that T is a continuous mapping.

(2) For $\mu \in (0,\lambda)$, this is a consequence of point (1); since C^β, $\beta \in (0,1)$ is dense in C^0, the claim is also true for $\mu = 0$. □

Proof of Proposition 5.1. First, we show the symmetry of T, that is: the operator $T|_X$ is symmetric relative to the natural dot product $(\cdot,\cdot)$ of L^2. Let $u, v \in X$; then:

$$\begin{aligned}(v,Tu)-(u,Tv) &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \frac{v(s)u(t)-u(s)v(t)}{|\sin\frac{t-s}{2}|}dtds \\ &= -\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \frac{v(t)u(s)-u(t)v(s)}{|\sin\frac{t-s}{2}|}dsdt \\ &= (u,Tv)-(v,Tu).\end{aligned}$$

Then, as in Ref. 9, we show that the complex extension of T, say T_C, to the space Π_C spanned by the complex valued trigonometric polynomials is essentially self-adjoint from the Hilbert space of complex square integrable functions L^2_C to itself. Obviously, Π_C is dense in L^2_C and T_C, is symmetric.

We have to show (see for example Ref. 18, corollary on p. 257) that the range $\mathrm{Ran}(T_C \pm i\mathbb{I})$ is dense in L^2_C; let $z = z_1 + z_2 \in \Pi_C$ be a complex trigonometric polynomial of degree N:

$$z = a_0 + ib_0 + \sum_{k=1}^{N}((a_k + ib_k)\cos kx + (c_k + id_k)\sin kx);$$

let us show that there exists

$$y = \overline{a}_0 + i\overline{b}_0 + \sum_{k=1}^{N}((\overline{a}_k + i\overline{b}_k)\cos kx + (\overline{c}_k + i\overline{d}_k)\sin kx) \in \Pi_C$$

such that $T_C y \pm iy = z$. According to Lemma 5.1, one has

$$T_C y \pm iy = (\alpha_0 \pm i)(\overline{a}_0 + i\overline{b}_0) + \sum_{k=1}^{N}(\alpha_k \pm i)((\overline{a}_k + i\overline{b}_k)\cos kx + (\overline{c}_k + i\overline{d}_k)\sin kx);$$

then, the condition $T_C y \pm iy = z$ is equivalent to the linear systems

$$\begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}\begin{pmatrix} \overline{a}_0 \\ \overline{b}_0 \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \quad \begin{pmatrix} \alpha_k & \mp 1 \\ \pm 1 & \alpha_k \end{pmatrix}\begin{pmatrix} \overline{a}_k & \overline{c}_k \\ \overline{b}_k & \overline{d}_k \end{pmatrix} = \begin{pmatrix} a_k & c_k \\ b_k & d_k \end{pmatrix};$$

the determinant of the matrices are equal to $1 + \alpha_k^2 > 0$; consequently, $(T_C \pm i\mathbb{I})\Pi_C = \Pi_C$.

The domain and spectrum of $\overline{T}$ are obtained from the Lemma on page 314 of Ref. 19. □

Proof of Proposition 5.2. From (5.23), we get

$$f_n = o\left(\frac{1}{n^{1+\mu}}\right),$$

which, according to Ref. 1, Chap. II, § 3, yields $f \in C^\mu$. Then, the Fourier coefficients of the unique solution φ of (5.21) satisfy

$$\varphi_n = O\left(\frac{1}{n^{1+\lambda}}\right)$$

and consequently $\varphi \in C^\lambda$. We know, thanks to Lemma 5.2, that the operator T is bounded from C^λ to C^μ and we can write

$$\begin{aligned}(c+T)\varphi &= (c+T)\lim_{|n|\to\infty}\sum_{|k|\le n}\varphi_k p_k = \lim_{|n|\to\infty}\sum_{|k|\le n}\varphi_k(c+T)p_k \\ &= \lim_{|n|\to\infty}\sum_{|k|\le n}\varphi_k(c+\overline{T})p_k = \sum_{k=-\infty}^{+\infty}\varphi_k(c+\alpha_k)p_k = f.\end{aligned}$$ □

Proof of Lemma 5.3. The matrix of the linear system (5.24) is given by

$$M_{ij} = \begin{cases} \mathcal{C} + 2\ln\dfrac{h}{2}, & i = j, \\ \ln\dfrac{j-i+\frac{1}{2}}{j-i-\frac{1}{2}}, & i \neq j, \quad |i-j| \leq p; \\ \dfrac{h}{2|\sin\frac{(i-j)h}{2}|}, & |i-j| > p. \end{cases} \tag{C.11}$$

We set $N = M + \alpha I$, $\alpha = -2\ln\frac{h}{2}$, and notice that $\alpha \to \infty$ when $h \to 0$ and that, for example with the maximum norm $\|N\|$ is bounded independently of h, we can write, for h sufficiently small:

$$M^{-1} = -\alpha^{-1}(I - \alpha^{-1}N) = -\alpha^{-1}\sum_{k=0}^{\infty}(\alpha^{-1}N)^k. \tag{C.12}$$

□

Proof of Lemma 5.4. From (5.26), we get:

$$\begin{aligned} b_i &= h\sum_{|j-i|>p}\frac{u(m_j)}{2|\sin\frac{m_j-m_i}{2}|} - \frac{1}{2}\int_{m_i+\delta}^{2\pi+m_i-\delta}\frac{u(t)}{|\sin\frac{t-m_i}{2}|}dt \\ &\quad + \sum_{|j-i|\leq p}u(m_j)\int_{s_{j-1}}^{s_j}\frac{dt}{|t-m_i|} - Pf\int_{m_i-\delta}^{m_i+\delta}\frac{u(t)}{|t-m_i|}dt \\ &\quad + Pf\int_{m_i-\delta}^{m_i+\delta}\frac{u(t)}{|t-m_i|}dt - \frac{1}{2}Pf\int_{m_i-\delta}^{m_i+\delta}\frac{u(t)}{|\sin\frac{t-m_i}{2}|}dt \\ &= \sum_{|j-i|>p}\left(\frac{hu(m_j)}{2|\sin\frac{m_j-m_i}{2}|} - \frac{1}{2}\int_{s_{j-1}}^{s_j}\frac{u(t)}{|\sin\frac{t-m_i}{2}|}dt\right) \\ &\quad + \sum_{\substack{j\neq i\\|j-i|\leq p}}\int_{s_{j-1}}^{s_j}\frac{u(mj)-u(t)}{|t-m_i|}dt + \left(u(m_i)Pf\int_{s_{i-1}}^{s_i}\frac{dt}{|t-m_i|} - Pf\int_{s_{i-1}}^{s_i}\frac{u(t)}{|t-m_i|}dt\right) \\ &\quad + Pf\int_{m_i-\delta}^{m_i+\delta}\frac{u(t)}{|t-m_i|}dt - \frac{1}{2}Pf\int_{m_i-\delta}^{m_i+\delta}\frac{u(t)}{|\sin\frac{t-m_i}{2}|}dt \\ &\doteqdot I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{C.13}$$

We rewrite I_1 as:

$$\begin{aligned} I_1 &= \sum_{|j-i|>p}\frac{1}{2}\int_{s_{j-1}}^{s_j}\frac{u(m_j)-u(t)}{|\sin\frac{t-m_i}{2}|}dt \\ &\quad + \sum_{|j-i|>p}\frac{u(m_j)}{2}\left(\frac{h}{|\sin\frac{m_j-m_i}{2}|} - \int_{s_{j-1}}^{s_j}\frac{1}{|\sin\frac{t-m_i}{2}|}dt\right) = I_{11} + I_{12}. \end{aligned}$$

The first sum can be bounded as follows:

$$|I_{11}| \leq k_u \left(\frac{h}{2}\right)^{\lambda} \frac{1}{2} \int_{m_i+\delta}^{2\pi+m_i-\delta} \frac{dt}{|\sin \frac{t-m_i}{2}|} = k_u \left(\frac{h}{2}\right)^{\lambda} \ln \frac{1+\cos\frac{\delta}{2}}{1-\cos\frac{\delta}{2}}$$

$$= k_u \left(\frac{h}{2}\right)^{\lambda} (-2\ln\delta + O(1)) \leq K_{11} h^{\lambda} |\ln h|;$$

for the second one, we use the error of the Gauss integration rule with the point in the middle of (s_{j-1}, s_j):

$$|I_{12}| \leq K_{12} \|u\|_{\infty} h^2 \sum_{|j-i|>p} 1 < K_{12} \|u\|_{\infty} h.$$

Hence, we have:

$$|I_1| \leq K_1 h^{\lambda} |\ln h|. \tag{C.14}$$

The Hölder continuity of u implies that, for small h:

$$|I_2| \leq k_u \left(\frac{h}{2}\right)^{\lambda} \sum_{\substack{j \neq i \\ |j-i| \leq p}} \int_{s_{j-1}}^{s_j} \frac{dt}{|t-m_i|} = k_u \left(\frac{h}{2}\right)^{\lambda} \int_{(m_i-\delta, m_i+\delta) \setminus (s_{i-1}, s_i)} \frac{dt}{|t-m_i|}$$

$$= k_u \left(\tfrac{h}{2}\right)^{\lambda} \left(2\ln\frac{2\delta}{h}\right) \leq K_2 h^{\lambda}. \tag{C.15}$$

One can check easily that:

$$|I_3| = \left| \int_{s_{i-1}}^{s_i} \frac{u(m_i) - u(t)}{|t-m_i|} dt \right| \leq \frac{2k_u}{\lambda} \left(\frac{h}{2}\right)^{\lambda}. \tag{C.16}$$

With the help of

$$k(t) = \left(\frac{1}{|t|} - \frac{1}{2|\sin\frac{t}{2}|}\right) = k(-t) \tag{C.17}$$

we write:

$$|I_4| \leq \|u\|_{\infty} \int_{m_i-\delta}^{m_i+\delta} |k(t-m_i)| dt = 2\|u\|_{\infty} \int_0^{\delta} |k(x)| dx = O(\delta^2)$$

which, together with (C.13)–(C.16) yields the result, since $\delta = (p+1/2)h$. □

References

1. N. K. Bary, A Treatise on Trigonometric Series, Vol. 1 (The Macmillan Company, 1964).
2. M. Carlen, Analysis and simulation of Stokes flow of knotted filaments, Master's thesis, EPF-Lausanne, Switzerland (2005).
3. M. Carlen, P. Caussignac and J. H. Maddocks, Simulation of the Stokesian sedimentation of ideal knots with a boundary element method, submitted (2008).
4. A. T. Chwang and T. Y. Wu, Hydromechanics of low Reynolds-number flow. Part 2. Singularity method for Stokes flow, *J. Fluid Mech.* **67** (1975) 787–815.

5. R. Estrada and R. P. Kanwal, *Singular Integral Equations* (Birkhäuser, 2000).
6. T. M. Fischer, An integral equation procedure for the exterior three-dimensional slow viscous flow, *Integr. Eqn. Oper. Th.* **5** (1982) 490−505.
7. G. Galdi, *On the Motion of a Rigid Body in a Viscous Liquid: A Mathematical Analysis with Applications*, Vol. 1, Chap. 7 (Elsevier, 2002).
8. T. Goetz, Interaction of fibers and flow: Asymptotics, theory and numerics, Ph.D. thesis, University of Kaiserslautern, Germany (2000).
9. T. Goetz and A. Unterreiter, Analysis and numerics of an integral equation model for slender bodies in low Reynolds-number flows, *J. Integral Eqs. Appl.* **12** (2000) 225−270.
10. O. Gonzalez, A. B. A. Graf and J. H. Maddocks, Dynamics of a rigid body in a Stokes fluid, *J. Fluid Mech.* **519** (2004) 133−160.
11. J. L. Guermond, A systematic formula for the asymptotic expansion of singular integrals, *J. Appl. Math. Phys.* (ZAMP) **38** (1987) 717−729.
12. W. Hackbusch, *Integral Equations: Theory and Numerical Treatment* (Birkhäuser, 1995).
13. R. E. Johnson, An improved slender-body theory for Stokes flow, *J. Fluid Mech.* **99** (1980) 411−431.
14. J. B. Keller and S. I. Rubinow, Slender-body theory for slow viscous flow, *J. Fluid Mech.* **75** (1976) 705−714.
15. O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow* (Gordon and Breach, 1969).
16. M. B. Mackaplow and E. S. G. Shaqfeh, A numerical study of the rheological properties of suspensions of rigid, non-Brownian fibers, *J. Fluid Mech.* **329** (1996) 155−186.
17. E. J. Rawdon, Can computers discover ideal knots, *Exp. Math.* **12** (2003) 287−302.
18. M. Reed and B. Simon, Functional analysis, in *Methods of Modern Mathematical Physics* (Academic Press, 1980).
19. F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Dover, 1990).
20. J. Rotne and S. Prager, Variational treatment of hydrodynamic interaction in polymers, *J. Chem. Phys.* **50** (1969) 4831−4837.
21. A. Sellier, Asymptotic solution of 2D and 3D boundary integral equations arising in fluid mechanics and electrostatics, *Comp. Mech.* **25** (2000) 600−612.
22. M. J. Shelley and T. Ueda, The Stokesian hydrodynamics of flexing, stretching filaments, *Physica D* **146** (2000) 221−245.
23. J. P. K. Tillet, Axial and transverse Stokes flow past slender axisymmetric bodies, *J. Fluid Mech.* **44** (1970) 401−417.
24. A.-K. Tornberg and M. J. Shelley, Simulating the dynamics and interactions of flexible fibers in Stokes flow, *J. Comput. Phys.* **196** (2004) 8−40.