Fourier approximation of symmetric ideal knots

M. Carlen, H. Gerlach

Institut de Mathématiques B, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland, {mathias.carlen, henryk.gerlach}@epfl.ch

ABSTRACT

Enforcing a specific symmetry group on a curve, knotted or not, is not trivial using standard interpolations such as polygons or splines. For a prescribed symmetry group we present a symmetrization process based on a Fourier description of a knot. The presence of symmetry groups implies a characteristic pattern in the Fourier coefficients. The relations between the coefficients are shown for five ideal knot shapes with their proposed symmetry groups.

Keywords: ideal knots, Fourier knots, symmetry of curves, Fourier coefficient pattern

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1. Introduction

In the field of geometric knot theory, various discretization techniques of curves have been adopted. In particular, for the problem of ideal knots – minimizing the length of a given knot with prescribed thickness – authors have used polygons[17,18], arcs of circles[3], biarcs[8] and Fourier knots[16]. These representations all have their own strengths and limitations, such as speed of computation, complexity, convergence or accurate approximation of curvature and torsion. Many computations suggest that some ideal knot shapes have inherent symmetries (see Figures 2, 3, 4, 5, 6 for examples).

In this paper we propose two results. First, a technique to exactly symmetrize an almost symmetric, closed curve, second, relationships in the pattern structure of the Fourier coefficients for several symmetric knots. The pattern structure can be used to reduce the number of degrees of freedom for either analytical or numerical treatments of Fourier representations.

In the first part of the paper we briefly review the notion of ideal knots and describe a Fourier representation for closed curves. Then we introduce the symmetrization process based on this parameterization. Given a symmetric knot, we show that its Fourier coefficients are not independent and list these relationships for the symmetric trefoil, figure eight knot, $5_1$, $8_{18}$ and $10_{123}$ knots. In the closing part of the paper we discuss practical implications of these relationships to the computations of ideal knots.
2. Ideal Fourier knots

To a continuous closed curve $\gamma : S \rightarrow \mathbb{R}^3$, ($S = \mathbb{R}/\mathbb{Z}$) we can assign a thickness $\Delta[\gamma] \in \mathbb{R}$, defined as the infimum of the radii of all circles passing through three points and corresponding to three distinct parameters, in the image of $\gamma$. Any thick curve (i.e. $\Delta[\gamma] > 0$) has a $C^1(S, \mathbb{R}^3)$ constant-speed parameterization [15]. For a fixed knot class we call minimizers of the ropelength $L[\cdot]/\Delta[\cdot]$ – i.e. length divided by thickness – ideal knots [19,14,6,13].

A periodic function $f(t)$ can be written as a Fourier series [21,9] where the linearly independent base functions are $\sin(kt)$ and $\cos(kt)$ for $t \in [0, 2\pi)$ and $k = 0, 1, \ldots$. For a closed curve $\gamma(t), t \in S$ embedded in $\mathbb{R}^3$ we can therefore define three Fourier series, one for each coordinate function of $\gamma$.

Definition 2.1. Let $C$ be a finite sequence of pairs of $\mathbb{R}^3$-vectors:

$$C = \{(a_i, b_i)\}_{i=1}^k, a_i, b_i \in \mathbb{R}^3.$$  

We can use such a sequence as Fourier coefficients and define

$$\gamma(t) := \sum_{i=1}^k (a_i \cos(f_i t) + b_i \sin(f_i t)), \quad t \in [0, 1]$$

as a curve in $C^\infty(S, \mathbb{R}^3)$ with frequencies $f_i = 2\pi i$. If the curve is injective, we call $\gamma$ a Fourier knot [10].

Remark 2.2. Note that the sum in the above definition starts at $i = 1$ and the constant $a_0$ is neglected. This term is independent of the parameter $t$ and therefore just a translation of the Fourier knot.

The Fourier representation of knots has been used in [16,22], where the emphasis is on finding simple Fourier shapes of given knot types. The following lemma justifies the use of Fourier knots to approximate ideal knot shapes.

Lemma 2.3. Let $\gamma \in C^{1,1}(S)$ be an ideal shape. Then for every $\varepsilon > 0$ there exists a finite set of coefficients $\{(a_i, b_i)\}_i$ such that the Fourier knot

$$\gamma_\varepsilon(t) := \sum_i (a_i \cos(f_i t) + b_i \sin(f_i t))$$

satisfies

$$\|\gamma - \gamma_\varepsilon\|_{C^1(S)} < \varepsilon \quad \text{and} \quad \Delta[\gamma] - \varepsilon < \Delta[\gamma_\varepsilon].$$

Proof. By [11] Corollary 3.3 there exists a curve $\gamma_{\infty, \varepsilon} \in C^\infty(S, \mathbb{R}^3)$ such that

$$\|\gamma - \gamma_{\infty, \varepsilon}\|_{C^1(S, \mathbb{R}^3)} < \varepsilon/2 \quad \text{and} \quad \Delta[\gamma] - \varepsilon/2 < \Delta[\gamma_{\infty, \varepsilon}].$$

By standard Fourier theory there exists a sequence $\{\gamma_i\}_i$ of Fourier curves – each represented by a finite number of coefficients – that approximate $\gamma_{\infty, \varepsilon}$ in a $C^2$-fashion. This approximation satisfies the hypotheses of [11] Lemma 3.2 and for a
large enough index $i^*$ we have

$$\|\gamma_{\infty, \varepsilon} - \tilde{\gamma}_{i^*}\|_{C^1(S)} < \varepsilon/2, \quad \Delta[\gamma_{\infty, \varepsilon}] - \varepsilon/2 < \Delta[\tilde{\gamma}_{i^*}].$$

Now $\gamma := \tilde{\gamma}_{i^*}$ is the sought Fourier knot. Ideality of $\gamma$ and $C^1$-convergence imply that indeed $L[\gamma_{\varepsilon}] \rightarrow L[\gamma]$ for $\varepsilon \rightarrow 0$.

### 3. Symmetry of curves

The definitions we give hereafter stress that a curve has a symmetry if and only if the resulting shape is identical to the original with respect to its parameterization. In other words, let $\gamma$ be the original curve and $\gamma_{\text{sym}}$ the shape after a symmetry has been applied. It is in fact only a symmetry if $\gamma(s) = \gamma_{\text{sym}}(s)$ for all $s$ (see Example 3.5).

**Definition 3.1.** Let $\gamma : S \rightarrow \mathbb{R}^N$ be a curve and $G \subset \text{Aut}(C^0(S, \mathbb{R}^N))$ a group acting on $C^0(S, \mathbb{R}^N)$. We call $\gamma$ $G$-symmetric if

$$g \gamma = \gamma \quad \forall g \in G.$$ 

We then call $G$ a symmetry-group of $\gamma$.

Here $\text{Aut}$ is the automorphism group, which is the set of bijective mappings from a mathematical object to itself. If $G$ is maximal in some respect, we may call it ‘the’ symmetry-group of $\gamma$. Just taking the maximal subgroup $G$ of $\text{Aut}(C^0(S, \mathbb{R}^3))$ such that $\gamma$ is $G$ symmetric is not what we want since it is too large. For example it would include all $\mathcal{H}$ defined for each homeomorphism $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $h \circ \gamma = \gamma$, such that $\mathcal{H}[\alpha] := h \circ \alpha$ for $\alpha \in C^0(S, \mathbb{R}^3)$.

Now we can state how to actually symmetrize a given closed curve (see Figure 7).

**Definition 3.2.** Let $\gamma \in C^0(S, \mathbb{R}^3)$ be some curve and $G \subset \text{Aut}(C^0(S, \mathbb{R}^3))$ a finite group. Then we call

$$\gamma_G := \frac{\sum_{g \in G} g \gamma}{|G|}$$

the $G$-symmetrization of $\gamma$.

As mentioned above, the parameterization of a knot plays an important role when dealing with symmetries. We now provide the tools needed to ensure that a curve and its symmetry image are identical.

**Definition 3.3.** For $x \in \mathbb{R}$ we define the parameter shift $\sigma_x : C^0(S, \mathbb{R}^3) \rightarrow C^0(S, \mathbb{R}^3)$ by $x$ of a curve $\gamma \in C^0(S, \mathbb{R}^3)$ as

$$\sigma_x[\gamma](t) := \gamma(t + x)$$

and the parameter reflection $\rho_x : C^0(S, \mathbb{R}^3) \rightarrow C^0(S, \mathbb{R}^3)$ around $x$ as

$$\rho_x[\gamma](t) := \gamma(2x - t).$$
Lemma 3.4. Shifts and reflections form a group

\[ \mathcal{P} := \{ \sigma_x : x \in \mathbb{R} \} \cup \{ \varrho_y : y \in \mathbb{R} \} \subset \text{Aut}(\mathcal{C}^0(\mathbb{S}, \mathbb{R}^N)) \]

and for \( x, y \in \mathbb{R} \)

(i) \( \sigma_x \circ \sigma_y = \sigma_{x+y} \),
(ii) \( \varrho_y \circ \varrho_x = \sigma_{y-x} \),
(iii) \( \varrho_x \circ \sigma_y = \varrho_{x-y/2} = \sigma_{-y} \circ \varrho_x \).

\[ \square \]

Note that the symmetry-groups of a curve are usually not just a subgroup of the orthogonal group \( O(3) \) but a subgroup of the product \( O(3) \) and the parameter shifts and reflections \( \mathcal{P} \). Neglecting the latter would only give a pointwise symmetrical shape, which is not appropriate for our purposes. To illuminate this difference consider the following example.

Example 3.5. Consider an egg shaped curve \( \gamma \) as in Figure 1. The point-set \( \gamma(S) \) is invariant under mirroring along the dashed line \( m \). But the curve is not, because \( \gamma(0) \neq m\gamma(0) \). Choose a \( t^\ast \) such that \( \gamma(t^\ast) = m\gamma(t^\ast) \) and define \( G := \{ \text{id}, m \circ \varrho_{t^\ast} \} \) then \( \gamma \) will be \( G \)-symmetric.

4. Symmetry of Fourier Knots

In the previous section we claimed that symmetry is easy to enforce on Fourier knots, but before we can compute the symmetrization given in Definition 3.2 we need to explain how the Fourier coefficients transform with respect to a given symmetry operation. To rotate or reflect a point in \( \mathbb{R}^3 \) we have the following matrices.

Definition 4.1. Let \( v \in \mathbb{R}^3, ||v|| = 1 \). We call \( M_v(w) = w - 2(v, w)v \) a reflection
Lemma 4.2. Let \( D_{v,\alpha} \) be a Fourier knot or shift its parameterization.

Let \( v, \alpha \) be a parameter reflection around \( v \) and \( \alpha \in \mathbb{R} \).

The next lemma describes how the coefficients in the Fourier representation transform when we reflect or rotate a Fourier knot or shift its parameterization.

**Lemma 4.2.** Let \( \gamma(t) = \sum (a_i \cos(f_i t) + b_i \sin(f_i t)) \) be a Fourier knot.

- Let \( M \in O(3) \) be an orthogonal matrix. Then
  \[
  (M \gamma)(t) = \sum_i (\tilde{a}_i \cos(f_i t) + \tilde{b}_i \sin(f_i t))
  \]
  with
  \[
  \tilde{a}_i = M a_i, \quad \tilde{b}_i = M b_i,
  \]
  or block-wise
  \[
  \begin{pmatrix} \tilde{a}_i \\ \tilde{b}_i \end{pmatrix} = U \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \quad \text{where} \quad U = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.
  \]

- Let \( \sigma_x \) be a parameter shift by some \( x \in \mathbb{R} \). Then
  \[
  (\sigma_x \gamma)(t) = \sum_i (\tilde{a}_i \cos(f_i t) + \tilde{b}_i \sin(f_i t))
  \]
  with
  \[
  \tilde{a}_i = \cos(f_i x) a_i + \sin(f_i x) b_i, \quad \tilde{b}_i = -\sin(f_i x) a_i + \cos(f_i x) b_i,
  \]
  or block-wise
  \[
  \begin{pmatrix} \tilde{a}_i \\ \tilde{b}_i \end{pmatrix} = S_x \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \quad \text{where} \quad S_x = \begin{pmatrix} \cos(f_i x)I & \sin(f_i x)I \\ -\sin(f_i x)I & \cos(f_i x)I \end{pmatrix}.
  \]

- Let \( g_x \) be a parameter reflection around \( x \in \mathbb{R} \). Then
  \[
  (g_x \gamma)(t) = \sum_i (\tilde{a}_i \cos(f_i t) + \tilde{b}_i \sin(f_i t))
  \]
  with
  \[
  \tilde{a}_i = \cos(2f_i x) a_i + \sin(2f_i x) b_i, \quad \tilde{b}_i = \sin(2f_i x) a_i - \cos(2f_i x) b_i,
  \]
  or block-wise
  \[
  \begin{pmatrix} \tilde{a}_i \\ \tilde{b}_i \end{pmatrix} = R_x \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \quad \text{where} \quad R_x = \begin{pmatrix} \cos(2f_i x)I & \sin(2f_i x)I \\ \sin(2f_i x)I & \cos(2f_i x)I \end{pmatrix}.
  \]

- A composition \( g \) of orthogonal maps, parameter shifts and reflections acts on the Fourier coefficients of frequency \( f_i \) by a matrix \( F_{g,i} \) that is a product of the corresponding block matrices \( U, S_x \) and \( R_x \) above.
Note that matrices $S_x$ and $R_x$ corresponding to the parameter shifts and reflections commute with the orthogonal matrices $U$ as expected.

A priori we do not know a symmetry group for an ideal knot since there is no analytic expression for most of them. However studying approximately ideal shapes obtained by the numerical computations suggests possible symmetries.

Conjecture 4.1. After a reparametrization, a translation and a rotation the following symmetry-groups are suggested by the numerical data:

- Trefoil $(3_1)$ \cite{11,18,20}:
  \[ G_{3_1} = \{(D_{(0\ 0\ 1)^t}, 2\pi/3 \cdot \sigma_{1/3})^i \circ (D_{(1\ 0\ 0)^t}, \pi \cdot \varrho_0)^j : i = 0,1,2, \ j = 0,1\}. \]
  This group has six elements and is isomorphic to the symmetric group of degree 3.

- Figure eight knot $(4_1)$ \cite{11,18,20}:
  \[ G_{4_1} = \{(D_{(1\ 0\ 1)^t}, \pi \cdot \sigma_{1/2})^i \circ (M_{(0\ 1\ 0)^t} \circ D_{(0\ 1\ 0)^t}, \pi/2 \cdot \sigma_{1/4})^j : i,j = 0,1\}. \]
  This group has $|G_{4_1}| = 4$ elements.

- The $5_1$-knot \cite{11} has only two elements in its symmetry group given by
  \[ G_{5_1} = \{(D_{(0\ 1\ 0)^t}, \varrho_0)^j : j = 0,1\}. \]

- The $8_{18}$-knot \cite{11} appears to be highly symmetric with 32 elements in its symmetry group given by
  \[ G_{8_{18}} = \{(D_{(0\ 0\ 1)^t}, \pi/2 \cdot \sigma_{-1/4})^i \circ (M_{(0\ 0\ 1)^t} \circ D_{(0\ 0\ 1)^t}, \pi/4 \cdot \sigma_{5/8})^j : i = 0,\ldots,3, j = 0,\ldots,7\}. \]

- The $10_{123}$-knot \cite{11} also appears to be highly symmetric with 50 elements in its symmetry group:
  \[ G_{10_{123}} = \{(D_{(0\ 0\ 1)^t}, \pi/5 \cdot \sigma_{-3/5})^i \circ (M_{(0\ 0\ 1)^t} \circ D_{(0\ 0\ 1)^t}, \pi/5 \cdot \sigma_{-3/10})^j : i = 0,\ldots,4, j = 0,\ldots,9\}. \]

The symmetries for these knots are visualized in Figures \cite{2,3,4,5,6}. The trefoil has a 3-symmetry by rotating the knot by an angle of $2\pi/3$ around a first symmetry axis. The second set of symmetries is a rotation by $\pi$ around the axis from the center of the knot through the middle of an ear as seen in Figure \cite{2}. The generators of the symmetry group are one of each type stated before. Figure \cite{7} illustrates the symmetrization process on a perturbed trefoil knot $\gamma_{3_1}$, the 6 intermediate shapes $g\gamma_{3_1}$ for $g \in G_{3_1}$ and the final symmetrized trefoil knot. The figure eight knot has a slightly less obvious symmetry group. A rotation by $\pi$ about the symmetry axis in Figure \cite{3} is a first symmetry. Another symmetry is obtained by rotating the knot by $\pi/2$ around the same axis and reflecting it through the transparent plane. Together they generate a symmetry group with four elements. Finally
the 5_1-knot can be mapped to itself by a rotation by \( \pi \) around the only symmetry axis as depicted in Figure 4. The authors in [1] briefly discuss the symmetries for the 8_{18} and 10_{123} knots. They do not explicitly state the symmetries which are first, a rotation by \( \pi/2 \) (\( 2\pi/5 \) for the 10_{123}) around the central symmetry axis and second, a rotation by \( \pi/8 \) (\( \pi/10 \) for the 10_{123}) followed by a reflection on the plane orthogonal to the symmetry axis. Note that we did not include the reparametrization in our explanation since it is intrinsic to the curve and not visualized in the figures.

**Theorem 4.3.** Let \( \gamma(t) = \sum_{i=1}^{k} (a_i \cos(f_i t) + b_i \sin(f_i t)) \), \( t \in [0, 1] \) be a curve in \( C^\infty(S, \mathbb{R}^3) \) with coefficients \( a_i, b_i \in \mathbb{R}^3 \), frequencies \( f_i = 2\pi i \) and let \( G \) be a finite subgroup of actions such that each action \( g \) can be represented for each frequency \( f_i \) by a block-matrix \( F_{g,i} \) as in Lemma 4.2. Then \( \gamma \) is \( G \)-symmetric iff

\[
\begin{pmatrix} a_i \\ b_i \end{pmatrix} = F_i \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \text{for all } i = 1, \ldots, k
\]  

(4.1)

where

\[
F_i := \frac{\sum_{g \in G} F_{g,i}}{|G|} \in \mathbb{R}^{6 \times 6}
\]  

(4.2)

is the frequency-wise average.

**Proof.** First, we note that two Fourier knots describe the same curve iff all coefficients coincide, since \( \sin(f_i t) \) and \( \cos(f_i t) \) form an orthogonal basis. Second, we note that some curve \( \gamma \) is \( G \)-symmetric iff \( \gamma = \gamma_G \) pointwise. Consequently, we have

\[
\begin{pmatrix} a_i \\ b_i \end{pmatrix} = F_i \begin{pmatrix} a_i \\ b_i \end{pmatrix}
\]  

(4.3)

for each frequency \( f_i \) iff \( \gamma \) is \( G \) symmetric as claimed. \( \square \)

**Lemma 4.4.** Let \( G = \{g_1, \ldots, g_n\} \) be a finite subgroup of \( \mathbb{R}^{3 \times 3} \times \mathcal{P} \).

(i) Then each element \( g_j \in G \) can be represented either as \( g_j = \alpha_j \circ \sigma_{p_j} \) or \( g_j = \alpha_j \circ \sigma_{q_j} \) with \( \alpha_j \in \mathbb{R}^{3 \times 3} \) and \( p_j \in \mathbb{Q}, q_j \in \mathbb{R} \).

(ii) If all parameter reflections occur around rational parameters, i.e. \( q_j \in \mathbb{Q} \) as in (i) for all \( j \), then there exists some \( k \in \mathbb{N} \) such that \( kq_j \) and \( kp_j \) are integers for all \( j \) and

\[
F_i := \frac{\sum_{g \in G} F_{g,i}}{|G|} = F_{i+k} \quad \text{for all } i \in \mathbb{N}.
\]

**Proof.**

(i) The representation is a direct consequence of Lemma 3.4. It remains to show that \( p_j \) is rational. Assume \( p_j \) was irrational. Then \( \{g_j : i \in \mathbb{N}\} \subset G \) would be infinite contradicting the finiteness of \( G \).
(ii) Since \( p_j \) and \( q_j \) are rational, define \( k \) as the least common multiple of all denominators. Then \( kp_j \) and \( kq_j \) are integers. Consequently in Lemma 4.2 we have 
\[
\cos(f_ip_j) = \cos(2\pi ip_j) = \cos(2\pi (i+k)p_j) = \cos(f_{i+k}x)
\]
and likewise \( \sin(f_ip_j) = \sin(f_{i+k}p_j) \) which implies \( F_i = F_{i+k} \).

**Remark 4.5.** If a curve \( \gamma \) is \( G \)-symmetric and \( F_i \) are the frequency-wise averages then \( \sigma_x \gamma \) is \( \sigma_x G \sigma_x \)-symmetric and the identity (4.3) becomes
\[
(a_ib_i) = S_x F_i S_{-x} (a_ib_i).
\]
In particular the identity remains unchanged if \( G \) does not contain parameter reflections. Similarly the coefficients of \( \rho_x \gamma \) satisfy
\[
(a_ib_i) = R_x F_i R_x (a_ib_i).
\]

As suggested in Conjecture 4.1, the trefoil knot, the figure eight, the 5_1, 8_18 and the 10_{123} are believed to have particular symmetry groups. We now derive the coefficient relations for these knots using equation 4.1 and the different matrices introduced in Lemma 4.2.

**Example 4.6.** Assume that symmetry Conjecture 4.1 is true. Then the Fourier coefficients must fulfill the following equations:

(i) Trefoil: Computing (4.2) yields:
\[
F_{3i+1} = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ \end{pmatrix},
\]
\[
F_{3i+2} = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ \end{pmatrix},
\]
\[
F_{3i+3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 \ \end{pmatrix},
\]
for \( i \in \mathbb{N} \) and by Lemma 4.4 it is enough to compute the first three matrices. By Theorem 4.3 we deduce that the only non-zero coefficients are:
\[
a_{3i+1,1} = -b_{3i+1,2}, \quad a_{3i+2,1} = b_{3i+2,2}, \quad b_{3i+3,3},
\]
with the notation \( a_{f,k} \), where \( f \) is the frequency and \( k \) the coordinate index.
These conventions will also be used for the following knots. For the trefoil, this means that there are 3 independent, non-zero coefficients out of 18.

(ii) 4_1: For \( i \in \mathbb{N} \):
\[
a_{4i+1,1} = b_{4i+1,3}, \quad a_{4i+1,3} = -b_{4i+1,1}, \quad a_{4i+2,2} = b_{4i+2,2}, \quad a_{4i+3,1} = -b_{4i+3,3}, \quad a_{4i+3,3} = b_{4i+3,1},
\]
with 6 independent coefficients out of 24.

(iii) 5_1: For \( i \in \mathbb{N} \):
\[
a_{i,2} = b_{i,1}, \quad b_{i,3},
\]
with 3 independent coefficients out of 6.
(iv) $8_{18}$: For $i \in \mathbb{N}$:

\[
\begin{align*}
  a_{8i+3,1} &= b_{8i+3,2}, & a_{8i+3,2} &= -b_{8i+3,1}, \\
  a_{8i+4,3} &= b_{8i+4,3}, & a_{8i+5,1} &= -b_{8i+5,2}, & a_{8i+5,2} &= b_{8i+5,1},
\end{align*}
\]

with 6 independent coefficients out of 48.

(v) $10_{123}$: For $i \in \mathbb{N}$:

\[
\begin{align*}
  a_{10i+3,1} &= -b_{10i+3,2}, & a_{10i+3,2} &= b_{10i+3,1}, \\
  a_{10i+5,3} &= b_{10i+5,3}, & a_{10i+7,1} &= b_{10i+7,2}, & a_{10i+7,2} &= -b_{10i+7,1},
\end{align*}
\]

with 6 independent coefficients out of 60.

5. Conclusion

We saw in Lemma 2.3 that ideal knots can be approximated by Fourier knots merely as a consequence of standard Fourier approximation theory. We then presented a procedure to symmetrize approximately symmetric ideal knot shapes for a Fourier representation. Specifically, for a curve with a certain symmetry group, we have derived a way to compute the characteristic pattern in the Fourier coefficients due to the symmetries (Lemma 4.6). In practice, the original shape used in the symmetrization process has to be close to ideal and close to the enforced symmetries, otherwise the resulting knot might not even be in the same isotopy class. A circle for example is an ideal shape, has all the above symmetries, and its Fourier coefficients do fulfill the relations in Lemma 4.6 however it is none of the knot types we discussed.

In Example 4.6 we have seen that symmetries of a curve imply identities of its Fourier coefficients. This could, for example, be used as a measure of how symmetric a curve is. More important, the converse is also true: if we enforce the identities, then we enforce the symmetry on the Fourier knot and at the same time reduce the degrees of freedom. For example only one sixth of the Fourier trefoil ($3_1$) coefficients are non-zero compared to the original shape without enforcing the symmetry. We used this to speed up the approximation of an ideal trefoil [7,10], improving the naturally slow simulated annealing algorithm [8].

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References

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Fig. 2. Trefoil symmetry axes for two different views. The axis with the prism on top is the 3-symmetry with rotation angles of $2\pi/3$. The three ellipsoid ended axis give the second symmetry, which is a rotation of angle $\pi$. The generators of the symmetry group are one element of both types\cite{4}.

Fig. 3. Three different views of the figure-eight knot symmetries. The first symmetry is a rotation of angle $\pi$ around the symmetry axis visualized in the image. The second generator of the symmetry group is a reflection through the plane and then a rotation of angle $\pi/2$ around the symmetry axis.

Fig. 4. The symmetry group for the $5_1$ knot has only a single generator, which is a rotation of angle $\pi$ about the symmetry axis shown in the image.
Fig. 5. The symmetries of the $8_{18}$ are generated by a $\pi/2$ rotation and the composition of a reflection and a $\pi/4$ rotation.

Fig. 6. The symmetries of the $10_{123}$ are generated by a $2\pi/5$ rotation and the composition of a reflection and a $\pi/5$ rotation.

Fig. 7. The symmetrization process of Definition 3.2 visualized. Starting from a not completely symmetric trefoil in the left corner, we apply each element of the presumed symmetry group to it and take the average to receive a symmetric shape in the lower right corner.