The Calogero–Sutherland–Moser (CSM) model as a quantum billiard

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Abstract. An interpretation of the CSM model as a one-parameter family of multi-dimensional soluble quantum billiards with potential is proposed. This interpretation is based on a specific ordering of the particles, on the separation of the centre-of-mass motion of the system and on the introduction of a multi-dimensional space generated by normal coordinates. The eigenvalues and eigenfunctions of the billiard Hamiltonian are given in terms of modified Sogo's solutions. Detailed calculations are presented for the particular case of the two-dimensional equilateral triangular billiard with potential.

Résumé. On propose une interprétation du modèle de CSM comme une famille à un paramètre de billiards avec potentiel, multidimensionnels, quantiques et solubles. Cette interprétation est basée sur un ordre spécifique des particules, sur la séparation du mouvement du centre de masse du système et sur l'introduction d'un espace multidimensionnel généré par des coordonnées normales. On donne les valeurs propres et les fonctions propres de l'Hamiltonien du billiard en termes de solutions de Sogo modifiées. On présente des calculs détaillés dans le cas particulier du billiard triangulaire équilatéral à deux dimensions avec potentiel.

1. Introduction and summary

A billiard is a system consisting of one particle confined in a multi-dimensional domain. In the classical case, the particle experiences elastic collisions to the walls, while in the quantum case all the eigenfunctions have to be zero on the boundaries of the domain. Moreover, the particle can be subject to a potential.

There is currently an increasing interest for both integrable and non-integrable (chaotic) quantum billiards [1, 2]. The purpose of this paper is to propose for investigation a new integrable billiard associated with the CSM model.

The CSM model is a one-dimensional soluble system of N particles interacting pairwise via Calogero's inverse square potential [3] and put on a ring. Sutherland [4] introduced it first in the quantum case and solved the associated Schrödinger equation. A systematic method to construct the periodic eigenfunctions of the system has been exposed by Sogo [5]. Forrester [6] has recently shown that these solutions can also be expressed as Jack polynomials.

The usual physical interpretation of this model is that of a quantum fluid, in particular since almost all its excited states carry a non-vanishing linear momentum. We wish to propose here another interpretation which is that of a one-parameter family of soluble quantum billiards with potentials defined in an adequately chosen $(N-1)$-dimensional simplex. We give explicitly their energy spectra as well as a procedure to construct their eigenfunctions.

This paper is organized as follows. Starting from the original CSM Hamiltonian, we establish in section 2 the billiard Hamiltonian. This is performed by considering an ordered configuration of particles, by separating the centre-of-mass motion and by introducing the $(N-1)$-dimensional space generated by the normal mode coordinates, a procedure already employed by the present authors [7] with the aim of exhibiting the phonon energies in the energy spectrum of the CSM model. The domain of the billiard is bounded by the envelope of the hyperplanes which express the binary collisions of the ordered sequence of particles on the ring.

From Sutherland and Sogo's results for the fluid, we derive in section 3 the solutions of the Schrödinger equation associated with the billiard. For the eigenfunctions, this is performed by eliminating the centre-of-mass dependence and, for the eigenenergies, by subtracting the centre-of-mass energy.

As an illustration of our general method, we compute explicitly in section 4 the first six eigenfunctions of the two-dimensional billiard.

Finally, integrable billiards without potential are briefly discussed in section 5.

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2. The billiard Hamiltonian

Consider a collection of N particles of mass m interacting pairwise via Calogero's inverse square potential and put on a ring of circumference L. Measure their positions in units of L/π, the energies in units of $\hbar^2\pi^2/mL^2$ and the coupling constant g in units of $\hbar^2/m$. The periodized Calogero potential becomes

$$\sum_{n=-\infty}^{\infty} \frac{g}{(r+n\pi)^2} = \frac{g}{\sin^2 r}. \tag{1}$$

With x_j, $j = 1, \ldots, N$, standing for the position of the jth particle, the dimensionless Hamiltonian of the CSM fluid reads

$$H = -\frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + g \sum_{1\le j<l\le N} \frac{1}{\sin^2(x_j - x_l)}. \tag{2}$$

As discussed by Sutherland [8], we need to suppose $g > -\frac{1}{4}$ in order to have 'physically acceptable' solutions of the problem. In addition, we exclude the case $g = 0$ which will be considered in section 5.

The potential is singular whenever $x_j = x_l + n\pi$, $j \neq l$, $n \in \mathbb{Z}$, with a $1/r^2$ singularity. As shown in [8], this implies that all the wavefunctions are zero at these places and that the configuration space is split into disconnected domains.

For our purpose, we need to consider one domain only corresponding to a definite ordering of the particles, and we choose the one defined by

$$x_1 \le x_2 \le \cdots \le x_N \le x_1 + \pi. \tag{3}$$

Furthermore, in order to see our billiard, we have to eliminate the centre-of-mass motion. Since the classical ground state of the CSM model is a perfect lattice of lattice constant π/N and since, once ordered, the particles preserve their order in the course of time, it is natural to borrow from solid-state physics the basis provided by the normal mode coordinates.

Let $X = (1/N)\sum_j x_j$ be the centre-of-mass coordinate. We translate the origin of the coordinates x_j to the classical equilibrium positions of the particles and introduce the real amplitudes q_n (q_{-n}), $n = 1, \ldots, M$ (M is given below), of the spatially even (odd) standing modes. At this point we have to consider the parity of the number N of particles. For N odd, $M = (N-1)/2$ and the transformation reads

$$\begin{aligned} x_j &= -\frac{\pi}{2} + \left(j - \frac{1}{2}\right)\frac{\pi}{N} + X \\ &\quad + \sqrt{\frac{2}{N}} \sum_{n=1}^{M} \left(q_n \cos\frac{2\pi n j}{N} - q_{-n} \sin\frac{2\pi n j}{N} \right), \\ &\quad j = 1, \ldots, N, \end{aligned} \tag{4}$$

while for N even, $M = N/2$ and

$$\begin{aligned} x_j &= -\frac{\pi}{2} + \left(j - \frac{1}{2}\right)\frac{\pi}{N} + X \\ &\quad + \sqrt{\frac{2}{N}} \sum_{n=1}^{M-1} \left(q_n \cos\frac{2\pi n j}{N} - q_{-n} \sin\frac{2\pi n j}{N} \right) \\ &\quad + \frac{1}{\sqrt{N}} q_M \cos \pi j, \qquad j = 1, \ldots, N, \end{aligned} \tag{5}$$

(there is no mode indexed by $-M$ in the latter case).

As the coordinate X will disappear later, we write $x_j = X + y_j$. In the new variables the CSM fluid Hamiltonian separates as

$$H = H_{\mathrm{CM}}(X) + H_{\mathrm{B}}(\{q_n\}), \tag{6}$$

with

$$\begin{aligned} H_{\mathrm{CM}}(X) &= -\frac{1}{2N}\frac{\partial^2}{\partial X^2}, \\ H_{\mathrm{B}}(\{q_n\}) &= -\frac{1}{2}\sum_n \frac{\partial^2}{\partial q_n^2} \\ &\quad + g \sum_{1\le j<l\le N} \frac{1}{\sin^2[y_j(\{q_n\}) - y_l(\{q_n\})]}, \end{aligned} \tag{7}$$

where $n = \pm 1, \ldots, \pm M$ for N odd and $n = \pm 1, \ldots, \pm(M-1), M$ for N even.

Our billiard is defined by the Hamiltonian H_{B} and the domain (3). It corresponds to a single particle in a $(N-1)$-dimensional space and confined in the simplex obtained by expressing conditions (3) in terms of the normal mode coordinates. The $1/r^2$ singularity of the potential forces all the wavefunctions to be zero on the boundaries of the domain. Moreover, this particle is subject to a convex potential when $g > 0$ and a concave one when $g < 0$.

In the particular case $N = 2$, the one-dimensional domain is given by $\pi/2\sqrt{2} \le q_1 \le \pi/2\sqrt{2}$ and the potential is $g/\sin^2((\pi/2) + \sqrt{2}q_1)$.

For $N = 3$, the two-dimensional domain is the equilateral triangle shown in figure 1 together with a few equipotential lines.

It is interesting to note that the cubic approximation of this potential reproduces exactly the Henon–Heiles potential [9].

For $N = 4$, the three-dimensional domain is the tetrahedron illustrated in figure 2.

3. Eigenvalues and eigenfunctions

Let us go back to the variables x_j in order to recall some of Sutherland and Sogo's results [4, 5] concerning the eigenvalues $E_{\boldsymbol{k}}$ and eigenfunctions $\psi_{\boldsymbol{k}}$ of the CSM problem $H\psi_{\boldsymbol{k}} = E_{\boldsymbol{k}}\psi_{\boldsymbol{k}}$ with the periodic boundary conditions

$$\begin{aligned} &\psi_{\boldsymbol{k}}(x_1, \ldots, x_j + \pi, \ldots, x_N) = \psi_{\boldsymbol{k}}(x_1, \ldots, x_j, \ldots, x_N), \\ &\quad j = 1, \ldots, N. \end{aligned} \tag{8}$$

Let $\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{1+4g}$. The ground-state wavefunction and energy are

$$\begin{aligned} \psi_0 &= \prod_{1\le j<l\le N} |\sin(x_j - x_l)|^{\lambda}, \\ E_0 &= \frac{1}{6}N(N^2-1)\lambda^2. \end{aligned} \tag{9}$$

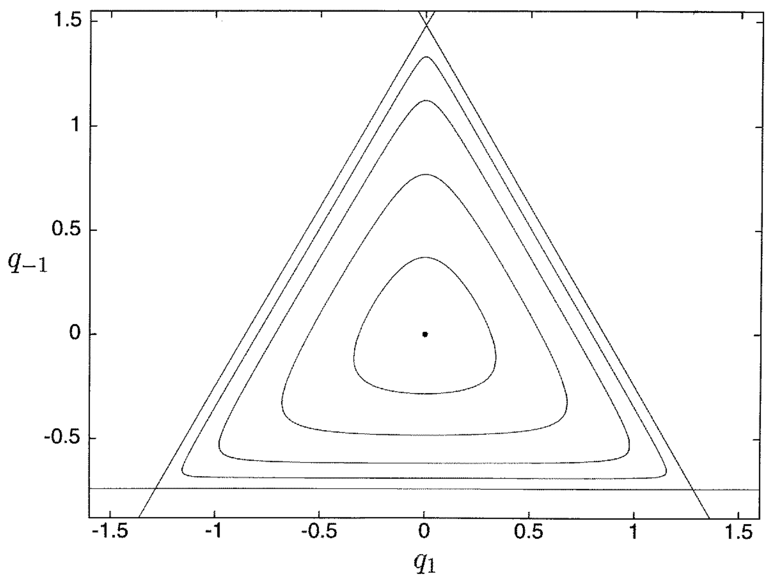

Figure 1. Contours of the potential in the triangle billiard. The central dot indicates the extremum of value $4g$. Starting from the centre, the concentric lines indicate the values $5g$, $10g$, $36g$ and $200g$, respectively.

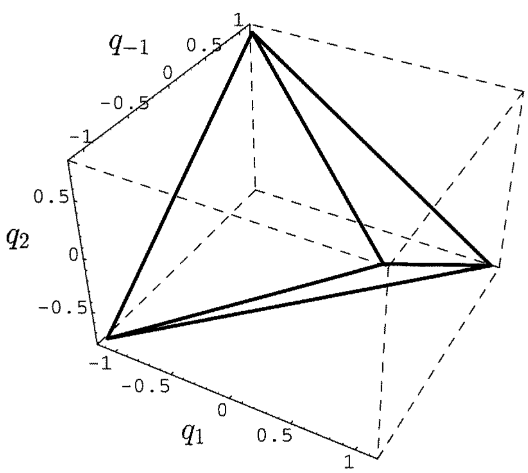

Figure 2. The tetrahedral billiard.

The quantum numbers indexing the excited states are the sets $\boldsymbol{k} = (k_1, \dots, k_N)$, where $k_1 \geq k_2 \geq \cdots \geq k_N$, and $k_j \in \mathbb{Z}$. The corresponding eigenfunctions are of the form

$$\psi_{\boldsymbol{k}} = \psi_0 \phi_{\boldsymbol{k}} \tag{10}$$

and the eigenenergies read

$$E_{\boldsymbol{k}} = E_0 + 2\sum_{j=1}^{N} [k_j^2 + \lambda k_j (N + 1 - 2j)]. \tag{11}$$

The functions $\phi_{\boldsymbol{k}}$ are constructed as follows. To each set $\boldsymbol{k}'$ of quantum numbers corresponds a Laurent polynomial $P_{\boldsymbol{k}'}$ in the variables $\exp(2\mathrm{i}x_j)$, $j = 1, \dots, N$:

$$P_{\boldsymbol{k}'} = \sum_{\pi} \exp\left(2\mathrm{i} \sum_{j=1}^{N} k'_{\pi_j} x_j\right), \tag{12}$$

where the π's are those permutations on the set $(1, 2, \dots, N)$ which give rise to distinct terms of the sum in (12). $P_{\boldsymbol{k}'}$ is said to be of order $K' = \sum_j k'_j$. Then the $\phi_{\boldsymbol{k}}$'s are linear combinations of Laurent polynomials of order K;

$$\phi_{\boldsymbol{k}} = \sum_{\substack{\boldsymbol{k}' \\ K' = K}} M_{\boldsymbol{k}}^{\boldsymbol{k}'} P_{\boldsymbol{k}'}, \tag{13}$$

where only a finite number of coefficients $M_{\boldsymbol{k}}^{\boldsymbol{k}'}$ are non-zero. A method for computing the coefficients $M_{\boldsymbol{k}}^{\boldsymbol{k}'}$ is given by Sogo [5].

From these results we wish to construct the eigenfunctions and the associated eigenvalues of the billiard Hamiltonian H_{B}. Let us consider the function

$$\chi_{\boldsymbol{k}} = \exp\left(-\mathrm{i}\frac{2}{N} K \sum_{j=1}^{N} x_j\right) \psi_{\boldsymbol{k}}. \tag{14}$$

Remembering that $\phi_{\boldsymbol{k}}$ is composed of Laurent polynomials of order K and using the coordinate transformation (4), (5), one easily verifies that $\chi_{\boldsymbol{k}}$ does not depend on the centre-of-mass coordinate X. Thus, in terms of the normal mode coordinates, we have

$$\psi_{\boldsymbol{k}}(X, \{q_n\}) = \exp(\mathrm{i}2KX) \chi_{\boldsymbol{k}}(\{q_n\}). \tag{15}$$

Introducing this expression together with (6) and (7) in the equation $H\psi_{\boldsymbol{k}} = E_{\boldsymbol{k}}\psi_{\boldsymbol{k}}$, we obtain

$$H_{\mathrm{B}} \chi_{\boldsymbol{k}} = \mathcal{E}_{\boldsymbol{k}} \chi_{\boldsymbol{k}}, \tag{16}$$

with

$$\mathcal{E}_{\boldsymbol{k}} = E_{\boldsymbol{k}} - \frac{2}{N}\left(\sum_{j=1}^{N} k_j\right)^2. \tag{17}$$

Now let $\boldsymbol{k}' = (k_1, \ldots, k_{N-1}, k_N)$ and $\boldsymbol{k} = (k_1 - k_N, \ldots, k_{N-1} - k_N, 0)$. From (10), (13) and (12), we verify that $\psi_{\boldsymbol{k}'} = \exp(2\mathrm{i}k_N \sum_j x_j)\psi_{\boldsymbol{k}}$. Then from (14) we have $\chi_{\boldsymbol{k}'} = \chi_{\boldsymbol{k}}$, since $K' = K + Nk_N$. It is therefore sufficient to consider only the quantum numbers of the form $\boldsymbol{k} = (k_1, \ldots, k_{N-1}, 0)$ for the solutions $\chi_{\boldsymbol{k}}$ of our problem with $N - 1$ degrees of freedom.

Since the $\psi_{\boldsymbol{k}}$'s form a complete set of wavefunctions of the CSM fluid with periodic boundary conditions, it is clear that our $\chi_{\boldsymbol{k}}$'s form a complete set of eigenfunctions of H_{B}. The energy spectrum of the billiard is therefore given by

$$\mathcal{E}_{\boldsymbol{k}} = \frac{1}{6}N(N^2-1)\lambda^2 + 2\sum_{j=1}^{N} k_j^2 - \frac{2}{N}\left(\sum_{j=1}^{N} k_j\right)^2 + 2\lambda\sum_{j=1}^{N} k_j(N+1-2j), \tag{18}$$

where

$$\boldsymbol{k} = (k_1, \ldots, k_{N-1}, 0), \qquad k_1 \geq \cdots \geq k_{N-1} \geq 0. \tag{19}$$

4. The two-dimensional billiard

The simplest billiard with a potential which is soluble with our method is the one-dimensional one and corresponds to $N = 2$. The eigenfunctions of this system have been expressed in an elegant form in terms of Gegenbauer polynomials by Olshanetsky and Perelomov [10].

We illustrate here our method in the more interesting case $N = 3$, corresponding to the two-dimensional triangular billiard.

We wish to compute explicitly the six eigenfunctions associated with the four lowest eigenenergies:

$$\begin{aligned} E_{(0,0,0)} &= 4\lambda^2, \\ E_{(1,0,0)} &= E_{(1,1,0)} = 4\lambda^2 + 4\lambda + \tfrac{4}{3}, \\ E_{(2,1,0)} &= 4\lambda^2 + 8\lambda + 4, \\ E_{(2,0,0)} &= E_{(2,2,0)} = 4\lambda^2 + 8\lambda + \tfrac{16}{3}. \end{aligned} \tag{20}$$

The corresponding ϕ functions are given by Sogo's method [5]. Up to a scalar factor, they read

$$\begin{aligned} \phi_{(0,0,0)} &= 1, \\ \phi_{(1,0,0)} &= P_{(1,0,0)}, \\ \phi_{(1,1,0)} &= P_{(1,1,0)}, \\ \phi_{(2,1,0)} &= (1+2\lambda)P_{(2,1,0)} + 6\lambda P_{(1,1,1)}, \\ \phi_{(2,0,0)} &= (1+\lambda)P_{(2,0,0)} + 2\lambda P_{(1,1,0)}, \\ \phi_{(2,2,0)} &= (1+\lambda)P_{(2,2,0)} + 2\lambda P_{(2,1,1)}. \end{aligned} \tag{21}$$

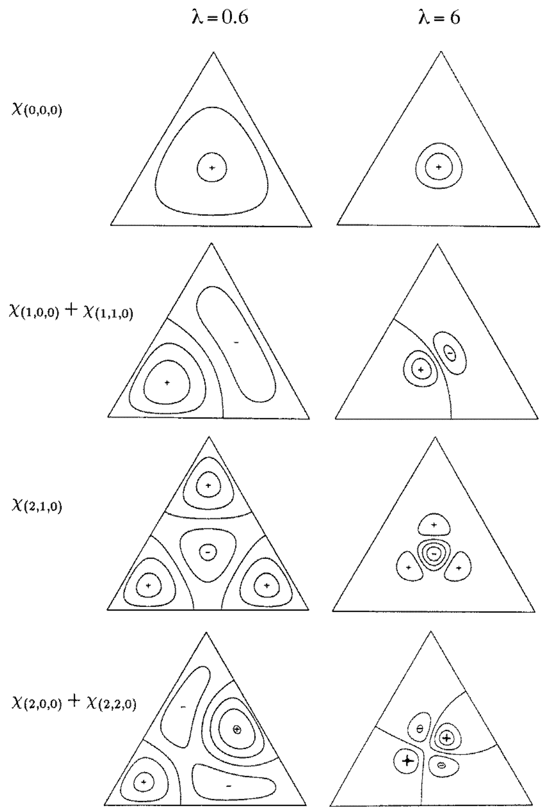

Figure 3. Contours of some (numerically normalized) eigenfunctions of the triangular billiard in the cases $\lambda = 0.6$ (left column) and $\lambda = 6$ (right column). For $\lambda = 0.6$, the lines indicate the values 0, $\pm\frac{1}{2}$, ± 1, and for $\lambda = 6$, they indicate the values 0, ± 1, ± 2. A + ($-$) sign marks a maximum (minimum).

The ground-state wavefunction is

$$\psi_0 = [\sin(x_3 - x_2)\sin(x_3 - x_1)\sin(x_2 - x_1)]^{\lambda}. \tag{22}$$

From these results, the eigenfunctions $\chi_{\boldsymbol{k}}$ are easily computed via (10), (14) and (12). For example

$$\begin{aligned} \chi_{(1,0,0)} &= \psi_0\, \mathrm{e}^{-\mathrm{i}\frac{2}{3}(x_1+x_2+x_3)}(\mathrm{e}^{2\mathrm{i}x_1} + \mathrm{e}^{2\mathrm{i}x_2} + \mathrm{e}^{2\mathrm{i}x_3}) \\ &= \psi_0(\mathrm{e}^{\mathrm{i}\frac{2}{3}(2x_1-x_2-x_3)} + \mathrm{e}^{\mathrm{i}\frac{2}{3}(-x_1+2x_2-x_3)} \\ &\quad + \mathrm{e}^{\mathrm{i}\frac{2}{3}(-x_1-x_2+2x_3)}). \end{aligned} \tag{23}$$

In the same way we find the other eigenfunctions

$$\begin{aligned} \chi_{(0,0,0)} &= \psi_0, \\ \chi_{(1,1,0)} &= \overline{\chi_{(1,0,0)}} = \psi_0(\mathrm{e}^{\mathrm{i}\frac{2}{3}(-2x_1+x_2+x_3)} + \mathrm{e}^{\mathrm{i}\frac{2}{3}(x_1-2x_2+x_3)} \\ &\quad + \mathrm{e}^{\mathrm{i}\frac{2}{3}(x_1+x_2-2x_3)}), \\ \chi_{(2,1,0)} &= \psi_0\{2(1+2\lambda)[\cos 2(x_1-x_2) + \cos 2(x_1-x_3) \\ &\quad + \cos 2(x_2-x_3)] + 6\lambda\}, \end{aligned}$$

$$\chi_{(2,2,0)} = \overline{\chi_{(2,0,0)}} = \psi_0[(1+\lambda)(e^{i\frac{4}{3}(-2x_1+x_2+x_3)} + e^{i\frac{4}{3}(x_1-2x_2+x_3)} + e^{i\frac{4}{3}(x_1+x_2-2x_3)}) + 2\lambda(e^{i\frac{2}{3}(-2x_1+x_2+x_3)} + e^{i\frac{2}{3}(x_1-2x_2+x_3)} + e^{i\frac{2}{3}(x_1+x_2-2x_3)})], \quad (24)$$

where the bar denotes complex conjugation. One observes that the complex conjugated eigenfunctions $\chi_{(1,0,0)}$ and $\chi_{(1,1,0)}$ are associated with the same eigenenergy. It is, therefore, useful to consider the functions $\chi_{(1,0,0)} + \chi_{(1,1,0)}$ and $i(\chi_{(1,0,0)} - \chi_{(1,1,0)})$ in order to have real eigenfunctions. For the same reason we consider $\chi_{(2,0,0)} + \chi_{(2,2,0)}$ and $i(\chi_{(2,0,0)} - \chi_{(2,2,0)})$.

These wavefunctions are readily expressed in terms of the normal coordinates by use of transformation (4), with $N = 3$. The contours of four of them are represented in figure 3 in the cases of a concave ($\lambda = 0.6$) and of a convex ($\lambda = 6$) potential.

Any eigenfunction of this billiard can be obtained in the same way.

5. Remark

We have exhibited a mapping from the N-particles CSM fluid to a $(N-1)$-dimensional billiard with potential. If we turn off the interaction, the CSM fluid becomes a one-dimensional system of free particles with 'hard cores' under periodic boundary conditions. This system is transformed by the same mapping into our $(N-1)$-dimensional billiard, but without potential. Since the eigenstates of the system of free particles with 'hard cores' are well known [11], the eigenstates of the $(N-1)$-dimensional billiards without potential are easily constructed.

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