A variational approach to loaded ply structures†

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Abstract

We show how an energy analysis can be used to derive the equilibrium equation and boundary conditions for an end-loaded variable ply much more efficiently than was done in previous works. Numerical results are then presented for a clamped balanced ply approaching lock-up. We also use the energy method to derive the equations for a more general ply made of imperfect anisotropic rods and briefly consider their helical solutions.

1 Introduction

Several papers have appeared recently on the problem of ply structures in one application or the other. In [3] a loaded ply of constant angle was considered for part of the configuration of a twisted textile yarn. In [2] a balanced (i.e., unloaded) ply of variable angle was studied in a model for supercoiled DNA plasmids, i.e., closed pieces of DNA. In [6] the equation for a more general loaded variable ply was derived by a direct balancing of forces and moments. The result specialises to the equations in the aforementioned papers in the appropriate limits. The result in turn is a special case of the general equation for a rod constrained to lie on a cylinder studied in [7], namely a cylinder of radius equal to the radius of the rod.

In this short paper we give an alternative variational derivation of the equation and boundary conditions for a loaded variable ply which is fully self-contained and remarkably simple when compared to the delicate balancing act performed in [6] and the careful manipulation with the surface constraint conducted in [7]. We use the result to study numerically the approach to lock-up [4, 5] of a clamped balanced ply subject to high twist. The variational method is also used to derive the equations for a more general ply made up of imperfect anisotropic rods.

2 Calculation of the energy contributions

In this section, by a ply we mean two strands of intrinsically-straight and prismatic rod of isotropic material and circular cross-section winding around a straight line segment of mutual

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contact, one strand providing a pressure force to a 180-degrees rotated copy of itself (in Section 6 some of these assumptions will be relaxed). We assume the contact to be frictionless. The (repulsive) pressure force will then be normal to the rod and workless, and will not figure explicitly in the following analysis. The ply is loaded by end forces and moments which are equivalent to an axial force F (positive for tension) and an axial twisting moment M. Let L be the length of one of the strands of the ply for which we choose the origin of arclength s at one of the ends. The total potential energy of the ply, V, can then be written as the sum of the bending energy, U_b, the torsional energy, U_t, the potential energy due to the end force, U_f, and the potential energy due to the end moment, U_m:

$$V = U_b + U_t + U_f + U_m = 2\int_0^L \left(\frac{1}{2}B\kappa^2 + \frac{1}{2}C\tau^2\right) ds - FL_p - MR. \tag{1}$$

Here κ and τ are the curvature and the twist of the ply strand, B and C are the bending and torsional stiffnesses, L_p is the length of the ply, and R is the relative end rotation of the ply, measured from the datum in which the two strands lie straight side by side (positive in the same direction as M). L_p and R are the displacements through which the end force and end moment do work, respectively. We shall now determine each of the four energy contributions individually.

Let $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ be a fixed right-handed orthonormal co-ordinate system. Since the centreline of each strand of the ply lies on an imaginary cylinder, it is convenient to introduce cylindrical co-ordinates (ρ, ψ, z) for the position vector,

$$\boldsymbol{r} = \rho\cos\psi\,\boldsymbol{i} + \rho\sin\psi\,\boldsymbol{j} + z\,\boldsymbol{k}, \tag{2}$$

where ρ is the radius and ψ the polar angle, and to define a cylindrical co-ordinate frame $\{\boldsymbol{e}_\rho, \boldsymbol{e}_\psi, \boldsymbol{e}_z\}$ with $\boldsymbol{e}_\rho$ taken normal to the cylinder, $\boldsymbol{e}_\psi$ in the circumferential direction, and $\boldsymbol{e}_z$ in the direction of the axis of the ply and the applied loads:

$$\begin{aligned} \boldsymbol{e}_\rho &= \cos\psi\,\boldsymbol{i} + \sin\psi\,\boldsymbol{j}, \\ \boldsymbol{e}_\psi &= -\sin\psi\,\boldsymbol{i} + \cos\psi\,\boldsymbol{j}, \\ \boldsymbol{e}_z &= \boldsymbol{k}. \end{aligned} \tag{3}$$

The position vector can then be written as $\boldsymbol{r} = \rho\,\boldsymbol{e}_\rho + z\,\boldsymbol{e}_z$ and we have the relations:

$$\begin{aligned} \boldsymbol{e}_\rho' &= \psi'\,\boldsymbol{e}_\psi, \\ \boldsymbol{e}_\psi' &= -\psi'\,\boldsymbol{e}_\rho, \end{aligned} \tag{4}$$

where the prime denotes differentiation with respect to arclength s.

In order to determine the torsional strain energy we need to consider the motion of a material frame as it moves along the rod. We choose a director frame $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$, with $\boldsymbol{d}_3$ tangential to the rod, i.e., $\boldsymbol{d}_3 = \boldsymbol{r}'$, and $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ along the principal axes of inertia in the normal cross-section of the rod (assumed to be inextensible and unshearable). We can parametrise the frame $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ by two angles, θ and ϕ, as follows:

$$\begin{aligned} \boldsymbol{d}_1 &= \sin\phi\,\boldsymbol{e}_\rho - \cos\phi\cos\theta\,\boldsymbol{e}_\psi + \cos\phi\sin\theta\,\boldsymbol{e}_z, \\ \boldsymbol{d}_2 &= \cos\phi\,\boldsymbol{e}_\rho + \sin\phi\cos\theta\,\boldsymbol{e}_\psi - \sin\phi\sin\theta\,\boldsymbol{e}_z, \\ \boldsymbol{d}_3 &= \sin\theta\,\boldsymbol{e}_\psi + \cos\theta\,\boldsymbol{e}_z. \end{aligned} \tag{5}$$

The angle θ measures the deviation from the straight configuration, while ϕ is the internal twist angle of $\boldsymbol{d}_2$ (say) about $\boldsymbol{d}_3$. Thus, a rod with no internal twist, $\phi = 0$, has its $\boldsymbol{d}_1$ lying in the tangent plane to the imaginary cylinder, and $\boldsymbol{d}_2$ normal to it, along its entire length (see Fig. 1(c)).

Now, since ρ is constant, say $\rho = r$, where r is the radius of the rod, we have $\boldsymbol{r}' = r\psi'\boldsymbol{e}_\psi + z'\boldsymbol{e}_z$, and hence, since $\boldsymbol{r}' = \boldsymbol{d}_3$,

$$\psi' = \frac{1}{r}\sin\theta. \tag{6}$$

The rate of change of the director frame is given by

$$\boldsymbol{d}_i' = \boldsymbol{u} \times \boldsymbol{d}_i \qquad (i = 1, 2, 3), \tag{7}$$

where $\boldsymbol{u}$ is the curvature vector, which can be written as $\boldsymbol{u} = \kappa_1\boldsymbol{d}_1 + \kappa_2\boldsymbol{d}_2 + \tau\boldsymbol{d}_3$, κ_1 and κ_2 being the curvatures about $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$, and τ being the twist about $\boldsymbol{d}_3$. Equation (7) can be inverted to give

$$\kappa_1 = \frac{1}{2}\left(\boldsymbol{d}_2' \cdot \boldsymbol{d}_3 - \boldsymbol{d}_3' \cdot \boldsymbol{d}_2\right), \qquad \kappa_2 = \frac{1}{2}\left(\boldsymbol{d}_3' \cdot \boldsymbol{d}_1 - \boldsymbol{d}_1' \cdot \boldsymbol{d}_3\right), \qquad \tau = \frac{1}{2}\left(\boldsymbol{d}_1' \cdot \boldsymbol{d}_2 - \boldsymbol{d}_2' \cdot \boldsymbol{d}_1\right). \tag{8}$$

Inserting (5) into (8) and using (4) and (6) yields

$$\begin{aligned} \kappa_1 &= \frac{1}{r}\sin^2\theta\cos\phi - \theta'\sin\phi, \\ \kappa_2 &= -\frac{1}{r}\sin^2\theta\sin\phi - \theta'\cos\phi, \\ \tau &= \phi' + \frac{1}{r}\sin\theta\cos\theta. \end{aligned} \tag{9}$$

So for the total curvature κ we find

$$\kappa^2 = \kappa_1^2 + \kappa_2^2 = \theta'^2 + \frac{1}{r^2}\sin^4\theta. \tag{10}$$

Finally, for the work terms we note that

$$L_p = \int_0^L \cos\theta\, ds \qquad \text{and} \qquad R = \frac{1}{r}\int_0^L \sin\theta\, ds = \psi(L) - \psi(0) \qquad \text{(by (6))}. \tag{11}$$

3 Derivation of the equilibrium equations

Collecting all energy terms of the previous section, we have

$$V = \int_0^L \left(B\kappa^2 + C\tau^2 - F\cos\theta - \frac{M}{r}\sin\theta\right) ds =: \int_0^L \mathcal{L}(\theta, \theta', \phi, \phi')\, ds, \tag{12}$$

with κ and τ given by (10) and $(9)_3$. So the kinematical state of the ply is specified by two generalised co-ordinates, θ and ϕ. Equilibrium solutions are stationary points of V, which are given by the Euler-Lagrange equations

$$\frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\theta'} = \frac{\partial\mathcal{L}}{\partial\theta} \qquad \text{and} \qquad \frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\phi'} = \frac{\partial\mathcal{L}}{\partial\phi}. \tag{13}$$

Since $\mathcal{L}$ is independent of ϕ (ϕ is an *ignorable* variable) the second equation yields $\tau = \text{const.}$, i.e., the twist is conserved along each strand of the ply. The first equation gives

$$r^2\theta'' = 2\sin^3\theta\cos\theta + \frac{r\tau C}{B}\cos 2\theta + \frac{r^2 F}{2B}\sin\theta - \frac{rM}{2B}\cos\theta, \tag{14}$$

as was found in [6] and [7].

4 Boundary conditions

A valid set of boundary conditions can be obtained by consideration of the boundary terms occurring in the variational calculation. These are here:

$$\left[\frac{d\mathcal{L}}{d\theta'}\,\delta\theta\right]_0^L = 2B\,[\theta'\,\delta\theta]_0^L \qquad \text{and} \qquad \left[\frac{d\mathcal{L}}{d\phi'}\,\delta\phi\right]_0^L = 2C\,[\tau\,\delta\phi]_0^L\,, \tag{15}$$

where $\delta\theta$ and $\delta\phi$ are the arbitrary variations of θ and ϕ. Since these boundary terms have to vanish we conclude that possible boundary conditions for θ are

	(A1)	θ fixed, θ' free:	$\theta(0) = \theta_0,\ \theta(L) = \theta_L$	(some θ_0, θ_L),
or	(A2)	θ free, θ' fixed:	$\theta'(0) = \theta'(L) = 0$,	

while possible boundary conditions for ϕ are

	(B1)	ϕ fixed, τ free:	$\phi(0) = \phi_0,\ \phi(L) = \phi_L$	(some ϕ_0, ϕ_L),
or	(B2)	ϕ free, τ fixed:	$\tau(0) = \tau(L) = 0$.	

The second alternatives (A2) and (B2) can also be written as $N(0) = N(L) = 0$ and $Q(0) = Q(L) = 0$, where $N = B\theta'$ is the tangential bending moment and $Q = C\tau$ the twisting moment in the rod. It is thus confirmed that the boundary conditions specify either a generalised force (*dead* loading) or its corresponding generalised displacement (*rigid* loading).

In many applications the natural boundary conditions will be fully rigid, i.e., (A1)+(B1) (*clamped* or *fixed-grip* boundary conditions).

5 Numerical solution

Fig. 1(a) shows a ply solution obtained by numerically solving the equilibrium equations subject to the clamped boundary conditions $\theta(0) = \theta(L) = 0$ and $\phi(0) = 0$, $\phi(L) = -6\pi$. Experimentally, this solution can be obtained by (*i*) placing two strands of straight and untwisted rod side by side, (*ii*) putting 3 full turns of (left-handed) twist into each of them, (*iii*) clipping the strands together at their ends, and (*iv*) letting go. Fig. 1(b) gives true 3D views of the pair of rods after the stages (*ii*) and (*iv*) (letting the right end go). The end views of these pairs in Fig. 1(c) illustrate that this 4-step process is correctly described by the above boundary conditions: after letting go of the clipped ends $\phi(L)$ remains unchanged while the final ply acquires an end rotation of $\psi(L) - \psi(0)$. Meanwhile, the end rotation of an individual strand, defined as the angle over which $\boldsymbol{d}_2$ (say) rotates along the ply, is given by $\psi(L) + \phi(L) - \psi(0) - \phi(0)$.

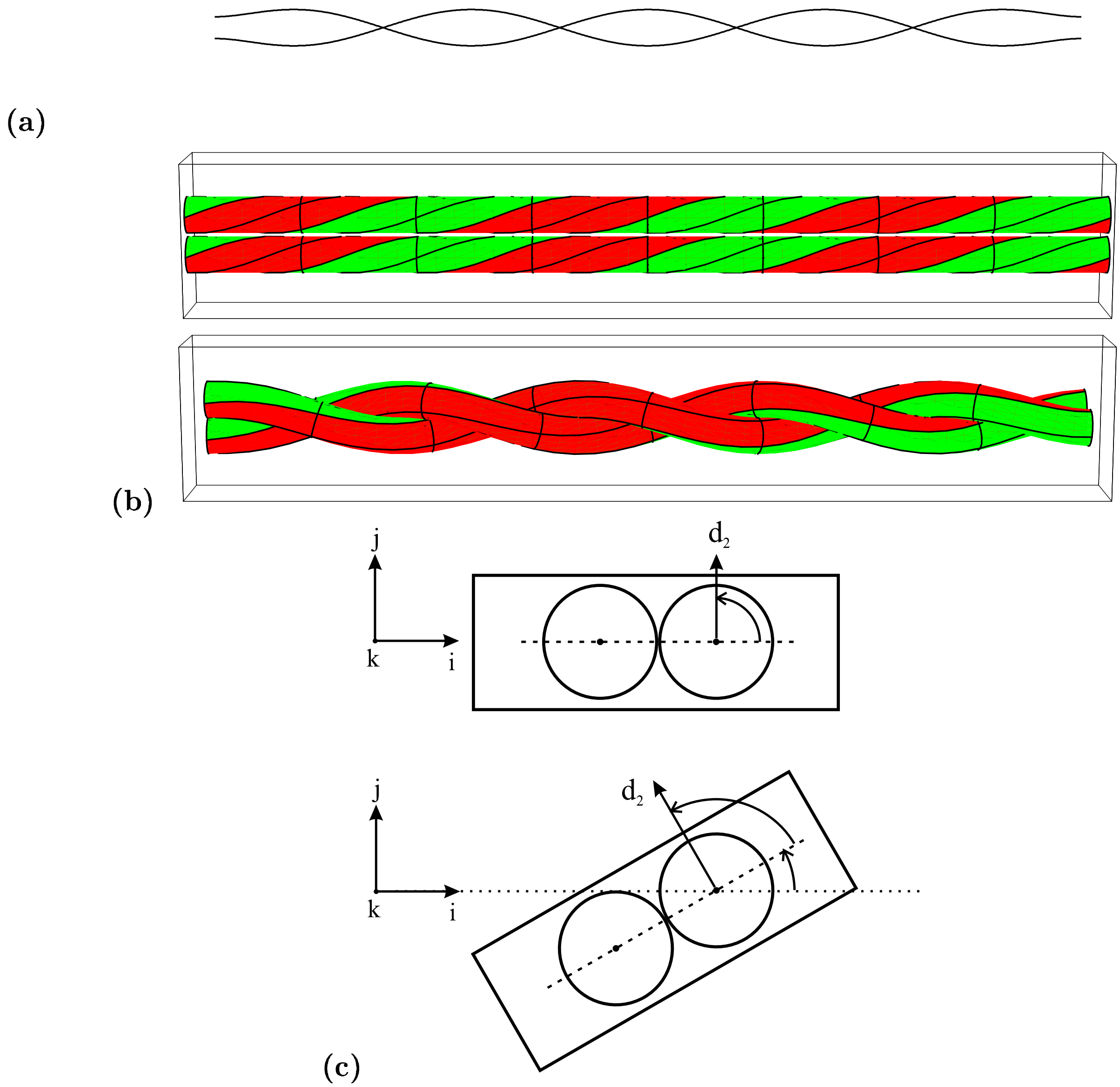

Figure 1: (a) True view of the centrelines of a clamped balanced ply subject to the boundary conditions $\theta(0) = \theta(L) = 0$ and $\phi(0) = 0$, $\phi(L) = -6\pi$. (b) Formation of this clamped balanced ply: a pair of pretwisted rods and the final plied structure they form after letting go of the right end. (c) End views of the two stages of ply formation. The parameters used are: $L/r = 50$, $C/B = 2/3$, $F = M = 0$, giving an approximately constant helical angle of 17.88°, a twist of $\tau L = -5.0889$, a ply length of $L_p/r = (z(L) - z(0))/r = 47.80$ and an end rotation of $R/(2\pi) = 2.2953$ turns. So the handedness of the helix is opposite to that of the twist in each of the strands.

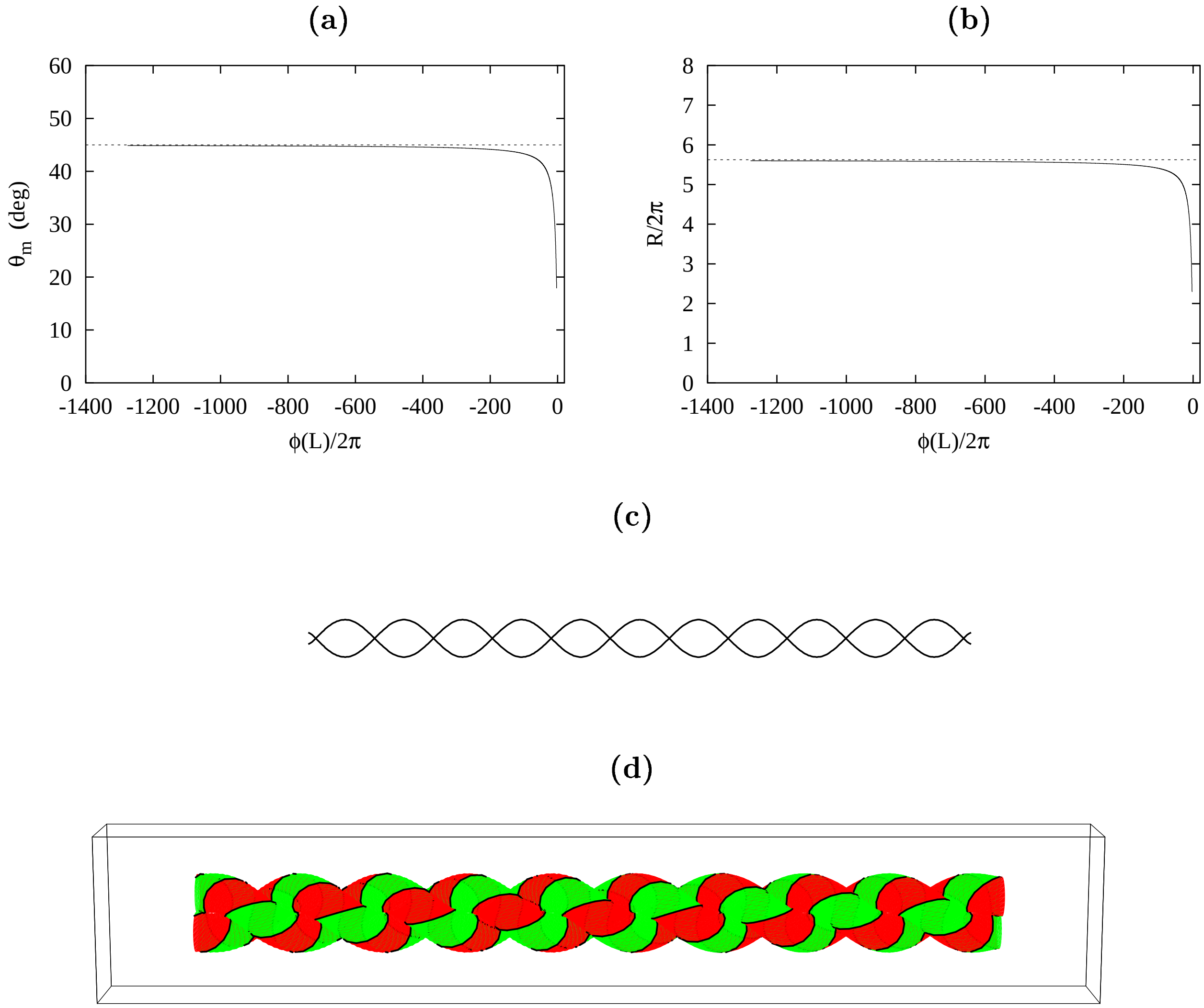

Figure 2: Winding up a clamped balanced ply: (a) midpoint helical angle, θ_m, and (b) ply end turn, $R/(2\pi)$, as a function of the strand end turn, $\phi(L)/(2\pi)$, put in initially. The starting point is the solution of Fig. 1. The ply is found to approach the lock-up state indicated by the dotted lines. In (c) a true view of the centrelines of the ply is shown at $\phi(L)/(2\pi) = -1273$. The ply then has a midpoint angle of $44.87°$, a twist of $\tau L = -7995$, a length of $L_p/r = 35.50$ and an end rotation of $R/(2\pi) = 5.5990$. (d) shows the ply at the point where its maximum curvature, attained at the end points, equals L/r. This solution has: $\phi(L)/(2\pi) = -17.78$, $\theta_m = 36.59°$, $\tau L = -88.18$, $L_p/r = 40.48$ and $R/(2\pi) = 4.6398$. ($L/r = 50,\ C/B = 2/3,\ F = M = 0$.)

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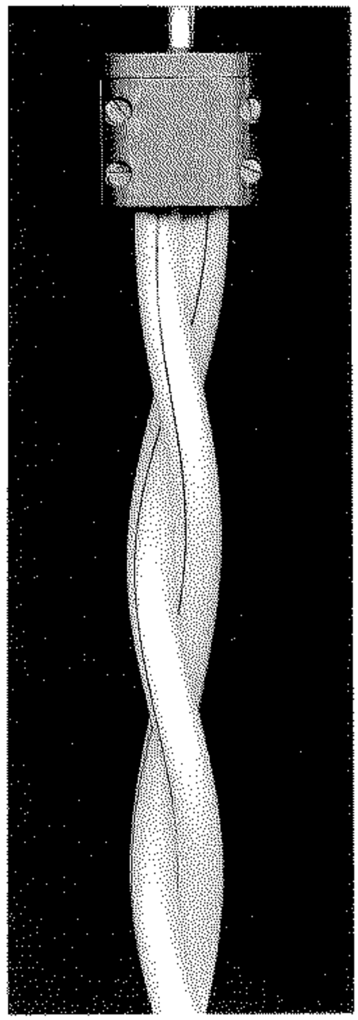

Figure 3: Photograph of a clamped ply in an experimental situation. The ply is made of two solid circular cross-section silicone rubber rods ($\gamma = 2/3$). Straight lines drawn on their surfaces visualise the twist in the ply.

Fig. 2 shows that as more and more turns of pretwist are put into the strands of the ply the ply angle approaches a uniform 45°. Remarkably, this is precisely the geometrical lock-up angle at which a uniform ply self-contacts, making larger angles kinematically impossible (see, e.g., [4], and also [5]). By (11), as $\theta \to 45°$, $R/(2\pi)$ tends to $L\sin\theta/(2\pi r) = \sqrt{2}(4\pi)^{-1}(L/r)$, as confirmed in (b), where, for $L/r = 50$, $R/(2\pi)$ approaches 5.6270. Note that this number is also precisely the (fractional) number of helical turns in a rod of length L bent into a helix of angle θ and radius r.

However, the dimensionless curvature $L\kappa$ of an incompressible rod cannot exceed L/r, and this value is reached at the clamped ends well before the 45°-state is reached, as shown in Fig. 2(d). So a clamped ply locks up at its ends well before the uniform asymptotic lock-up state is reached. It should be noted, however, that large-strain effects place even the solution in Fig. 2(d) outside the realm of the rod theory used (for instance, at clamp lock-up the inner fibre of the rod is infinitely compressed!).

Fig. 3 shows a photograph of a clamped ply.

6 A note on plies made of imperfect and anisotropic rods

In Section 2 the only place where a perfect (i.e., intrinsically straight and prismatic) and isotropic rod is assumed is in writing down the total potential energy V in Eq. (1). The rest of the analysis is kinematics, which remains equally valid for more general rods. We can therefore derive the equations for a ply made of an imperfect anisotropic rod (which must have an external circular profile, e.g., a tape cast inside a rubber hose) by replacing (12) by

$$\begin{aligned}\bar{V} &= \int_0^L \left(B_1(\kappa_1 - \hat{\kappa}_1)^2 + B_2(\kappa_2 - \hat{\kappa}_2)^2 + C(\tau - \hat{\tau})^2 - F\cos\theta - \frac{M}{r}\sin\theta \right) ds \\ &=: \int_0^L \bar{\mathcal{L}}(\theta, \theta', \phi, \phi')\, ds,\end{aligned} \tag{16}$$

where B_1 and B_2 are now the bending stiffnesses about $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$, respectively, while $\hat{\kappa}_1$, $\hat{\kappa}_2$ and $\hat{\tau}$ are the intrinsic (or *initial*) curvatures and twist. After inserting (9) we obtain the following Euler-Lagrange equations

$$\begin{aligned}
&\frac{d}{ds}\left[2\theta'\left(B_1\sin^2\phi + B_2\cos^2\phi\right) - \frac{B_1 - B_2}{r}\sin^2\theta\sin 2\phi + 2B_1\hat{\kappa}_1\sin\phi + 2B_2\hat{\kappa}_2\cos\phi \right] = \\
&\quad \frac{4}{r^2}\sin^3\theta\cos\theta\left(B_1\cos^2\phi + B_2\sin^2\phi\right) - \frac{2}{r}\sin 2\theta\left(B_1\hat{\kappa}_1\cos\phi - B_2\hat{\kappa}_2\sin\phi\right) \\
&\quad -\frac{B_1 - B_2}{r}\theta'\sin 2\theta\sin 2\phi + \frac{2C}{r}\left(\phi' + \frac{1}{2r}\sin 2\theta - \hat{\tau}\right)\cos 2\theta + F\sin\theta - \frac{M}{r}\cos\theta, \\
&\frac{d}{ds}\left[2C\left(\phi' + \frac{1}{2r}\sin 2\theta - \hat{\tau}\right)\right] = -\frac{B_1 - B_2}{r^2}\sin^4\theta\sin 2\phi + (B_1 - B_2)\theta'^2\sin 2\phi \\
&\quad -\frac{2(B_1 - B_2)}{r}\theta'\sin^2\theta\cos 2\phi + \frac{2}{r}\sin^2\theta\left(B_1\hat{\kappa}_1\sin\phi + B_2\hat{\kappa}_2\cos\phi\right) \\
&\quad +2\theta'\left(B_1\hat{\kappa}_1\cos\phi - B_2\hat{\kappa}_2\sin\phi\right),
\end{aligned} \tag{17}$$

which is a set of two coupled second-order differential equations. This means that, when written as a set of first-order equations, (17), for the perfect rod ($\hat{\kappa}_1 = \hat{\kappa}_2 = \hat{\tau} = 0$), represents an explicit four-dimensional reduction of the general and rather complicated set of equations derived in [7]. Note that in (17) B_1, B_2, C, $\hat{\kappa}_1$, $\hat{\kappa}_2$ and $\hat{\tau}$ may be functions of arclength.

For helical plies ($\theta = \text{const.}$) and constant parameters B_1, B_2, C, $\hat{\kappa}_1$, $\hat{\kappa}_2$, $\hat{\tau}$, (17) reduces to

$$\begin{aligned} -\frac{2(B_1 - B_2)}{r}\phi' \sin^2\theta \cos 2\phi + 2\phi'\left(B_1\hat{\kappa}_1 \cos\phi - B_2\hat{\kappa}_2 \sin\phi\right) = \\ \frac{4}{r^2}\sin^3\theta\cos\theta\left(B_1\cos^2\phi + B_2\sin^2\phi\right) + \frac{2C}{r}\left(\phi' + \frac{1}{2r}\sin 2\theta - \hat{\tau}\right)\cos 2\theta \\ -\frac{2}{r}\sin 2\theta\left(B_1\hat{\kappa}_1\cos\phi - B_2\hat{\kappa}_2\sin\phi\right) + F\sin\theta - \frac{M}{r}\cos\theta, \\ C\phi'' = -\frac{B_1 - B_2}{2r^2}\sin^4\theta\sin 2\phi + \frac{1}{r}\sin^2\theta\left(B_1\hat{\kappa}_1\sin\phi + B_2\hat{\kappa}_2\cos\phi\right). \end{aligned} \tag{18}$$

It follows in particular from the second equation in (18) that for perfect rods uniform helical plies, i.e., plies which in addition have a constant internal twist ($\phi' = \text{const.}$), need either $B_1 = B_2$ (isotropic rod) or have $\phi = 0$ or $\pi/2$, which means they are winding as a tape on a cylinder (either standing or lying) with no internal twist and bending only in one direction (cf. (9)). Such solutions we have previously called *one-twist-per-wave* solutions [1, 8]. From $(18)_1$ it follows that their helical angle θ satisfies

$$\frac{4B}{r^2}\sin^3\theta\cos\theta + \frac{C}{r^2}\sin 2\theta\cos 2\theta + F\sin\theta - \frac{M}{r}\cos\theta = 0, \tag{19}$$

where $B = B_1$ if $\phi = 0$ and $B = B_2$ if $\phi = \pi/2$.

For isotropic but imperfect rods, on the other hand, $(18)_2$ yields that a uniform helical ply has an internal twist angle given by

$$\tan\phi = -\frac{\hat{\kappa}_2}{\hat{\kappa}_1}. \tag{20}$$

Eq. $(18)_1$ then again provides an equation for the helical angle. Boundary conditions follow from the boundary terms as in (15) ($d\bar{\mathcal{L}}/d\theta'$ and $d\bar{\mathcal{L}}/d\phi'$ are given by the expressions inside the square brackets in (17)).

Notice that Eqs (18)–(20) for helical solutions are also true for individual rods which do not form a ply, as those are just special cases with zero pressure!

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