A Constructive Approach to Modelling the Tight Shapes of Some Linked Structures

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(Received December 12, 2002; Accepted November 27, 2003)

Keywords: Ideal Knot and Link, Borromean Rings, Tight Clasp

Abstract. A variational approach is used to find the shortest curves connecting two pairs of points. The curves are to be separated by a constant distance and they are confined to lie in two orthogonal planes. The method is applicable to constructing tight shapes of linked structures each component of which is known to be planar. Two particular examples are considered and explicit solutions are presented for the Borromean rings, and for two clapsed pieces of a rope that provide the minimum of the centreline length for a fixed diameter of the cross-section. A concept of tight periodic structures is introduced and discussed.

1. Introduction

The notion of an ideal shape of a knot or link has emerged recently as a result of attempts to introduce a quantity that, first, is well-defined and, second, can be measured on a physical realization of the knotted structure (Moffatt, 1990, 1996; Katritch et al., 1996; Grosberg et al., 1996). The idea is to ascribe a uniform thickness to (closed) curve(s) embedded in space, and then to look for a configuration of the given topology that provides the minimal length-to-thickness ratio under the constraint of self-impenetrability. There is a hope that this ratio would give a unique number for every knot or link type, thus providing a new means of classification. A number of works have appeared where the properties of the ideal representations are shown to be related to the behaviour of physical systems such as DNA knots (Stasiak et al., 1996) and catenanes (links) (Laurie et al., 1998) or chemical dissipative structures (Malevanets and Kapral, 1998).

Although some first results have been recently obtained in the theory of ideal knots and links (Gonzalez and Maddocks, 1999; Litherland et al., 1999; Gonzalez et al., 2002; Cantarella et al., 2002), a surprising fact is that little is known about the shapes themselves. Proven analytical solutions known so far, include only links with the centrelines

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made up of straight line intervals and circular arcs. No explicit closed-form description has been found of an ideal configuration with a fragment where the contact generates a one-to-one mapping of the points on the centrelines. A number of approximations have been computed by means of the numerical algorithms (both stochastic and deterministic) that discretize the centreline and represent it as a polygonal line (Laurie, 1998; Pieranski, 1998). The work of these algorithms models the evolution of the shape as the length-to-thickness ratio decreases.

Here a new approach to construct extremal tight shapes is presented. Although its application is currently limited to links with planar centres, it has enabled non-obvious shapes to be described by explicit closed-form expressions. There is reason to believe that both the method and the examples will provide an insight into the general properties of the ideal knots and links.

2. A Constrained Variational Approach

Consider a curve \( r_1(\sigma) = (x_1(\sigma), y_1(\sigma), 0) \) of class \( C^2 \) in the \( xy \)-plane, \( \sigma \) being a parameter, \( \sigma \in [0, \Lambda] \) (Fig. 1). Let \( r_2(\sigma) = (x_2(\sigma), d, z_2(\sigma)) \) be another curve of class \( C^2 \) lying in the orthogonal plane \( y = d \) and parametrized with the same \( \sigma \). We assume that the curves \( r_1 \) and \( r_2 \) represent the centrelines of two pieces of an incompressible rope of constant diameter \( d \). The rope pieces touch each other continuously so that, for every \( \sigma_0 \in [0, \Lambda] \), the points \( r_1(\sigma_0) \) and \( r_2(\sigma_0) \) are the closest ones, i.e. the distance between them is equal to \( d \). Without loss of generality we may set \( d = 1 \).

We are seeking shapes of the curves \( r_1 \) and \( r_2 \) connecting two pairs of points such that the sum of their lengths is minimal.

Fig. 1. The curves \( r_1(\sigma) \) and \( r_2(\sigma) \) lie in the orthogonal planes. The points \( A = (x_1, y_1, 0) \) and \( B = (x_2, d, z_2) \) belong to curves such that the chord \( AB \) is perpendicular to each of the curves with the distance \( AB \) is constant and equal to \( d \).
where \( \Phi(x'_1, y'_1, x'_2, z'_2) = \sqrt{(x'_1)^2 + (y'_1)^2} + \sqrt{(x'_2)^2 + (z'_2)^2} \), the prime denoting the derivative with respect to \( \sigma \).

The distance between the two curves should be constant, therefore we require

\[
 r'_1(r_1 - r_2) = 0, \quad \sigma \in [0, \Lambda], \quad (2)
\]

and

\[
 r'_2(r_1 - r_2) = 0, \quad \sigma \in [0, \Lambda], \quad (3)
\]

or, in coordinate form,

\[
 x'_1(x_1 - x_2) + y'_1(y_1 - 1) = 0, \quad (4)
\]

and

\[
 x'_2(x_1 - x_2) - z'_2z_2 = 0. \quad (5)
\]

If we subtract Eq. (5) from Eq. (4) and then integrate the result, we obtain \( \|r_1 - r_2\|^2 = \text{const} \).

It should be pointed out that, since we want to model the tight shape of a perfect rope, the curvature of the centreline may not exceed \( 2/d \). Nevertheless, no constraint on the curvature is explicitly included in the formulation of the variational problem. Instead, in the construction of the entire shape, we shall only choose extremal solutions that comply with the curvature constraint.

Thus, we look for a minimizer of the functional

\[
 \int_0^\Lambda \Psi(x_1, y_1, x_2, z_2, x'_1, y'_1, x'_2, z'_2, \lambda_1, \lambda_2) d\sigma \quad (6)
\]

with

\[
 \Psi(x_1, y_1, x_2, z_2, x'_1, y'_1, x'_2, z'_2, \lambda_1, \lambda_2) = \Phi(x'_1, y'_1, x'_2, z'_2) - \lambda_1 \left[ x'_1(x_1 - x_2) + y'_1(y_1 - 1) \right] - \lambda_2 \left[ x'_2(x_1 - x_2) + z'_2z_2 \right],
\]

where \( \lambda_1(\sigma) \) and \( \lambda_2(\sigma) \) are Lagrange multipliers associated with the pointwise constraints (2) and (3).

The corresponding Euler-Lagrange equations are

\[
 \frac{d}{d\sigma} \frac{\partial \Phi}{\partial x'_1} - \lambda_1'(x_1 - x_2) + x'_1(\lambda_1 + \lambda_2) = 0, \quad (7a)
\]
\[ \frac{d}{d\sigma} \frac{\partial \Phi}{\partial y_j} - \lambda'_1(y_1 - 1) = 0, \]  
(7b) 
\[ \frac{d}{d\sigma} \frac{\partial \Phi}{\partial x_2} - \lambda'_2(x_1 - x_2) - x'_1(\lambda_1 + \lambda_2) = 0, \]  
(7c) 
\[ \frac{d}{d\sigma} \frac{\partial \Phi}{\partial z_2} + \lambda'_2 z_2 = 0. \]  
(7d) 

We notice that

\[ \frac{d}{d\sigma} \frac{\partial \Phi}{\partial x_1} = -y'\kappa_1, \]
\[ \frac{d}{d\sigma} \frac{\partial \Phi}{\partial y_1} = x'_1\kappa_1, \]
\[ \frac{d}{d\sigma} \frac{\partial \Phi}{\partial x_2} = -z'_2\kappa_2, \]
\[ \frac{d}{d\sigma} \frac{\partial \Phi}{\partial z_2} = x'_2\kappa_2, \]

where

\[ \kappa_1 = \frac{x'_1y''_1 - x''_1y'_1}{\left[(x'_1)^2 + (y'_1)^2\right]^{3/2}} \]  
and
\[ \kappa_2 = \frac{x''_2z'_2 - x'_2z''_2}{\left[(x'_2)^2 + (z'_2)^2\right]^{3/2}} \]

are the curvatures of the first and second curves, respectively. The system (7) then takes the form

\[ -y'\kappa_1 - \lambda'_1(x_1 - x_2) + x'_1(\lambda_1 + \lambda_2) = 0, \]  
(8a) 
\[ x'_1\kappa_1 - \lambda'_1(y_1 - 1) = 0, \]  
(8b) 
\[ -z'_2\kappa_2 - \lambda'_2(x_1 - x_2) - x'_1(\lambda_1 + \lambda_2) = 0, \]  
(8c) 
\[ x'_2\kappa_2 + \lambda'_2 z_2 = 0. \]  
(8d) 

We can eliminate \( \kappa_1 \) (respectively \( \kappa_2 \)) from Eqs. (8a) and (8b) (Eqs. (8c) and (8d)). The resulting equation is

\[ x'_2\lambda_1(x_1 + \lambda_2) = 0, \]  
(9) 

where we also took into account constraint (4) or (5).

Equation (9) is formally satisfied if at least one of the following equalities holds: a) \( x_1 = x_{10} = \text{const} \), b) \( x_2 = x_{20} = \text{const} \), c) \( \lambda_1 + \lambda_2 = 0 \).
In case a), Eq. (4) implies \( y_1 = y_{10} = \text{const} \) which means that the fragment of the first curve reduces to a single point and the parametrization is singular. If the length of \( r_{2a} \) is non-zero, then the second curve \( r_{2a} \) may only be a circular arc of radius one with centre \((x_{10}, y_{10}, 0)\), which should belong to the plane \( y = 1 \). Thus, we have \( r_{1a} = \{(x_{10}, 1, 0)\}, r_{2a} = \{(x_2 - x_{10})^2 + z_2^2 = 1, y_2 = 1\} \).

In case b), we have from Eq. (5) \( z_2 = \text{const} \), and now the curve \( r_{2b} \) reduces to a point. The common parametrization is singular for the second curve. The situation is similar to case a) with the curves interchanged: the curve \( r_{1b} \) is a circular arc and \( z_2 = 0 \), i.e. \( r_{1b} = \{(x_i - x_{20})^2 + (y_i - 1)^2 = 1, z_1 = 0\}, r_{2b} = \{(x_{20}, 1, 0)\} \).

Let \( r_{2a} \) and \( r_{1b} \) be complete circles. If \(|x_{10} - x_{20}| = 1\), then \( r_{1a} \in r_{1b} \), which implies \( r_{2b} \in r_{2a} \) and we have the Hopf link which is known to be ideal as the solution of the Gehring link problem (Cantarella et al., 2002, and references therein).

In the regular case we may reduce Eq. (9) to \( \lambda_1 + \lambda_2 = 0 \). Now we add Eqs. (8a) and (8c) with \( \lambda_1 = -\lambda_2 \) to get

\[
y_1'\kappa_1 + z_2'\kappa_2 = 0.
\]

Let us introduce an angle \( \phi_1(\sigma) \) between the tangent to the curve \( r_1 \) and the \( x \)-axis. Similarly, \( \phi_2(\sigma) \) will be an angle between the tangent to \( r_2 \) and the \( x \)-axis (see Fig. 1). Then

\[
\sin \phi_1 = \frac{y_1'}{\sqrt{(x_1')^2 + (y_1')^2}}, \quad \cos \phi_1 = \frac{x_1'}{\sqrt{(x_1')^2 + (y_1')^2}},
\]

\[
\sin \phi_2 = \frac{z_2'}{\sqrt{(x_2')^2 + (z_2')^2}}, \quad \cos \phi_2 = \frac{x_2'}{\sqrt{(x_2')^2 + (z_2')^2}}.
\]

and

\[
\kappa_1 = \frac{\phi_1'}{\sqrt{(x_1')^2 + (y_1')^2}}, \quad \kappa_2 = \frac{\phi_2'}{\sqrt{(x_2')^2 + (z_2')^2}}.
\]

Equation (10) may be put in the form

\[
\phi_1' \sin \phi_1 + \phi_2' \sin \phi_2 = 0,
\]

which can be integrated to yield

\[
\cos \phi_1 + \cos \phi_2 = C, \quad C = \text{const}, \quad |C| \leq 2.
\]

The integration constant \( C \) is specified by the tangents in a pair of corresponding points.
Equation (13) allows for the constant solutions $\phi_1 = \phi_2 = 0$ with $C = 2$, and $\phi_1 = \phi_2 = \pi$ with $C = -2$. These trivial solutions are just a pair of parallel straight lines.

With the help of Eqs. (4) and (5), we may eliminate the derivatives from Eqs. (11) and (12):

$$\sin \phi_1 = \pm \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}, \quad \cos \phi_1 = \mp \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}},$$

$$\sin \phi_2 = \pm \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta z)^2}}, \quad \cos \phi_2 = \mp \frac{\Delta z}{\sqrt{(\Delta x)^2 + (\Delta z)^2}},$$

where we denote $\Delta x = x_2 - x_1$, $\Delta y = 1 - y_1$ and $\Delta z = z_2$. The property of constant distance between the two curves implies $(\Delta x)^2 = 1 - (\Delta y)^2 - (\Delta z)^2$, and we obtain from Eq. (13)

$$\mp \frac{\Delta y}{\sqrt{1 - (\Delta z)^2}} \mp \frac{\Delta z}{\sqrt{1 - (\Delta y)^2}} = C. \quad (14)$$

Let us now introduce an angle $\theta_1$, $\theta_1 \in [0, \pi]$, between the vector $r_2 - r_1 = (\Delta x, \Delta y, \Delta z)$, $||r_2 - r_1|| = AB$, and the z-axis. Since the tangent to the first curve at point A is orthogonal to AB, we can interpret the angles $\theta_1$ and $\phi_1$, $\phi_1 \in [0, 2\pi]$ as the spherical coordinates of the difference vector $r_2 - r_1$ (see Fig. 2):

$$\Delta x = -\sin \theta_1 \sin \phi_1, \quad \Delta y = \sin \theta_1 \cos \phi_1, \quad \Delta z = \cos \theta_1. \quad (15)$$

Conversely, we may define an angle $\theta_2$, $\theta_2 \in [0, \pi]$, between the vector $r_2 - r_1$ and the ($-y$)-axis. The pair $\theta_2$ and $\phi_2$, $\phi_2 \in [0, 2\pi]$, may also serve as the spherical angles for the direction $r_2 - r_1$, because the tangent to the second curve at point B is always orthogonal to AB:

$$\Delta x = -\sin \theta_2 \sin \phi_2, \quad \Delta y = \cos \theta_2, \quad \Delta z = \sin \theta_2 \cos \phi_2. \quad (16)$$

We equate the expressions for the differences $\Delta x$ and $\Delta z$ from Eq. (15) and Eq. (16) and eliminate the angle $\theta_2$ from the resulting equations. Then we are left with one equation for the angle $\theta_1$ which can be written as

$$\sin^2 \theta_1 = \frac{\sin^2 \phi_2}{1 - \cos^2 \phi_1 \cos^2 \phi_2}.$$  

It is then possible to express the coordinates as functions of $\phi_1$ and $\phi_2$:
where $\varepsilon = \pm 1$. The last expressions suggest that $\Delta x(\phi_1, \phi_2) = \Delta x(\phi_2, \phi_1)$ and $\Delta y(\phi_1, \phi_2) = \Delta z(\phi_2, \phi_1)$.

There exist symmetries:

- $\phi_1 \to \pi - \phi_1, \phi_2 \to \phi_2 + \pi, \varepsilon \to \varepsilon, C \to -C, \Delta x \to -\Delta x, \Delta y \to \Delta y, \Delta z \to -\Delta z$ (rotation around the y-axis through $\pi$);
- $\phi_1 \to \phi_1 + \pi, \phi_2 \to \pi - \phi_2, \varepsilon \to \varepsilon, C \to -C, \Delta x \to -\Delta x, \Delta y \to -\Delta y, \Delta z \to \Delta z$ (rotation around the z-axis through $\pi$);
- $\phi_1 \to \phi_1, \phi_2 \to 2\pi - \phi_2, C \to C, \varepsilon \to -\varepsilon, \Delta x \to \Delta x, \Delta y \to \Delta y, \Delta z \to -\Delta z$ (reflection about the xy-plane).

Due to these symmetries, we may restrict ourselves, without loss of generality, to consideration of only one octant $\phi_1 \in [0, \pi/2], \phi_2 \in [0, \pi/2]$ with $\varepsilon = 1$, where $\Delta x \leq 0, \Delta y \geq 0, \Delta z \geq 0$.

Now we can choose either $\phi_1$ or $\phi_2$ as an independent variable, and, with the help of Eq. (13), we obtain the parametric representations of the coordinates of the curves $r_1$ and $r_2$:
\[ x_1 - x_2 = \epsilon \frac{1 - (C - \cos \phi_1)^2}{\sqrt{1 - (C - \cos \phi_1)^2 \cos^2 \phi_1}} \sin \phi_1, \quad (20) \]

\[ y_1 = 1 - \epsilon \frac{1 - (C - \cos \phi_1)^2}{\sqrt{1 - (C - \cos \phi_1)^2 \cos^2 \phi_1}} \cos \phi_1, \quad (21) \]

\[ z_2 = \epsilon \frac{(C - \cos \phi_1) \sin \phi_1}{\sqrt{1 - (C - \cos \phi_1)^2 \cos^2 \phi_1}}, \quad (22) \]

and

\[ x_1 - x_2 = \epsilon \frac{1 - (C - \cos \phi_2)^2}{\sqrt{1 - (C - \cos \phi_2)^2 \cos^2 \phi_2}} \sin \phi_2, \quad (23) \]

\[ y_1 = 1 - \epsilon \frac{(C - \cos \phi_1) \sin \phi_2}{\sqrt{1 - (C - \cos \phi_2)^2 \cos^2 \phi_2}}, \quad (24) \]

\[ z_2 = \epsilon \frac{1 - (C - \cos \phi_2)^2}{\sqrt{1 - (C - \cos \phi_2)^2 \cos^2 \phi_2}} \cos \phi_2, \quad (25) \]

We observe that when \( \phi_1 = \pi/2 \), then \( x_1 - x_2 = \sqrt{1 - C^2} \), \( y_1 = 1 \) and \( z_2 = C \), i.e. the curve \( r_1 \) intersects the plane of \( r_2 \) at a right angle. Similarly, if \( \phi_2 = \pi/2 \), then \( x_1 - x_2 = \sqrt{1 - C^2} \), \( y_1 = 1 - C \) and \( z_2 = 0 \), i.e. the second curve crosses the \( xy \)-plane orthogonally.

If \(|C| > 1\), then each of the curves may cross the other plane non-orthogonally, namely, \( r_2 \) crosses the \( xy \)-plane, when \( \cos \phi_1 = \pm 1 \), \( \cos \phi_2 = C \mp 1 \) and \( r_1 \) crosses the plane \( y = 1 \), if \( \cos \phi_1 = C \pm 1 \), \( \cos \phi_2 = \mp 1 \).

Unfortunately, Eq. (20) gives us only the difference of the \( x \)-coordinates. We can find \( x_1 \) from the differential equation

\[ \cot \phi_1 = \frac{d y_1}{d \phi_1}, \quad (26) \]
where $\phi_1$ is treated as an independent variable and $y_1$ is assumed to be defined by Eq. (21). The last equation may be rewritten in an integral form as

$$x_1(\phi_1) = \int_{\phi_{10}}^{\phi_1} \cot \phi_1 \frac{dy_1}{d\phi_1} d\phi_1 + x_1(\phi_{10}).$$

(27)

Integration by parts yields

$$x_1(\phi_1) - x_1(\phi_{10}) = \cot \phi_1 y_1(\tilde{\phi}_1) \bigg|_{\phi_{10}}^{\phi_1} + \int_{\phi_{10}}^{\phi_1} \frac{y_1(\tilde{\phi}_1)}{\sin^2 \phi_1} d\tilde{\phi}_1.$$ 

(28)

Thus the $x_1$-coordinate is found as a quadrature. As to the $x_2$-coordinate, the above approach may be applied as well. This time we treat the relationship

$$\cot \phi_2 = \frac{\frac{dx_2}{d\phi_2}}{\frac{dz_2}{d\phi_2}}$$

(29)

as a differential equation for $x_2(\phi_2)$. Its integration gives

Fig. 3. The curvature $\kappa_1$ as function of $\phi_1$ for various $C$. 

Note that the integrals in Eq. (28) and Eq. (30) are hyperelliptic integrals.

We can now see that the curves \( r_1(\phi_1) \) and \( r_2(\phi_2) \) are described by essentially the same expressions, which implies the following symmetry property: if \( r_1(\phi_1) \) is known, then the corresponding second curve \( r_2(\phi_2) \) may be obtained by 1) rotating the \( xy \)-plane through \( \pi/2 \) about the axis \( y = 1, z = 0 \) to place \( r_1 \) into the plane \( y = 1 \), 2) reflecting the new curve about the \( yz \)-plane, and 3) shifting the curve along the line \( y = 1, z = 0 \) to satisfy the condition of constant distance between the curves. The symmetry property holds true if we consider the entire curves for given \( C \).

In order to find an expression for the curvature \( \kappa_1 \), we again make use of Eq. (26)

\[
\kappa_1(\phi_1) = \frac{1}{\sqrt{\left( \frac{dx}{d\phi_1} \right)^2 + \left( \frac{d\phi_1}{dy_1} \right)^2}} = \left| \sin \phi_1 \left( \frac{dy_1}{d\phi_1} \right)^{-1} \right| = \left( 1 - \frac{(C - \cos \phi_1)^2}{1 - C^2 + \cos \phi_1 \left[ C + (2 - \cos^2 \phi_1)(C - \cos \phi_1) \right]} \right)^{3/2}.
\]

The function in Eq. (31) is singular at \( C = 1, \cos \phi_1 = 0 \) and at \( C = 0, \cos \phi_1 = 1 \). The graph of this function for some values of \( C \) is presented in Fig. 3.

As we have seen, for \( \phi_1 = \pi/2 \), when the curve \( r_1 \) passes through the plane of the second curve, the last formula reduces to

\[
\kappa_1(\pi/2) = \frac{1}{\sqrt{1 - C^2}},
\]

which has sense only for \(|C| < 1\).

The length of the fragment of the curve with non-vanishing curvature may be found from the quadrature

\[
L = \int_{\phi_{10}}^{\phi_{11}} \frac{d\phi_1}{\kappa_1(\phi_1)}.
\]

Assuming both \( y'(\phi) \geq 0 \) and \( \sin \phi_1 > 0 \), the last expression may be integrated by parts to take the form
L = \frac{y_1(\phi_1)}{\sin \phi_1} + \int_{\phi_1}^{\phi_2} \frac{y_1(\phi_1) \cos \phi_1}{\sin^2 \phi_1} d\phi_1.

One further improvement can be made. The very formulation of the constrained variational problem Eq. (6) does not guarantee that the solutions found provide the minimal distance between two centreline curves. The conditions Eq. (4) and Eq. (5) only stand for an extremality of the distance. Therefore, we have to consider how to distinguish its minima.

Let \( D(\sigma_1, \sigma_2) = ||r_1(\sigma_1) - r_2(\sigma_2)||^2 \) be the squared distance between the points on two curves which are parametrized with \( \sigma_1 \) and \( \sigma_2 \). Then, for the solution of the problem Eq. (6), \( (\partial D)/(\partial \sigma_1) \equiv (\partial D)/(\partial \sigma_2) \equiv 0 \). Consider a pair of closest points on two curves. Let their parameters be \( \sigma_1^*, \sigma_2^* \). Then, for \( \sigma_1^*, \sigma_2^* \), the matrix of second derivatives \( (\partial^2 D)/(\partial \sigma_i \partial \sigma_j) \), \( i, j = 1, 2 \), always has one zero eigenvalue corresponding to the closest point parametrization. The positive sign of the second eigenvalue indicates that the distance between two centrelines reaches its minimal value for the closest-point pair with \( \sigma_1^*, \sigma_2^* \).

Since we know the explicit parametric expressions for the centreline coordinates Eqs. (20)–(22) or Eqs. (23)–(25), it is easy to compute the second derivatives of the distance \( D \) considered as a function of \( \sigma_1 = \cos \phi_1 \) and \( \sigma_2 = \cos \phi_2 \) for \( \phi_{1,2} \in [0, \pi/2] \). In order that the distance be minimal, the following condition should be satisfied:

\[
S(\sigma_1, C) \equiv Tr \left( \frac{\partial^2 D}{\partial \sigma_1 \partial \sigma_2} \right)_{\sigma_1 + \sigma_2 = C} \equiv \frac{\partial^2 D}{\partial \sigma_1^2} \sigma_1 + \frac{\partial^2 D}{\partial \sigma_2^2} \sigma_2 \bigg|_{\sigma_1 + \sigma_2 = C} > 0.
\]

Fig. 4. The region is shaded where the extremal distance between the centrelines is not minimal.
(Here the constraint $\sigma_1 + \sigma_2 = C$ is just Eq. (13).) Thus, an explicit algebraic expression for $S(\sigma_1, C)$ may be obtained. Note that, in contrast to the expressions for the $x_{1,2}$-coordinates, it involves no quadrature. The final form of $S$ is too unwieldy to be presented here, but it may be readily manipulated with the help of a symbolic algebra software like MAPLE to justify the following properties:

1. $S(\sigma_1, \sigma_1) = 0$ for $\sigma_1 \in [0, 1]$,
2. $S(0, C) = 0$ for $C \in [0, 1]$,
3. $S(\sigma_1, C) > 0$ for $\sigma_1 \in [0, 1]$ and $C \in (0, \sigma_1)$,
4. $\lim_{\sigma_1 \to 1} S(\sigma_1, C) = +\infty$ for $C \in (0,1)$.

Properties (2) and (4) suggest that, for every $C \in (0, 1)$, there exists a value $\sigma_1 \in (C, 1)$ such that $S(\sigma_1, C) = 0$. Indeed, numerical computation reveals the second zero-value curve for $S$ in addition to the diagonal (Fig. 4). Therefore, the solutions do not provide the minimal distance if the parameters fall within the shaded region in the figure.

In the next sections we apply the obtained results to the construction of tight shapes of some linked structures. The set of building blocks includes: 1) (contact-free) straight line pieces; 2) circular arcs of unit radius, which are in correspondence with a single point on the other piece; 3) circular arcs of one-half radius (self-contacted pieces); 4) non-straight, non-circular extremal curves as described above.

Fig. 5. The piecewise circular approximation of the ideal Borromean rings.
The tight shapes will be built by joining the fragments so that to provide continuity of tangents in the switching points.

3. Tight Borromean Rings

The Borromean rings (Figs. 5 and 6) are made of three parts which are linked in such a way that once any of them is removed, then the other two are not linked (Cromwell et al., 1998). We assume that each ring is actually a closed incompressible tube of the same length and same uniform diameter. The ideal Borromean rings have, by definition, the minimal possible length-to-thickness ratio. In Cantarella et al. (2002), an approximation of the ideal Borromean rings was proposed. The whole structure was composed of three identical rings each having a plane centreline made of four circular arcs. It is easy to find the length-to-radius ratio for that shape to be \( l_{\text{circ}} = 12(\pi + 4\arcsin(\sqrt{7} - 1)/4) \).

There are good reasons to believe that the ideal configuration has to be very close to this piecewise circular one. Here we try to improve it by applying the approach developed in the previous section. We assume that all three rings are congruent and have planar centrelines. Moreover, each ring has two orthogonal planes of symmetry, each plane being orthogonal to the centreline plane. In other words, each ring could be made up of four congruent parts. Without loss of generality, the radius of tube may be taken 1/2, in concordance with unit distance between the centrelines \( d = 1 \).

![Fig. 6. Borromean rings. The same shape as in Fig. 5 but with smaller thickness of the tube.](image)
The quarters of three centrelines $A_iB_iC_iD_iE_iF_i$, $i = 1, 2, 3$, are shown in Fig. 7 (not all points are marked). The origin of the reference frame $O\xi\eta\zeta$ is placed in the centre of symmetry of the whole structure, and the coordinate planes coincide with the planes of the centrelines. The origin of the reference frame $Dxyz$ is located in the point $D = D_1 = (1, a, 0)$, $a = OA_i$. At point $A_i$, the $i$-th centreline passes orthogonally through the $(i-1)$-th plane (where we identify the zeroth plane with the third one). Suppose $C_1 = (\cos \chi, a - \sin \chi, 0)$ is the first point where the tube 1 comes into contact with the second ring. Thus, $C_1$ corresponds to $A_2$. The following piece in the direction of the point $D_1$ could only be circular of radius 1. Let the circular arc end in some point $E_1 = (\cos \phi_{10}, a + \sin \phi_{10}, 0)$. It could happen that before the point $C_1$ the tube 1 is either contact-free or it touches the third ring. We continue our consideration under assumption of the second case, though a similar analysis may be done with account for a contact-free segment of positive length, and it may be shown that the shortest centreline is achieved when this straight segment vanishes. Thus, in point $C_1$ the first tube switches its contact between the third ($F_3$) and the second ($A_2$) ones. It means that the distances $A_2C_1$ and $C_1F_3$ are unit, and all three points lie on the same straight line as $A_2C_1$ and $C_1F_3$ are two perpendiculars to the tangent in the point $C_1$. For some angle $\psi$, a fragment of the first tube may have the circular centreline going around point $F_3$ and the piece $A_1B_1$ is supposed to be an extremal non-circular solution of continuous contact between the first and the third rings. Thus, we have introduced three unknown angles $\chi \in [0, \pi/2]$, $\psi \in [0, \chi]$ and $\phi_{10} \in [-\chi, \pi/2]$.

The centreline switches from a circular to a non-circular extremal in the point $E_1$. If we assume that the piece $E_1F_1$ loses continuous contact with $A_2B_2$ at some point $F_1'$, then the centreline beginning from $F_1'$ would be either circular (which is evidently impossible) or straight, followed by a circular arc around $F_1$ with maximal curvature. The second case
may be analyzed in the same way as will be done in the next section for a similar problem. It is shown there that the angle $\phi_{10}$ should not exceed $\pi/6$, otherwise the configurations would have longer centrelines. It may be demonstrated that, in the case of the Borromean rings, the assembly constraints prevent $\phi_{10}$ from being less than $\pi/6$. Therefore, we accept that starting at $E_1$ the centreline proceeds at a constant distance to the piece $A_2B_2$, and the shape of pieces $E_iF_i$ and $A_{i+1}B_{i+1}$ can be found by applying the formulas developed in the previous section.

The parameters $a$, $\chi$, $\psi$ and $\phi_{10}$ should satisfy the following constraints. First of all, it is clear that $a = 2\sin \chi$. We also have

$$a + x_{1F} = 2\cos \chi,$$

where $x_{1F}$ is the $x$-coordinate of point $F_1$.

Since the directions of the tangents are determined in the pair of corresponding points $E_1$ and $A_2$, we can find the parameter $C$ from Eq. (13): $C = \cos \phi_{10}$. The other pair of corresponding points are $F_1$ and $B_2$, but the direction of the continuous tangent at point $B_2$ is specified by the angle $\chi - \psi$, so we come to the equation $\sin(\chi - \psi) = \cos \phi_{10}$, which means that $\chi + \phi_{10} = \pi/2 + \psi$.

With the last relationship we can eliminate $\chi$ and then substitute $a = 2\sin \chi$ into Eq. (35) to obtain

$$x_{1F} = 2[\sin(\phi_{10} - \psi) - \cos(\phi_{10} - \psi)].$$

By using Eq. (28), we find

$$x_{1F} = \csc \phi_{10} - \int_{\phi_{10}}^{\pi/2} \frac{1 - y_1(\tilde{\phi}_1)}{\sin^2 \tilde{\phi}_1} d\tilde{\phi}_1,$$  

where $y_1$ is computed by Eq. (21). Finally, Eq. (36) may be given the form

$$\int_{\phi_{10}}^{\pi/2} \frac{\cos \tilde{\phi}_1}{\sin^2 \tilde{\phi}_1} \left[ 1 - \left( \frac{\cos \phi_{10} - \cos \tilde{\phi}_1}{\csc \phi_{10} - \cos \tilde{\phi}_1} \right)^2 \right] d\tilde{\phi}_1 = \csc \phi_{10} + 2\left[ \cos(\phi_{10} - \psi) - \sin(\phi_{10} - \psi) \right].$$

The last integral equation may be considered as a constraint on the values of $\psi$ and $\phi_{10}$. It can be solved numerically to obtain a one-parameter family of possible shapes. There is only one branch of solutions in the range $0 < \phi_{10} \leq \pi/2$, $\psi \geq 0$. For every configuration, we compute its length-to-radius ratio by using Eqs. (34) and (21). In our case

$$l = 12(4\psi + \pi + 2l_c),$$

where
\[ l_e = 2 \int_{\phi_{10}}^{\pi/2} \frac{d\phi}{k_1(\phi)} = 2 \left( \cot \phi_{10} - \int_{\phi_{10}}^{\pi/2} \frac{1 - y_1(\phi)}{\sin^2 \phi} d\phi \right) \]

is the normalized length of extremal arc computed by Eq. (33).

In Fig. 8 the dependence of the length-to-radius ratio is presented as a function of \( \psi \). We see that the tightest shape is achieved for \( \psi = 0 \), i.e. when the arc \( B_C \) disappears. As \( \psi \) grows to its maximum \( \psi = \chi = \arcsin(\sqrt{7} - 1)/4 \), the length-to-radius ratio approaches the limit where there are no non-circular arcs \( A_B \) and \( E_F \) and \( \phi_{10} = \pi/2 \).

For the tightest configuration, \( \phi_{10} = 1.14723118 \) (65.73\(^\circ\)). The distance \( OA \), is \( a = 2\cos \phi_{10} = 0.822026 \). The numerical value of the length-to-radius ratio will be \( l = 58.0399908 \). As compared to the “circular” configuration with \( l_{\text{circ}} = 58.0526017 \), the new shape is shorter by \( \Delta l = l_{\text{circ}} - l = 0.0126109 \). (The numerical simulation gives a result which is less optimal than the “circular” one: \( l_{120} = 58.3052 \), due to LAURIE (2002) where each ring was modelled as a 40-gon.)

We see that the difference between the two shapes is so small that it could hardly be visible to the naked eye. Figure 9 shows the actual arrangement of the rings in space. The thickness of the tubes is reduced by half to make the contact line visible. Only two
fragments of the latter are drawn: 1) a circular one formed as the fragment with the centreline $C_1D_1E_1$ goes around the cross-section at point $A_2$, and 2) the contact curve between the pieces corresponding to $E_1F_1$ and $A_2C_2$ (see Fig. 7).

The dependence of the curvature on the angle $\phi_1$ is shown in Fig. 10 for the extremal non-circular arc $E_iF_i$.

4. On the Shape of the Clasped Rope of Minimal Length

As another example we consider the linked structure formed by two pieces of incompressible inextensible rope (Fig. 11). Let us fix the diameter of the cross-section to 1. The entire structure is located between two parallel planes $\alpha$ and $\beta$, and each piece of the rope is attached to its own plane at the points $G_1 \in \alpha, H_1 \in \alpha$ and $G_2 \in \beta, H_2 \in \beta$. The straight line segments $G_1H_1$ and $G_2H_2$ are orthogonal to each other and they have the same length $N, N \geq 1$. Denote the distance between the parallel planes by $M, M \geq 2$.

We assume that each loop of the rope has a planar centreline and the length of each piece is equal to $L/2$. The problem is to find the shape of the centrelines such that their total length $L$ be minimal. The centrelines should have a continuous tangent everywhere. In fact, we search for a configuration of only one piece of rope because the second one is assumed to have the same geometry due to the symmetry of the problem.
Figure 12 shows the section of the two clasped ropes by the plane of the first centreline $G_1A_1H_1$. It is symmetric about the axis $K_1A_1 \perp \alpha$, $G_1K_1 = K_1H_1$. $A_1$ ($A_2$) is the most distant point on the first (second) centreline from the plane $\alpha$ ($\beta$). Denote $x_{1A} = A_1A_2$ and $l_1 = K_1A_2$.

Each half of the centreline includes a straight fragment which does not touch anything, and a curvilinear part which is in contact with the other piece. We concentrate on one half of the 1-st piece $G_1C_1A_1$. $G_1C_1$ is straight and $C_1$ is the first point where the contact occurs, i.e. $G_1C_1 \perp C_1A_2$, $C_1A_2 = 1$. Therefore, the entire length of both centrelines is

$$L = 4(L_{GC} + L_{CA}).$$

(40)

Let $\chi$ be an angle between $G_1C_1$ and the normal to the plane $\alpha$. Clearly,

$$L_{GC} = l_1 \sec \chi + \tan \chi.$$  

(41)

We also have the two relationships:

$$M = 2l_1 + x_{1A}$$

(42)

and
We continue our analysis assuming that every cross-section along $C_1A_1$ touches the second rope. Then the non-straight part of the centreline may include two fragments: a circular arc $C_1E_1$ and the remaining non-circular fragment $E_1A_1$ (Fig. 13). The piece of rope corresponding to $C_1E_1$ touches the cross-sectional slice of the second rope with the centre $A_2$. The part $E_1A_1$ is in contact with the piece $A_2E_2$.

Let the location of the point $E_1$ be specified by the angle $\phi_{10}$. Then, since the tangent to the second centreline at point $A_2$ is orthogonal to the $xy$-plane, we have from Eq. (13) that $C = \cos \phi_{10}$ and the tangent is continuous in the switching point $E_1$.

However, the curvature of the centreline is not allowed to be greater than 2, otherwise the tubular neighbourhood of radius 1/2 is not embedded. From Eq. (32) we deduce that, for $\phi_{10} \geq \pi/6$, the curvature at $A_1$ does not exceed 2. It is also easy to check that the curvature is maximal at $A_1$ for $\phi_{10} \geq \pi/6$. 

\[
\frac{N}{2} = \sec \chi + l_1 \tan \chi. \tag{43}
\]
Fig. 12. The central section of the two clasped ropes.

Fig. 13. Construction of tight clasped ropes.
For $\phi_{10} < \pi/6$, the extremal non-circular solution cannot proceed up to the point $A_1$: it should terminate earlier at a point denoted by $U_1$ in Fig. 14. For the complementary centreline, this automatically means that the point $E_1$ would be in correspondence not with $A_2$ but with another point $V_2$ which is closer to $E_2$. Then $A_2V_2$ would be a circular arc of the
minimal radius 1/2. Since \( E_1 \) belongs to the circular arc \( C_1E_1 \) of radius 1, it is impossible for \( E_1V_2 \) to be orthogonal to both tangents at the points \( E_1 \) and \( V_2 \). The only possibility to keep a non-zero arc \( A_2V_2 \) is to assume that in points \( E_1 \) and \( V_2 \) the ropes lose contact. More precisely, we introduce two contact-free (hence, straight) fragments \( E_1P_1 \) and \( V_2U_2 \) such that \( P_1 \) is in correspondence with \( U_2 \) and the extremal solution describes the fragments \( P_1U_1 \) (and \( U_2P_2 \)).

Let the arc \( A_1V_1 \) (and \( A_2V_2 \)) be measured by angle \( \gamma \geq 0 \). We denote the straight segments \( E_1P_1 \) and \( V_2U_2 \) by \( r_1 = (X_1, Y_1, Z_1) \) and \( r_2 = (X_2, Y_2, Z_2) \), respectively. Their parametric representation is as follows

\[
X_1 = \sin \phi_{10} + t \cos \phi_{10}, \quad Y_1 = 1 - \cos \phi_{10} + t \sin \phi_{10}, \quad Z_1 = 0;
\]
\[
X_2 = \frac{1 - \cos \gamma}{2} + s \sin \gamma, \quad Y_2 = 1, \quad Z_2 = \frac{1}{2} \sin \gamma + s \cos \gamma,
\]

where \( t \geq 0, \) and \( s \geq 0, \) are parameters. Clearly, the tangents to the curves are continuous at the joining points \( E_i \) and \( V_i \).

We are looking for values of \( t = t_1 \) and \( s = s_1 \) such that

\[
\|r_1(t_1) - r_2(s_1)\| = 1, \quad \frac{dr_1(t_1)}{dt} \cdot (r_1(t_1) - r_2(s_1)) = 0, \quad \frac{dr_2(s_1)}{ds} \cdot (r_1(t_1) - r_2(s_1)) = 0. \quad (44)
\]

The existence of positive solutions to Eq. (44) provides the possibility to join the straight fragments to the extremal curves without jumps of tangents.

We can express the values \( t_1 \) and \( s_1 \) from the last two conditions Eq. (44)

\[
t_1 = [\cos \gamma (\cos \gamma - 1)/2 + \sin^2 \gamma \sin \phi_{10} \cos \phi_{10}/(1 - \sin^2 \gamma \cos^2 \phi_{10})],
\]
\[
s_1 = [\sin \phi_{10} - (\cos \gamma \cos^2 \phi_{10} + \sin^2 \phi_{10})/2] \sin \gamma/(1 - \sin^2 \gamma \cos^2 \phi_{10}).
\]

Substitution of them into the first condition of Eq. (44) gives an equation for \( \phi_{10} \) and \( \gamma \) which defines implicitly the function \( \gamma(\phi_{10}) \) (Fig. 15). The value of \( \gamma \) vanishes for \( \phi_{10} = \pi/6 \), and it increases, as \( \phi_{10} \) becomes smaller.

Now we can compute the length of one half of one centreline. It includes the following components. The length of the circular arc \( C_1E_1 \) is simply \( L_{CE} = \phi_{10} - \chi \geq 0 \) (note that it does not make sense to consider the case \( \phi_{10} < \chi \)). The length of the remaining part \( L_{EA}(\phi_{10}) \), \( \phi_{10} \in [\chi, \pi/2] \), is the sum \( L_{EA} = L_{EP} + L_{PU} + L_{UV} + L_{VA} \), where

\[
L_{EP} = \begin{cases} t_1, & \text{if } \phi_{10} < \pi/6, \\ 0, & \text{otherwise,} \end{cases} \quad L_{UV} = \begin{cases} s_1, & \text{if } \phi_{10} < \pi/6, \\ 0, & \text{otherwise,} \end{cases} \quad L_{VA} = \begin{cases} \gamma/2, & \text{if } \phi_{10} < \pi/6, \\ 0, & \text{otherwise,} \end{cases}
\]

and \( L_{PU} \) may be computed by Eq. (33) with \( \phi_{11} = (\pi/2) - \gamma \) and \( C = \cos \phi_{10} + \sin \gamma \).
Fig. 16. The gain in length $\Delta L$ for $M = 3$ as a function of $N$ and $\phi_{10}$. 
With the help of Eqs. (21) and (28), we find the coordinates of the central lines; in particular,

\[ x_{1A}(\phi_{10}) = \sin \phi_{10} + t_1 \cos \phi_{10} + x_1(\phi_{11}) - x_1(\phi_{10}) + s_1 \cos \phi_{11} + \frac{1}{2}(1 - \sin \phi_{11}). \]  

(45)

Now we can express \( l_1 \) from Eq. (42), insert it into Eq. (41) and then further substitute \( L_{GC} \) into Eq. (40) to obtain
Equation (43) together with Eq. (42) allow us to express the angle $\chi$ as a function of $M, N$ and $\phi_{10}$:

$$
\chi(M, N, \phi_{10}) = 2 \arctan \left( \frac{x_{1A}(\phi_{10}) - M + \sqrt{[M - x_{1A}(\phi_{10})]^2 + N^2 - 4}}{N + 2} \right). \quad (47)
$$

Combining Eq. (47) with Eq. (46), we obtain the entire length as a function of $M, N$ and $\phi_{10}$. This length can be compared with the one for $\phi_{10} = \pi/2$ which corresponds to the shapes containing no non-circular arc. Figure 16 shows the difference of lengths $\Delta L = L(M, N, \phi_{10}) - L(M, N, \pi/2)$ for fixed $M = 3$. Computations reveal that the absolute minimum is reached for $N = 2$, i.e. for the ropes with parallel straight line parts ($\chi = 0$). In this case, $\Delta L$ does not depend on the distance $M$ between the parallel planes. Another property of $\Delta L$ is that, for particular $N$ and for $\phi_{10} \geq \pi/6$, the optimal value of $\Delta L$ takes place for $\phi_{10}^* = |\chi|$. Actually, the values of $N$ smaller than 1 mean that the ends of the rope interpenetrate each other. However, we include these values just to demonstrate the global behaviour of $\Delta L$ and the connection with the ideal Hopf link which corresponds to $N = 0, M = 3$ and $\phi_{10} = \pi/2$. 

Fig. 19. The tightest clasp in the parallel case ($N = 2$). One half of each piece of rope is shown. Three fragments of the contact line are displayed. The thickness radius of the tubes is drawn as 1/4 instead of 1/2 to make the structure easier to understand.
Now consider the parallel case in more detail. Figure 17 shows how $\Delta L$ changes with $\phi_{10}$. The minimal value $\Delta L = -0.0201801598$ is reached for $\phi_{10} = \phi_{10}^* = 0.5216625925$ and $\gamma = 0.1335013510$. Note that the contact-free segments are extremely short: $L_{E_P} = 0.0038832246$, $L_{U_V} = 0.0002248152$. Figure 18 presents the comparison of the centreline for $\phi_{10} = \phi_{10}^*$ with the circular arc ($\phi_{10} = \pi/2$). We see that their difference is rather small. The contact set is presented in the next Fig. 19. The graph of the curvature of the centreline $C_1A_1$ is shown in Fig. 20. The curvature is discontinuous because of the presence of straight line segments, it equals 1 at the circular arc $C_1E_1$, then it jumps to zero for a short interval, after that it behaves smoothly at the segment $P_1U_1$, then it jumps again for the second short straight line segment, and finally it takes on the maximally allowable value of 2 near the tip point $A_1$.

5. Periodic Tight Structures

The above solution for the tight clasp of two pieces of the perfect rope can be used to create periodic structures in space. Let us start with the parallel case. As before, we fix the diameter of the cross section to 1. We identify the plane $\alpha$ with the plane $\beta$ rotated through $\pi/2$ around the common normal to both planes. Then the shape of the linked ropes in case $N = 2$ gives us the periodic chain of period $2M$, $M \geq 2$. It makes little sense to consider such a shape without imposing additional constraints. The linkage density may serve as a good characteristic number to be fixed. We assume first that the chain of $n$ links of the rope connects two parallel planes (Fig. 21). Then the linkage density may be defined as the ratio $\lambda = 2(n - 1)/M$, $\lambda \leq 1$. If $\lambda = 1$ then the shortest chain contains stadium-like rings with circular arcs (Cantarella et al., 2002). For smaller values of $\lambda$, the shortest configuration includes non-circular curvilinear fragments as described in the previous section. It is easy to see that the parallel clasp problem is the simplest nontrivial case of the chain with $n = 2$. Now we can get rid of the parallel planes at the ends of the chain and consider an infinitely long chain with a specified linking density $\lambda$. Its shortest shape is described in exactly the same manner.

It is worth noting that the problem of finding most compact shapes of an one-dimensional periodic structure was considered by Piérański (1998) for double-strand
Fig. 21. A tight chain between two parallel walls.

regular helices, and, for the single-strand helix, by Maritan et al. (2000) and by Przybyl and Pierański (2001). As pointed out by Stasiak and Maddocks (2000), in both cases it is necessary to have an additional “compactness condition” on the centreline in order to compensate for the absence of the constraints of closure and knotting. Specification of the linkage density serves the same purpose in the above-described chain.

The rope structures can also be periodic in two dimensions. Again, we start with the case of two clasped perfect ropes and formally set $M = 0$. The last condition means that all four ending points $G_i, H_i, i = 1, 2$, belong to the same plane. Now consider a square of size $N$, lying in the same plane $\alpha = \beta$, the sides of which are parallel to $G_iH_i$ and contain the end points. The entire plane $\alpha = \beta$ may be covered with square cells $N \times N$, and each cell represents a two-piece clasped rope associated with it. We arrange the ropes in such a way as to make the tangents continuous at the points $G_i, H_i$. This is accomplished if every two squares having a common side, contain the clasped rope pairs which are reflections of each other in the plane $\alpha$. Finally, we obtain a two-dimensional periodic structure of periods $2N$ resembling an interwoven textile (Fig. 22). To fix the scale, we set the diameter of the rope’s cross section to 1.
Let us consider the shape of the clasp for $M = 0$. Figure 23 represents the difference of the length between the extremal non-circular and circular solutions as a function of the angle $\phi_{10}$ for various $N$. For every $N$, the best result is achieved when $\phi_{10}$ takes on its minimal possible value $\phi_{10} = \chi$.

The question now arises of how small the semi-period $N$ could be. The smallest $N$ corresponds to the requirement that $C_i \in \alpha$, i.e. $C_j = G_j$. Then $\sin \chi = x_{1A}/2$, $\tan \chi = x_{1A}/N$. Taking $\chi = \phi_{10}$, we come to the equation for $\phi_{10}$

$$x_{1A}(\phi_{10}) = 2\sin \phi_{10},$$

which has the solution $\phi_{10} = 0.5395100925$. The minimal semi-period is $N = 2\cos \phi_{10} = 1.715920915$ and the length of two centrelines per one cell is $L_{\min} = 4.163380172$.

We can compare this solution to the shortest shape made up with circular arcs. For the latter, we have $N_{\text{circ}} = \sqrt{3} = 1.732050808$, $L_{\text{circ}} = 4\pi/3 = 4.188790204$. Therefore, we conclude that the extremal non-circular solution provides the minima both of the cell size $N$ and of the centreline length per period. In that sense this periodic interwoven structure may be called a tight web.
Fig. 23. The gain in length $\Delta L$ for $M = 0$ as a function of $\phi_{10}$ for several values of $N$.

6. Concluding Remarks

Although we have not proved that the solutions we have found realize ideal shapes (i.e. those with minimal length-to-thickness ratio among all possible configurations of the same topology), the above considerations as well as numerical results suggest that ideal knots and links generically include pieces of rope that contact each other so that the points on the centrelines are in one-to-one correspondence.

It is Sullivan who has first pointed to the fact that the shortest shape of two (parallel) clasped ropes is achieved for non-circular centrelines. According to him, there are fragments of a non-zero length in the vicinity of the climax points $A_i$ where the curvature is constant and takes its maximally allowed value of 2. Although his numerical result (see the graph for curvature in Sullivan (2003)) looks differently from what is presented in this paper, the difference is likely due to the limited accuracy of numerical computation.

Generally, the problem of the tight clasp of linked ropes is different in the very formulation from the ideal shapes. In the latter, there exist only two natural lengths: the diameter of the rope and its length which means that only its dimensionless ratio is essential for the shape. In the tight clasp problem, we have introduced two more lengths, $N$ and $M$ and, as a consequence, the tight shapes constitute a two-parameter family. We can take another way and specify more constraints in the problem statement. One natural example is the clasp of ropes whose free fragments are constrained to be parallel. If $d$ is the diameter of the cross-section, then the parallelism means the constraint $N = 2d$. As we have seen, the
distance \( M \) between the parallel planes does not matter in this particular case. It is possible then to compute the length-to-diameter ratio excluding the long straight pieces of the rope. Thus, we come to the slightly paradoxical concept of the ideal (i.e. having the minimal length-to-diameter ratio) shape of the link consisting of non-closed parts. The parallelism constraint serves as a boundary condition instead of the closure requirement.

In a similar way, the constraint that comes from the periodicity in two dimensions, also replaces the closure property of the components.

The approach presented is developed for the configurations with the centrelines lying in orthogonal planes. It seems to be a straightforward task to adapt the method to centrelines confined to belong to a priori known surfaces. However, it remains unclear how to find those surfaces for ideal knots or links.

Already the special examples (in particular, the Borromean rings) allow us to make a conclusion that the numerical computation of the tightest shapes presents a difficult and delicate problem: in order to obtain a conclusive evidence of some fine features of the extremal shape, the numerical accuracy should be improved by at least two orders. Since the existing algorithms work with discretized centrelines represented as polygonal lines, this means a significant increase of a number of segments that, in turn, will cause an enormous growth of the computational work. A possible way out could be a method that does not use the polygonal approximation. There are good grounds to think that the further development of the variational approach will eventually lead to the design of an efficient and accurate numerical algorithm.

Deep gratitude is expressed to John Maddocks for his support and help with the language and style. The author also thanks Jana Smutny who gave useful feedback on an earlier version of the manuscript. The figures were produced with MAPLE and POV-RAY. Most of the figures appear in colour in the electronic version of the articles.

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