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Physica A 347 (2005) 268–288

PHYSICA A

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Non-Poisson processes: regression to equilibrium versus equilibrium correlation functions

Paolo Allegrini[a], Paolo Grigolini[b,c,d], Luigi Palatella[e], Angelo Rosa[f,*], Bruce J. West[g]

[a] *INFM, unità di Como, via Valleggio 11, 22100 Como, Italy*
[b] *Center for Nonlinear Science, University of North Texas, P.O. Box 311427, Denton, TX 76203-1427, USA*
[c] *Dipartimento di Fisica dell'Università di Pisa and INFM, Via Buonarroti 2, 56127 Pisa, Italy*
[d] *Istituto dei Processi Chimico Fisici del CNR, Area della Ricerca di Pisa, Via G. Moruzzi 1, 56124 Pisa, Italy*
[e] *Dipartimento di Fisica and INFM, Center for Statistical Mechanics and Complexity, Università di Roma "La Sapienza", P.le A. Moro 2, 00185 Rome, Italy*
[f] *Institut de Mathématiques B, Faculté des Sciences de Base, École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland*
[g] *Mathematics Division, Army Research Office, Research Triangle Park, NC 27709, USA*

Received 4 June 2004; received in revised form 7 July 2004
Available online 8 September 2004

Abstract

We study the response to perturbation of non-Poisson dichotomous fluctuations that generate super-diffusion. We adopt the Liouville perspective and with it a quantum-like approach based on splitting the density distribution into a symmetric and an anti-symmetric component. To accomodate the equilibrium condition behind the stationary correlation function, we study the time evolution of the anti-symmetric component, while keeping the symmetric component at equilibrium. For any realistic form of the perturbed distribution density we expect a breakdown of the Onsager principle, namely, of the property that the subsequent regression of the perturbation to equilibrium is identical to the corresponding equilibrium correlation function. We find the directions to follow for the calculation of

*Corresponding author. Tel.: +41-21-69-30358; fax: +41-21-69-35530.
E-mail address: angelo.rosa@epfl.ch (A. Rosa).

doi:10.1016/j.physa.2004.08.004

higher-order correlation functions, an unsettled problem, which has been addressed in the past by means of approximations yielding quite different physical effects.

PACS: 05.40.−a; 89.75.−k; 02.50.Ey

Keywords: Stochastic processes; Non-Poisson processes; Liouville and Liouville-like equations; Correlation function; Regression to equilibrium

1. Introduction

Complex physical systems typically have both nonlinear dynamical and stochastic components, with neither one dominating. The response of such systems to external perturbations determines the measurable characteristics of phenomena from the beating of the human heart to the relaxation of stressed polymers. What distinguishes such complex phenomena from processes successfully studied using equilibrium statistical mechanics is how these systems internalize and respond to environmental changes. Consequently, it is of broad interest to determine which of the prescriptions from equilibrium statistical physics is still applicable to complex dynamical phenomena and which are not. Herein we address the breakdown of one of these fundamental relations, that is, the Onsager Principle. The authors of Ref. [1] discussed how to make the generalized master equation (GME) compatible with the Onsager principle, leaving open, however, the practical problem of the response to realistic perturbations that, as we shall see, cause the breakdown of this important property.

It is worthwhile to briefly review this issue in the standard cases. In the case of ordinary statistical mechanics an exhaustive treatment of the relaxation of perturbations to equilibrium can be found in Ref. [2]. Let us consider as a prototype of ordinary statistical mechanics the case when the stochastic variable under study $\xi(t)$ is described by the linear Langevin equation

$$\frac{d\xi}{dt} = -\gamma\xi(t) + \eta(t) , \tag{1}$$

where the random driving force $\eta(t)$ is white noise. Let us imagine that $\xi(t)$ is the velocity of a particle with unit mass and a given electrical charge. Furthermore, we assume that this system reaches the condition of equilibrium, and at a given time $t = 0$, we apply an electrical field $E(t)$. The external field $E(t)$ is an arbitrary function of time, fitting the condition that $E(t) = 0$, for $t < 0$. The adoption of linear response theory yields the prescription for the mean response of the system to the external field

$$\langle\xi(t)\rangle = \int_0^t \Phi_\xi(t')E(t-t')\,dt' , \tag{2}$$

where $\Phi_{\xi}(t)$ is the equilibrium correlation function of ξ. In this familiar case, of dissipative Brownian motion, the correlation function of the particle velocity is the exponential $\exp(-\gamma t)$.

There are two limiting cases of time dependence of the perturbation: (a) the electric field is proportional to the Heaviside step function, $E(t) = K\Theta(t)$, and is therefore a constant field after it is turned on at $t = 0$; (b) the electric field is proportional to the Dirac delta function, $E(t) = K\delta(t)$, and is consequently an initial pulse that perturbs the particle velocity. In these two limiting cases we obtain for the velocity of the Brownian particle

$$\langle \xi(t) \rangle = K \int_0^t \Phi_{\xi}(t')\,\mathrm{d}t' \tag{3}$$

and

$$\langle \xi(t) \rangle = K\Phi_{\xi}(t) \; , \tag{4}$$

respectively. These two limiting cases show that in the case of ordinary statistical mechanics the system's response to an external perturbation is expressed in terms of the unperturbed correlation function. We shall refer to the conditions of Eqs. (3) and (4) as the Green–Kubo relation and the Onsager relation, respectively.

The search for a dynamical derivation of anomalous diffusion, that is, where the mean square value of the dynamic variable is not linear in time, has been a subject of great interest in recent years. There are two main theoretical perspectives on how to explain the origin of anomalous diffusion. The first perspective is based on the assumption that there are unpredictable events, that the occurrence of these events obey non-Poisson statistics, and is related to the pioneering paper by Montroll and Weiss [3]. For more recent research work on this subject we refer the reader to the excellent review papers of Refs. [4,5]. The other perspective rests on the assumption that single diffusion trajectories have an infinite memory. The prototype of the latter perspective is the concept of fractional Brownian motion introduced by Mandelbrot [6]. A problem worthy of investigation is whether or not, in the case of anomalous diffusion, the response to external perturbation departs from the predictions of Eqs. (3) and (4). In the last few years, this problem has been addressed by some investigators [7–12]. These authors have discussed the Green–Kubo relation of Eq. (3). Notice that in the special case of ordinary statistical mechanics this relation can also be written in the following form

$$\langle x(t) \rangle = \frac{K}{2\langle \xi^2 \rangle} \langle x^2(t) \rangle_0 \; . \tag{5}$$

To understand how to derive this equation, originally proposed by Bouchaud and George [13], we refer to the following equation of motion:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \xi(t) \; . \tag{6}$$

Since $\xi(t)$ is a fluctuating velocity, it generates spatial diffusion and we denote by $x(t)$ the position of the corresponding diffusing particle. The external field affects the velocity fluctuation and, consequently, the diffusion process generated by these fluctuations. In the absence of perturbations, the second moment of the diffusing particle, $\langle x(t)^2\rangle_0$, obeys the prescription

$$\langle x(t)^2\rangle_0 = 2\langle \xi^2\rangle \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \Phi_\xi(t'') \ . \tag{7}$$

It is straightforward to prove that Eq. (3) yields Eq. (5). This is done by considering the mean value of Eq. (6)

$$\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = \langle \xi(t)\rangle \ . \tag{8}$$

The time integration of the left-hand term of this equation yields the left-hand term of Eq. (5) and the time integration of the right-hand term of Eq. (8), using Eqs. (3) and (7), yields the right-hand term of Eq. (5).

The relation of Eq. (5) is denoted as *generalized Einstein relation*, because it might hold true also when the equilibrium correlation function does not exist [10]. However, in the case of ordinary statistical mechanics Eq. (5) becomes equivalent to the Green–Kubo property. In this generalized sense we can state that the authors of Refs. [7–12] studied the Green–Kubo relation of Eq. (3), all of them but the authors of Ref. [7], devoting their attention to the sub-diffusional case, that is, the mean-square average displacement is proportional to t^γ with $\gamma < 1$. It is worth mentioning that there exists experimental evidence in favor of this relation [14,15].

Herein we focus our attention on the Onsager relation of Eq. (4). The only earlier work on how to extend the Onsager principle to the case of non-Poisson statistics, known to us, is that of Ref. [1]. In this paper the problem is addressed analyzing a virtually infinite symbolic sequence, of +'s and −'s, corresponding to the two values of $\xi(t)$, which, in fact, is supposed to be a dichotomous variable. They made the ergodic and stationary assumption and proved that this assumption yields the Onsager's regression principle. Adopting the same view with a Gibbs set of infinitely many independent sequences, the prescription of Ref. [1] would work as follows. We set the preparation at $t = -\infty$. This ensures the equilibrium condition, if, as assumed in Ref. [1], the non-Poisson statistics is compatible with ergodicity. Then at a given time $t > 0$, we select only the systems where $\xi = +$. From this time on, we follow the time evolution of these systems and we prove that $\langle \xi(t)\rangle$ is proportional to the correlation function $\Phi_\xi(t)$, implying the validity of Onsager's principle.

This conclusion is correct, but the choice done to prepare the initial out-of-equilibrium condition state implies that we have a detailed information on the single constituents of the Gibbs ensemble, an ideal condition difficult, if not impossible, to realize in practice. Here we plan to address the same problem from a perspective compatible, in principle, with experiment, by adopting a Liouville-like approach. This approach will allow us to establish some general conclusions about the issue raised by Bologna et al. [16] about a possible conflict between the density and trajectory picture. For this reason, we shall illustrate the rules for the calculation of the correlation function $\Phi_\xi(t)$, with a prescription that can be in principle extended to correlation functions of any order. We expect that these prescriptions might lead to a successful evaluation of the fourth-order correlation function, which has been studied so far by means of a factorization assumption which is violated by the non-Poisson statistics.

2. An idealized model of intermittent randomness and the corresponding density equation

As done in Ref. [16], let us focus on the following dynamical system. Let us consider a variable y moving within the interval $I = [0, 2]$. The interval is defined over an overdamped potential V, with a cusp-like minimum located at $y = 1$. If the initial condition of the particle is $y(0) > 1$, the particle moves from the right to the left toward the potential minimum. If the initial condition is $y(0) < 1$, then the motion of the particle toward the potential minimum takes place from the left to the right. When the particle reaches the potential bottom it is injected to an initial condition, different from $y = 1$, chosen in a random manner. We thus realize a mixture of randomness and slow deterministic dynamics. The left and right portions of the potential $V(y)$ correspond to the laminar regions of turbulent dynamics, while randomness is concentrated at $y = 1$. In other words, this is an idealization of the map used by Zumofen and Klafter [17], which does not affect the long-time dynamics of the process, yielding only the benefit of a clear distinction between random and deterministic dynamics. Note that the waiting time distribution in the two laminar phases of the reduced form has the same time asymptotic form as

$$\psi(t) = (\mu - 1)\frac{T^{\mu-1}}{(t+T)^{\mu}} \ . \tag{9}$$

We select this form as the simplest possible way to ensure the normalization condition

$$\int_0^{\infty} \mathrm{d}t\psi(t) = 1 \ . \tag{10}$$

We note that Eq. (10) implies $\mu > 1$. The condition $\mu > 2$ corresponds to the existence of a finite mean sojourn time, and, thus, to the possibility itself of defining the stationary correlation function of the fluctuation ξ, which, with the choice of Eq. (9) reads [18]

$$\Phi_{\xi}(t) \equiv \frac{\langle \xi(t)\xi(0)\rangle}{\langle \xi^2 \rangle} = \left[\frac{T}{t+T}\right]^{\mu-2} \ . \tag{11}$$

From within the perspective of a single trajectory this dynamical model has the form

$$\dot{y} = \lambda[\Theta(1-y)y^z - \Theta(y-1)(2-y)^z] + \frac{\Delta_y(t)}{\tau_{random}}\delta(y-1) \ . \tag{12}$$

The function $\Theta(x)$ is the ordinary Heaviside step function, $\Delta_y(t)$ is a random function of time that can achieve any value on the interval $[-1, +1]$, and τ_{random} is the injection time that must fulfill the condition of being infinitely smaller than the time of sojourn in one of the laminar phases. Note that z is a real number fitting the condition $z > 1$. In fact, the equality

$$z = \frac{\mu}{(\mu - 1)} \tag{13}$$

relates the dynamics of Eq. (12) to the distribution Eq. (9). The Poisson condition is recovered in the limit $z \to 1$, namely, in the limit $\mu \to \infty$. Thus, in a sense, the whole region $z > 1$ ($\mu < \infty$) corresponds to anomalous statistical mechanics. However, the deviation from normal statistical mechanics is especially evident when $z > 1.5$, a condition implying that the second moment diverges. In the case $z > 2$ the departure from ordinary statistical mechanics becomes even more dramatic, due to the fact that the first moment also diverges and, as we shall see in this Section, the process becomes non-ergodic.

Let us now move to the density picture, namely, to a formulation of Eq. (12) from within the Gibbs perspective. The form of this equation of evolution for the probability density is

$$\frac{\partial}{\partial t} p(y,t) = -\lambda \frac{\partial}{\partial y} \left\{ [\Theta(1-y)y^z - \Theta(y-1)(2-y)^z] p(y,t) \right\} + C(t) , \tag{14}$$

where

$$C(t) = \lambda p(1,t) . \tag{15}$$

It is important to stress that we are forced to set the equality of Eq. (15) to fulfill the following physical conditions:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{I=[0,2]} p(y,t)\,\mathrm{d}y = \int_{I=[0,2]} \frac{\partial}{\partial t} p(y,t)\,\mathrm{d}y = 0 , \tag{16}$$

which, in turn, ensures the conservation of probability. We assume the ordinary normalization condition

$$\int_{I=[0,2]} p(y,t)\,\mathrm{d}y = 1 , \tag{17}$$

which is kept constant in time, as a consequence of Eq. (16). It is evident that the inhomogeneous term $C(t)$ corresponds to the action of the stochastic term, namely, the second term on the right-hand side of Eq. (12).

It is important to point out that our dynamic perspective allowed us to describe the intermittent process through the Liouville-like equation

$$\frac{\partial p(y,t)}{\partial t} = \mathscr{R} p(y,t) , \tag{18}$$

where y denotes a continuous variable moving either in the right or in the left laminar region, with ξ getting the values W or $-W$, correspondingly, and the operator $\mathscr{R}$ reading

$$\mathscr{R} = -\lambda \frac{\partial}{\partial y}[\Theta(1-y)y^z - \Theta(y-1)(2-y)^z] + \lambda \int_0^2 \mathrm{d}y \delta(y-1) . \tag{19}$$

This operator departs from the conventional form of a differential operator, since the last term corresponds to the unusual role of an injection process, which is random rather than being deterministic. In the ordinary Fokker–Planck approach the role of the stochastic force is played by a second-order derivative, which is not as unusual as the integral operator of Eq. (19). Here the role of randomness is played by the back-injection process, which, from within the density perspective is described by an operator that selects from all possible values $p(y,t)$, the specific value of $p(y,t)$ at $y=1$. The idealization that we have adopted, of reducing the size of the chaotic region to zero, with the choice of the process of back injection located at $y=1$, makes it possible for us to use the continuous time representation and the equation of motion Eq. (18) rather than the conventional Frobenius–Perron representation. This representation will allow us to obtain analytical results. However, the same physical conclusions would be reached, albeit with more extensive algebra, using the conventional maps and the Frobenius–Perron procedure described in the recent book by Driebe [19].

We note that the equilibrium probability density solving Eq. (14) is given by

$$p_0(y) = \frac{2-z}{2}\left[\frac{\Theta(1-y)}{y^{z-1}} + \frac{\Theta(y-1)}{(2-y)^{z-1}}\right] . \tag{20}$$

This equilibrium density becomes negative for $z>2$. This fact, by itself, is not a big problem. As we shall see in Section 5, we might define the distribution density as $-p_0(y)$, thereby settling this problem. There is, however, a deeper reason why we cannot rest in Eq. (20), when $z>2$. This is the fact that the distribution cannot be normalized, thereby signaling the important fact that for $z>2$ there no longer exists an invariant distribution. The lack of an invariant distribution accounts for the non-ergodicity in the fluorescence of single nanocrystals, modeled as a dichotomous process, recently pointed out by Brokmann et al. [20].

For the purposes of calculation in the next few sections, it is convenient to split the density $p(y,t)$ into a symmetric and an anti-symmetric part with respect to $y=1$,

$$p(y,t) = p_S(y,t) + p_A(y,t) . \tag{21}$$

This separation based on symmetry yields the following two equations from Eq. (14):

$$\frac{\partial}{\partial t}p_S(y,t) = -\lambda\Theta(1-y)\frac{\partial}{\partial y}[y^z p_S(y,t)] + \lambda\Theta(y-1)\frac{\partial}{\partial y}[(2-y)^z p_S(y,t)] + C(t) \tag{22}$$

and

$$\frac{\partial}{\partial t}p_A(y,t) = -\lambda\Theta(1-y)\frac{\partial}{\partial y}[y^z p_A(y,t)] + \lambda\Theta(y-1)\frac{\partial}{\partial y}[(2-y)^z p_A(y,t)] \,. \quad (23)$$

We note that the anti-symmetric part of the density is driven by a conventional differential operator, which we denote by Γ. Thus, we rewrite Eq. (23) as follows:

$$\frac{\partial}{\partial t}p_A(y,t) = \Gamma p_A(y,t) \,, \quad (24)$$

where

$$\Gamma \equiv -\lambda\Theta(1-y)\frac{\partial}{\partial y}y^z + \lambda\Theta(y-1)\frac{\partial}{\partial y}(2-y)^z \,. \quad (25)$$

Then, it is convenient to define

$$p_{S,A}(y=1,t) = \frac{1}{2}\left(\lim_{y\to 1^+} p_{S,A}(y,t) + \lim_{y\to 1^-} p_{S,A}(y,t)\right) \,, \quad (26)$$

which implies $p_A(y=1,t)=0$.

The reader might be disturbed by the fact that these results have been derived by dealing with the step function as a constant, while it apparently yields a significant discontinuity at $y=1$.

Actually, as we shall see in Section 5, the discontinuity affects only the anti-symmetric part, and this discontinuity is justified by the fact that the right and left portions of the anti-symmetric part correspond to two independent dynamic processes of particles and holes.

The operator with the unusual form, containing $C(t)$, is only responsible for the time evolution of the symmetric part of the probability density. We notice, on the other hand, that any physical effect producing a departure of $C(t)$ from its equilibrium value, if this exists, namely, if $z<2$, implies a departure from equilibrium. A stationary correlation function can be evaluated, as we shall see in the next few sections, using only Eq. (23), without forcing Eq. (22) to depart from the equilibrium condition.

As we shall see in Section 6, the evaluation of correlation functions of order higher than the second cannot be done without producing a departure of $C(t)$ from its equilibrium value. This might generate the impression that the correlation functions of order higher than the second cannot be evaluated without internal inconsistencies, if we use only the density picture. The evaluation of these higher-order correlation functions was done in Ref. [21], by using a procedure based on the time evolution of single trajectories. Actually, as we shall see in Section 6, the density approach should yield the same result. However, we think that deriving this result using only the Liouville-like equation of this section is a hard task, which was bypassed in the past by means of the factorization approximation [16,25], violated by the non-Poisson case.

3. The correlation function of the dichotomous fluctuation from the trajectory picture

Let us focus our attention on Eq. (12) and consider the initial condition $y_0 \in [0, 1]$. Then, it is straightforward to prove that the solution, for $y < 1$ is,

$$y(t) = y_0\left(1 - \lambda(z-1)y_0^{z-1}t\right)^{-1/(z-1)} . \tag{27}$$

From (27) and imposing the condition $y(T) = 1$, we can find the time at which the trajectory reaches the point $y = 1$, which is

$$T = T(y_0) = \frac{1 - y_0^{z-1}}{\lambda(z-1)y_0^{z-1}} . \tag{28}$$

Since we have to find $\xi(t)$ and $\xi(t) = \xi(y(t))$, from the general form of Eq. (28) we obtain:

$$\begin{aligned}\frac{\xi(t)}{W} &= [\Theta(1-y_0)\Theta(T(y_0)-t) - \Theta(y_0-1)\Theta(T(2-y_0)-t)] \\ &\quad - \sum_{i=0}^{+\infty} \operatorname{sign}\left[\Delta_y\left(\sum_{k=0}^{i}\tau_k\right)\right]\left[\Theta\left(\sum_{k=0}^{i+1}\tau_k - t\right) - \Theta\left(\sum_{k=0}^{i}\tau_k - t\right)\right] ,\end{aligned} \tag{29}$$

where the time increments are given by

$$\begin{aligned}\tau_0 &= T(y_0) = \frac{1-y_0^{z-1}}{\lambda(z-1)y_0^{z-1}}\Theta(1-y_0) + \frac{1-(2-y_0)^{z-1}}{\lambda(z-1)(2-y_0)^{z-1}}\Theta(y_0-1) \\ \tau_{i\geqslant 1} &= \frac{1-[1+\Delta_y(\tau_i)]^{z-1}}{\lambda(z-1)[1+\Delta_y(\tau_i)]^{z-1}}\Theta(-\Delta_y(\tau_i)) + \frac{1-[1-\Delta_y(\tau_i)]^{z-1}}{\lambda(z-1)[1-\Delta_y(\tau_i)]^{z-1}}\Theta(\Delta_y(\tau_i)) .\end{aligned} \tag{30}$$

Then, for the correlation function we obtain the following expression:

$$\begin{aligned}\frac{\langle\xi(t)\xi(0)\rangle}{W^2} &= \langle[\Theta(1-y_0)\Theta(T(y_0)-t) + \Theta(y_0-1)\Theta(T(2-y_0)-t)]\rangle \\ &\quad + \sum_{i=0}^{+\infty}\left\langle \operatorname{sign}(y_0-1)\operatorname{sign}\left[\Delta_y\left(\sum_{k=0}^{i}\tau_k\right)\right]\right. \\ &\quad \left.\times\left[\Theta\left(\sum_{k=0}^{i+1}\tau_k - t\right) - \Theta\left(\sum_{k=0}^{i}\tau_k - t\right)\right]\right\rangle .\end{aligned} \tag{31}$$

As pointed out in Section 2, the calculation of the correlation function rests on averaging on the invariant distribution given by Eq. (20). As a consequence of this averaging, the second term in Eq. (31) vanishes. In fact, the quantity to average is anti-symmetric, whereas the statistical weight is symmetric.

It is possible to write the surviving term in the correlation as

$$\begin{aligned}\frac{\langle \xi(t)\xi(0)\rangle}{W^2} &= (2-z)\int_0^1 \Theta\left(\frac{1-y^{z-1}}{\lambda(z-1)y^{z-1}} - t\right)\frac{1}{y^{z-1}}\,\mathrm{d}y \\ &= (2-z)\int_0^{(1+\lambda(z-1)t)^{-1/(z-1)}} y^{-z+1}\,\mathrm{d}y \\ &= (1+\lambda(z-1)t)^{-(2-z)/(z-1)} \\ &\equiv (1+\lambda(z-1)t)^{-\beta}\,, \end{aligned} \tag{32}$$

with

$$\beta = \frac{2-z}{z-1}\ . \tag{33}$$

Since we focus our attention on $0<\beta<1$, we have to consider $\frac{3}{2}<z<2$. Note that the region $1<z<\frac{3}{2}$ does not produce evident signs of deviation from ordinary statistics. However, as we shall see in Section 6, the higher-order correlation functions can be easily evaluated only in the case $z=1$, when the correlation function becomes identical to the exponential function $\exp(-\lambda t)$, where the factorization assumption holds true. If $z>1$, the calculation of the higher-order correlation functions from within the Liouville-like approach becomes a very difficult task. Note also that Eq. (32) becomes identical to Eq. (11) after setting the condition

$$\lambda(z-1) = \frac{1}{T}\ . \tag{34}$$

4. The correlation function of the dichotomous fluctuation from the density picture

The result of the preceding section is reassuring, since it establishes that the intermittent model we are using generates the wanted inverse power law form for the correlation function of the dichotomous variable $\xi(t)$. In the present section we show that exactly the same result can be derived from the adoption of the Frobenius–Perron form of Eq. (14).

To fix ideas, let us concisely review the results obtained in earlier work [22,23], which refers to a dynamic system that can be identified with the left part of the model of Section 2. A particle in the interval $[0,1]$ moves toward $y=1$ following the prescription $\dot{y} = \lambda y^z$ and when it reaches $y=1$ it is injected backwards at a random position in the interval. The evolution equation obeyed by the densities defined on this interval is the same as Eq. (14), with $C(t) = \lambda p(1,t)$. This dynamic problem was addressed in Refs. [22,23], and solved using the method of characteristics as detailed in Ref. [24]. It is important to stress that this approach is the requisite price for adopting the idealized version of intermittency. The adoption of the more conventional reduced map of Ref. [17] would have made it possible for us to adopt the elegant prescriptions of Driebe [19], as done in Ref. [22].

Let us remind the reader that the solution afforded by the method of characteristics, in the case of this simple non-linear equation with stochastic boundary conditions is

$$p(y,t) = \int_0^t \frac{\lambda p(1,\xi)}{g_z((t-\xi)y^{z-1})}\,\mathrm{d}\xi + p\left(\left[\frac{y}{[g_z(y^{z-1}t)]^{1/z}}\right],0\right)\frac{1}{g_z(y^{z-1}t)} \tag{35}$$

where

$$g_z(x) \equiv (1+\lambda(z-1)x)^{z/(z-1)} \,. \tag{36}$$

As shown later in this section, to find the correlation function $\langle\xi(t)\xi(0)\rangle$, and the corresponding functions of higher order as well, using only densities, we have to solve Eqs. (22) and (23), which are equations of the same form as that yielding Eq. (35). For this reason we adopt the method of characteristics again. To do the calculation in this case, it is convenient to adopt a frame symmetric with respect to $y=1$. Then, let us define $Y=y-1$. Using the new variable and Eq. (35), we find for Eqs. (22) and (23) the following solution:

$$\begin{aligned} p_S(Y,t) = {} & \int_0^t \frac{\lambda p_S(0,\tau)}{g_z((t-\tau)(1-|Y|)^{z-1})}\,\mathrm{d}\tau \\ & + p_S\left(\left[1-\frac{1-|Y|}{[g_z((1-|Y|)^{z-1}t)]^{1/z}}\right],0\right)\frac{1}{g_z((1-|Y|)^{z-1}t)} \end{aligned} \tag{37}$$

and

$$\begin{aligned} p_A(Y,t) = {} & p_A\left(\mathrm{sign}(Y)\left[1-\frac{1-|Y|}{[g_z((1-|Y|)^{z-1}t)]^{1/z}}\right],0\right) \\ & \times\frac{1}{g((1-|Y|)^{z-1}t)} \,. \end{aligned} \tag{38}$$

Then, the solution consists of two terms: (1) the former is an *even* term and is responsible for the long-time limit of the distribution evolution and (2) the latter is an *odd* term which disappears in the long-time limit. We note that (2) is a desirable property because Eq. (23) does not contain the injection term $C(t)$ and the equilibrium density (20) is an *even* function, independently of the symmetry of the initial distribution.

Let us illustrate the rules that we adopt in this paper to evaluate the equilibrium correlation function $\langle\xi(t_n)\xi(t_{n-1})\cdots\xi(t_1)\rangle$, with $t_n \geqslant t_{n-1} \geqslant \cdots \geqslant t_1 \geqslant 0$. We assume the system to be prepared in the equilibrium state $p_0(Y)$, which is identified with the ket state $|0\rangle$. We apply to it the time evolution operator $\exp(\mathscr{R}t)$, with the operator $\mathscr{R}$ defined by Eq. (18). This does not have any effect, since formally $|0\rangle$ is an eigenstate of $\mathscr{R}$, with vanishing eigenvalue $\alpha=0$. Thus, $\mathscr{R}|0\rangle = \alpha|0\rangle = 0$. At time $t=t_1$ we apply to $|0\rangle$ the sign operator, which yields 1 if $Y<0$ and -1 if $Y>0$. The eigenvalues of this operator coincide with the values of the stochastic fluctuation $\xi(t)$. For the sake of simplicity we assign to this operator the same symbol ξ as the stochastic fluctuation.

The application of ξ to $|0\rangle$ generates an anti-symmetric distribution $p_A(Y)$. Then we let the system evolve from time $t = t_1$ to t_2, with the time evolution obeying Eq. (23), which is formally given by $\exp(\Gamma(t_2 - t_1))p_A(Y)$. At time $t = t_2$ we apply to this distribution the operator ξ again, with the effect of changing it in the symmetric distribution $p_S(Y)$. At this stage we let this distribution evolve in time, according to Eq. (14), namely, as $\exp(\mathscr{R}(t_3 - t_2))p_S(Y)$, and so on. The calculation is concluded with an average over the equilibrium distribution $p_0(Y)$.

In the ordinary quantum mechanical case where the dynamic operators are super-operator associated to Hermitian Hamiltonian, these rules yield results equivalent to the ordinary prescription, with the observable motion expressed by means of the Heisenberg representation.

As pointed out in Section 2, according to these rules the two-time correlation function is determined by the anti-symmetric part of the probability density alone, and its explicit expression becomes:

$$\langle \xi(t)\xi(0)\rangle = (2-z)\int_0^1 \frac{1}{(1-\tau)^{z-1}}\bigg|_{\tau=1-(1-Y)/[g_z((1-Y)^{z-1}t)]^{1/z}} \frac{1}{g_z((1-Y)^{z-1}t)}\,\mathrm{d}y \tag{39}$$

The integral (39) is exactly solvable and leads to the expression

$$\langle \xi(t)\xi(0)\rangle = (1+\lambda(z-1)t)^{-(2-z)/(z-1)}\ , \tag{40}$$

which is the same result as that found in Section 3, using trajectories rather than the probability densities. In a similar way, it is possible to calculate the correlation $\langle Y(t)Y(0)\rangle$ and determine that its temporal behavior is an inverse power law with the same exponent as that in Eq. (40).

5. Onsager regression to equilibrium

In conclusion, in the two preceding sections we have established that the Liouville-like representation of Eq. (18) yields, as expected, the equilibrium correlation function of Eq. (11) with

$$T \equiv \frac{\mu-1}{\lambda}\ . \tag{41}$$

What about Onsager's regression to equilibrium? We have to create an out-of-equilibrium condition and we have to assess under which condition the regression to equilibrium fits the prescription of Eq. (4). This discussion will shed light into the quantum-like formalism used in Section 4 to evaluate the equilibrium correlation function $\Phi_\xi(t)$. On the other hand, the formal approach of Section 4 will make it possible for us to prove that the Onsager regression is difficult to realize in practice.

We have seen that the time evolution of the correlation function $\Phi_\xi(t)$ is determined only by the operator Γ responsible for the time evolution of the anti-symmetric part of the distribution density. This means that the process of back injection taken into account by $C(t)$, see Eq. (22), is not activated. From a careful consideration of Eq. (24) the reader can convince him/herself that the time evolution of $p_A(y,t)$, with $y<1$ is disconnected from the time evolution of $p_A(y,t)$, with $y>1$. This is an important fact that explains why there is no need to ensure for the anti-symmetric part the same continuity conditions as those made necessary to assign a physical meaning to the symmetric part. In conclusion, we are allowed to replace the formal picture of Eqs. (23) and (24) with a reduced picture only involving the region $[0,1]$,

$$\frac{\partial}{\partial t} p_{red}(y,t) = \Gamma_{red} p_{red}(y,t) \; , \tag{42}$$

where

$$\Gamma_{red} \equiv -\lambda \frac{\partial}{\partial y} y^z \; . \tag{43}$$

This corresponds to the time evolution of trajectories moving from a given initial position $y<1$ up to $y=1$, where the particle is instantaneously absorbed.

To shed light into the somewhat disconcerting negative probability distribution, concerning the right part of the anti-symmetric probability distribution, we can imagine the particles on the right as being holes. The holes move from $y>1$ to the left, toward $y=1$, while the particles move from $y<1$, to the right, toward $y=1$, again. Particles and holes annihilate one another at $y=1$. This particle–hole representation only refers to the anti-symmetric part of the probability distribution. In this case, by definition, the number of particles is identical to the number of holes, so making the conservation of the number of particles depend only on Eq. (22). Eq. (22) ensures the conservation of the number of particles by means of the back injection process, which assigns to the right and to the left the same number of particles per unit of time. The total number of particles and holes decreases due to the mutual annihilation process, thereby yielding the relaxation of the mean value of ξ, $\langle\xi\rangle$. This variable gets the value 1 for a particle, and the value -1 for a hole. Thus, $\langle\xi\rangle$ is proportional to the total number of particles and holes, thereby undergoing a relaxation process. In other words, rather than introducing the disturbing concept of negative probability distribution for the anti-symmetric part, we introduce the notion of hole. Both particles and holes are characterized by positive probability distributions, with the particles moving from the left to the right and the holes in the opposite direction. All this is formal, and serves the purpose of clarifying the twofold nature of the equilibrium correlation function, an equilibrium property taking care of an out-of-equilibrium process such as the regression to equilibrium subsequent to a perturbation.

Let us first of all discuss a physical condition where the prescription of Eq. (4) is fulfilled. Let us consider the initial distribution $p(y,0)$ defined as follows:

$$p(y,0) = 2p_0(y), \quad y<1 \; , \tag{44}$$

where $p_0(y)$ denotes the equilibrium distribution of Eq. (20)

$$p(y,0) = 0, \quad y>1 \; . \tag{45}$$

Note that the factor of 2 is made necessary by the fact that we are moving the particles located on the right region, at $2-y$, to the left, at y, with $y<1$. This has the effect of creating an asymmetric initial condition, where the number of particles is not compensated by the number of holes. The anti-symmetric part of this out-of-equilibrium initial condition, where the number of particles is identical to the number of holes, coincides with the anti-symmetric part of the equilibrium distribution. With this merely formal trick we make easier the problem of evaluating $\langle \xi \rangle$, which is equal to the population difference between the left and the right state, denoted by us as $\Delta P(t)$. It is evident that

$$\Delta P(t) = \int_0^1 \mathrm{d}y p(y,t) , \tag{46}$$

where $p(y,t)$ is driven by the reduced picture of Eq. (43). It is also evident that

$$\Delta P(t) = \int_0^1 \exp(\Gamma_{red} t) p(y,0)\,\mathrm{d}y = 2\int_0^1 \exp(\Gamma t) p_A^{(eq)}(y,0)\,\mathrm{d}y , \tag{47}$$

where $p_A^{(eq)}(y,0)$ denotes the left portion of the anti-symmetric distribution corresponding to the equilibrium distribution. On the basis of the results of Section 4, we know that $\int_0^1 \exp(\Gamma t) p_A^{(eq)}(y,0)\,\mathrm{d}y$ is the equilibrium correlation function. On the other hand, as pointed out earlier, $\Delta P(t) = \langle \xi \rangle$. Of course, this means that $y<1$ corresponds to $\xi = 1$ and $y>1$ to $\xi = -1$. Thus we conclude that in this case the Onsager condition of Eq. (4) is exactly fulfilled, with $K = 2$.

What about the regression to equilibrium in general? For any initial distribution $p(y,0)$ we adopt the same approach as the one earlier used. We create a new initial distribution, with the right part empty, such that the anti-symmetric portion of the new out-of-equilibrium distribution is identical to the anti-symmetric portion of the original distribution. Thus, we prove that

$$\Delta P(t) = 2\int_0^1 \exp(\Gamma t) p_A^{(noneq)}(y,0)\,\mathrm{d}y . \tag{48}$$

Note that with $p_A^{(noneq)}(y,0)$ we denote the left portion of the anti-symmetric part of the initial distribution $p(y,0)$ when this is not the equilibrium condition. How to realize this condition in such a way as to fulfill the Onsager principle? A possible answer can be found in Ref. [7]. The authors of this paper adopted a deterministic intermittent map, of which our dynamic model of Section 2 is an idealization. The geometric perturbation prescription adopted by these authors is equivalent to imagine that at a time $t_a<0$ the parameter λ of left portion is abruptly turned into $\lambda' \neq \lambda$. If $t_a = -\infty$, the system has enough time to regress to the corresponding equilibrium. In this equilibrium condition, the left and right distribution have the same analytical form, but different statistical weight. This can be described by

$$p_y = (2-z)\left[p_L \frac{\Theta(1-y)}{y^{z-1}} + p_R \frac{\Theta(y-1)}{(2-y)^{z-1}} \right] , \tag{49}$$

with

$$p_R + p_L = 1 \tag{50}$$

and

$$p_R \neq p_L \ . \tag{51}$$

It is evident that this special initial condition fulfills the Onsager principle. In fact in this case

$$p_A^{(noneq)}(y,0) = p_0(y) \ . \tag{52}$$

However, for this property to hold true we have to assume that the perturbed condition applies from $t = t_a = -\infty$ to $t = 0$. What about making $t_a > \infty$ and closer and closer to $t = 0$? We argue that this yields the breakdown of the Onsager principle. Note that

$$\frac{\mathrm{d}\Delta P(t)}{\mathrm{d}t} = 2\int_0^1 \Gamma_{red} \exp(\Gamma_{red} t) p_{red}(y,0)\, \mathrm{d}y = p_{red}(1,t) \ . \tag{53}$$

This suggests that the relaxation time corresponds to the time of transit from $y<1$ to $y=1$, where the particle is instantaneously annihilated. On the other hand, an averaging process must be carried out, which has to take into account the distribution of particles at $t=0$. We think that making t_a finite has the effect of rendering incomplete the evolution toward the equilibrium distribution in the presence of perturbation. If $p_{red}(y,0) = p_0(y)$, the regression of $p_{red}(1,t)$ is proportional to the equilibrium correlation function. However, if the regression to equilibrium in the presence of perturbation is incomplete, the time evolution toward equilibrium must be evaluated in a quite different way. We assume that in the regions close to the border $y=0$, the initial distribution deviates from the equilibrium prescription of being proportional to $1/y^{z-1}$. More precisely, we assume that for $\varepsilon < y < 1$, the distribution $p(y,t)$ is proportional to $1/y^{z-1}$, whereas for $y \leqslant \varepsilon$ the distribution might or might not depart from equilibrium. In the case where it does, the distribution can be, for instance, independent of y. In the ideal case $t_a = -\infty$, $\varepsilon = 0$, with no breakdown of the Onsager principle.

To study the breakdown of this principle, we evaluate the asymptotic behavior of $\Delta P(t)$. We select a set of initial conditions $y(0)$'s with $0 \leqslant y(0) \leqslant \varepsilon$. We call M the number of particles located in this region at $t=0$. The asymptotic behavior of $\Delta P(t)$ is determined by the time necessary for these trajectories to reach the border $y=1$. The first trajectory will reach the border after a given time $t = T_{first}$, after which the number M will begin decreasing, thereby determining the decay of $\Delta P(t)$ in this time-asymptotic region. It is straightforward to prove that the time of arrival at $y=1$ of the trajectory with initial condition $y(0)$, called t, is related to $y(0)$ by

$$y(0;t) = \frac{1}{[1+(z-1)\lambda t]^{1/(z-1)}} \ . \tag{54}$$

Thus we obtain

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}y(0;t)}{\mathrm{d}t} p(y(0;t)) \ . \tag{55}$$

First of all, let us show that this procedure recovers the result of the earlier two sections. Let us assume that $p(y(0;t))$ is always proportional to $1/y(0;t)^{z-1}$. In this case we obtain

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{1}{[1+(z-1)\lambda t]^{\mu-1}} . \tag{56}$$

Then we evaluate $M(t)$ from

$$M(t) = 1 + \int_0^t \mathrm{d}t' \frac{\mathrm{d}M}{\mathrm{d}t'} . \tag{57}$$

By using Eq. (57) we recover the Onsager principle, with $M(t) \propto$ to $1/t^{\mu-2}$. In the case where $p(y(0;t))$ is independent of $y(0;t)$, the same calculation procedure yields $M(t) \propto 1/t^{\mu-1}$, which sanctions the breakdown of the Onsager principle.

6. Higher-order correlation functions

We now show that the calculation of higher order correlation functions, though difficult, can be done by establishing an even deeper connection with the quantum mechanical perspective. Let us address the problem of evaluating the fourth-order correlation function $\langle \xi(t_4)\xi(t_3)\xi(t_2)\xi(t_1)\rangle$. As shown in Section 4, we move from the equilibrium distribution and let it evolve for a time t_1. The distribution selected is in equilibrium. Therefore it will remain unchanged. At time $t = t_1$ we apply the operator ξ to the distribution. Since the equilibrium distribution is symmetric, the application of the sign operator changes it into the anti-symmetric distribution. We let this distribution evolve for the time $t_2 - t_1$. This has the effect of yielding

$$\Phi_\xi(t_2 - t_1)p_A^{(eq)}(y,0) + p_A^{(noneq)}(y,t_2 - t_1) .$$

At this stage there are two possibilities:

(a) $p_A^{(noneq)}(y,t_2 - t_1) = 0$;
(b) $p_A^{(noneq)}(y,t_2 - t_1) \neq 0$.

Let us consider case (a) first. In this case, we proceed as follows. At time t_2 we apply to the distribution the operator ξ, and we change it into the original equilibrium distribution. This means that the time evolution from t_2 to t_3 leaves it unchanged. At time t_3 we apply to it the operator ξ and we turn it into $p_A^{(eq)}$ again. We let this distribution evolve till time t_4. At this time we apply to it the operator ξ again and make the final average. The result of condition (a) yields

$$\langle \xi(t_4)\xi(t_3)\xi(t_2)\xi(t_1)\rangle = \langle \xi(t_4)\xi(t_3)\rangle\langle \xi(t_2)\xi(t_1)\rangle . \tag{58}$$

However, condition (a) applies only when $z = 1$. In this case it is straightforward to prove that the equilibrium distribution is flat, the anti-symmetric component consists of two flat components, positive, $y > 1$, and negative, $y < 1$, which are left unchanged by the application of the operator Γ of Eq. (25).

In the case $z > 1$ condition (a) is violated and with it the factorization condition of Eq. (58). This agrees with the recent work (21), where an explicit expression for the fourth-order correlation function in terms of $\Phi_\xi(t)$ was given by means of an approximated approach that nevertheless leads to an accurate reproduction of numerical results. The demonstration was made by applying the method of conditional probabilities to the study of single trajectories. Thus we are forced to consider case (b).

The problem with condition (b) is that it yields a distribution with the symmetric component departing from equilibrium. This departure is in an apparent conflict with the assumption that the correlation function is calculated using the equilibrium condition. In fact it seems to be equivalent to stating that the calculation of an equilibrium correlation function generates an out-of-equilibrium condition. Let us see why this appears to be true.

In the case of the two-time correlation function we apply the operator ξ to the density distribution twice. The first application allows us to observe the time evolution of the anti-symmetric component distribution density, with no conflict with equilibrium, given the fact that the symmetric component remains at equilibrium. The second application of ξ turns

$$\Phi_\xi(t_2 - t_1) p_A^{(eq)}(y, 0) + p_A^{(noneq)}(y, t_2 - t_1)$$

into

$$\Phi_\xi(t_2 - t_1) p_S^{(eq)}(y, 0) + p_S^{(noneq)}(y, t_2 - t_1) .$$

The calculation done in Section 4 proves that $\mathrm{Tr}[p_S^{(noneq)}(y, t_2 - t_1)] = 0$, with the symbol Tr denoting, for simplicity, the integration over y from 0 to 2. To evaluate the fourth-order correlation function, after applying the operator ξ for the second time, we must study the time evolution of $p_S^{(noneq)}(y, t_2 - t_1)$ from t_2 to t_3, yielding $p_S^{(noneq)}(y, t_3, t_2, t_2 - t_1)$. This is a contribution generating some concern, since it activates again the back injection process, which we have seen to be intimately related to the deviation from equilibrium. However, the compatibility with equilibrium condition is ensured by the property $\mathrm{Tr}[p_S^{(noneq)}(y, t_3, t_2, t_2 - t_1)] = 0$.

At time t_3 we have to apply the operator ξ again, and this allows us to make an excursion in the anti-symmetric representation, with the time evolution given by the operator $\exp(\Gamma(t_4 - t_3))$. At time t_4 we apply the operator ξ again, we go back to the symmetric representation and we conclude the calculation by means of trace operation.

In conclusion, the compatibility with equilibrium is guaranteed by the fact that at the intermediate steps of the calculation $\mathrm{Tr}[p] = 0$ (the final step, of course, generates the density-generated correlation function, thereby implying $\mathrm{Tr}[p] \neq 0$). If the intermediate p is anti-symmetric, this condition is obvious. If the intermediate p is symmetric, the vanishing trace condition generates the apparently unphysical property that the symmetric contribution gets negative values over some portions of the interval I. We have to stress, however, that the Liouville-like approach illustrated in this paper keeps the distribution density $p(y, t)$ positive definite, as it must. The generation of a negative distribution density refers to the calculation of equilibrium correlation functions, of any order, and it is a quantum-like property that must be adopted to guarantee that the genuine distribution density remains in the equilibrium condition.

It is worth pointing out that, although these arguments show that it is possible to evaluate the fourth-order correlation function, we could not find its explicit expression, and as a consequence it is not yet possible for us to assess whether or not the explicit result of Ref. [21] can be recovered. Furthermore, the explicit result of Ref. [21] is not exact. Consequently, we have to conclude that the evaluation of the fourth-order correlation function in the non-Poisson case is still an open problem.

7. Comments on the quantum-like formalism

The reader might be disconcerted by the properties of the quantum-like formalism adopted in this paper, and especially by those illustrated in Section 6, such as $\mathrm{Tr}[p] = 0$, implying that also the symmetric part of the probability distribution might become negative, as well as the anti-symmetric part, as discussed in Section 4. Actually, we did not derive these properties from the Liouville-like discussed in this paper. We have been rather forced to make the plausible conjecture that these properties are true by the wish of extending the trajectory-density equivalence from Poisson to non-Poisson statistics.

The authors of Ref. [16] questioned the equivalence between density and trajectory picture on the basis of a generalized diffusion equation (GDE) derived from Eq. (14), thought of as being the generator of diffusion. The GDE of Ref. [16] yields a dramatic conflict with the adoption of the trajectory perspective. The reason for this conflict is that the authors of Ref. [16] adopted condition (a) of Section 6 implicitly. If we adopt condition (b) of Section 6 instead, to avoid the trajectory-density conflict emerging from (a), then we have to accept all the disconcerting properties of the distribution density formalism. In Section 4 we have adopted the particle–hole perspective to justify the anti-symmetric part of the distribution density becoming negative. It is not enough. The correct evaluation of higher-order correlation functions implies that also the symmetric component must be non-positive definite, although in the virtual sense discussed in Section 6.

We think that all this might help to understand, by analogy, the strange and apparently paradoxical properties of quantum mechanics, whose statistical treatments is based on the quantum Liouville equation. The counterpart of this quantum treatment is given here by Eq. (14). To establish a satisfactory connection between Liouville and CTRW formalism, we have to prove the plausible conjectures of Section 6, in spite of their disconcerting nature. The only possible alternative would be to abandon the Liouville approach (and quantum mechanics with it?) and to adopt the CTRW formalism instead.

8. Concluding remarks

In conclusion, this paper shows how to derive the equilibrium correlation function using only information afforded by the Liouville-like approach. The intriguing problem to solve was how to use the Liouville equation, without conflicting with the equilibrium condition. The solution to this intriguing problem is obtained by splitting the Liouville equation into two independent components, the symmetric component corresponding to Eq. (22) and the anti-symmetric component corresponding to Eq. (23). This splitting allows us to study the regression to equilibrium of the correlation function $\Phi_\xi(t)$, through Eq. (23), without ever departing from equilibrium, a physical condition that is controlled by the independent equation of motion for the symmetric component, Eq. (22). This is very formal, and Section 5 makes it possible for us to reconcile it with physical intuition. We imagine that the Liouville equation is used to evaluate the difference between the population of the left and right state. This makes it possible to establish a direct connection between the experiment of regression to equilibrium and the formalism of Sections 2 and 4. We have to create an asymmetrical initial condition, with more population on the left than on the right. The time evolution of $\Delta P(t)$ depends only on the time evolution of the anti-symmetric component of the distribution density, and consequently only on the operator Γ. This is the reason why it is possible, in principle, to connect regression from an out-of-equilibrium initial condition to the equilibrium correlation function, a condition that implies no deviation from equilibrium distribution. However, creating in a finite time an out-of-equilibrium condition such that the left part of the anti-symmetric component is identical to the left part of the equilibrium distribution is impossible. This is the reason why we predict the breakdown of the Onsager principle in general.

Section 6 explains why in the literature on dichotomous fluctuations the factorization assumption of Eq. (58) is often made regardless of the Poisson or non-Poisson nature of the underlying process. See, for instance, the work of Fulinski [25] as well as Ref. [16]. In fact, if condition (a) applies, the higher-order correlation functions are factorized, thereby making their calculation easy. However, this assumption conflicts with the trajectory arguments of Ref. [21], which prove that the factorization condition is violated by the non-Poisson condition. A rigorous use of the Liouville equation shows that condition (a) does not apply, and that we have to use condition (b) instead. The calculation of the fourth-order correlation function is not straightforward, and this is the reason why, to the best of our knowledge, it was never done using the Liouville approach. Furthermore, as earlier pointed out in Section 7, the rules discussed in Section 6 are conjectured rather than proved, and correspond to the mathematical properties to guarantee the density-trajectory equivalence.

This is a fact of some relevance for the creation of master equations with memory. There are two major classes of GME. The first class is discussed, for instance, in Ref. [1]. The master equations of this class are equivalent to the CTRW of Montroll and Weiss [3] and are based on the waiting time distribution $\psi(t)$. The second class of master equations is based on the correlation function $\Phi_\xi(t)$. Recent examples of this second class can be found in Ref. [16] and in Ref. [26]. Due to the direct dependence on the correlation function $\Phi_{\xi}(t)$, the derivation of the master equations of this second class is made easy by the factorization assumption. It must be pointed out, on the other hand, that the factorization property, which is not legitimate with renewal non-Poisson processes, is a correct property if the deviation from the exponential relaxation is obtained by time modulation of a Poisson process [26]. Beck [27] is the advocate of the modulation process as generator of complexity. Thus, we find that the master equations of the first class are generated by the renewal perspective of Montroll and Weiss [3] and those of the second class are the appropriate tool to study complexity along the lines advocated by Beck [27]. We think that the results of the present paper might help the investigators in the field of complexity to make the proper choice, either modulation or renewal [28], or a mixture of the two conditions.

As a final remark, let us go back to the issue raised by Ref. [16] about the trajectory-density conflict. On the basis of the results of this paper we can conclude that, at first sight, this conflict can be judged to be apparent, insofar as the conclusions of the authors of this paper were based on the assumption that the Liouville-like approach yields the factorization property of Eq. (58), which is correct only in the Poisson case. However, it is not yet possible to express a final verdict on this important issue. This is so because the trajectory method [21] does not yield yet an exact expression for the fourth-order correlation function, and using the arguments of Section 6 we could not arrive yet at any expression, either exact or approximated, from the density perspective. In Section 6 we have seen that the calculation of the fourth-order correlation function activates the contribution $C(t)$, which is responsible for an aging effect [1]. On the other hand the surprising effects [28] discovered by Sokolov, Blumen and Klafter [12] indicate that the non-Poisson property-induced aging effect might lead us to adopt with caution the Liouville-like approach. Similar conclusions have been recently reached by Aquino and coworkers [30] who have been forced to study the absorption process of radiation by a non-Poisson system with a method entirely based on the CTRW, given the fact that the ordinary Liouville-like approach cannot properly take into account the aging effect. In conclusion, we cannot rule out the possibility that the issue of the trajectory-density conflict raised by Ref. [16] might signal the symptoms of a disease that can only be cured by moving from the Liouville-like to the CTRW perspective.

Acknowledgements

PG thankfully acknowledges the Army Research Office for the financial support through Grant DAAD19-02-0037.

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[28] In the case where the renewal condition applies, we want to point out that, as shown in (1), the GME can be derived from the CTRW [3]. It is still unknown how to do that using the Liouville-like approach of Section 2, probably due to the complexity of the problem illustrated in Section 6. Furthermore, the authors of Ref. [12] made the following impressive discovery. In the non-Poisson case even if a correct GME is available [29], its response to external perturbations departs from the correct prediction that one can obtain by perturbing the CTRW instead. Thus the trajectory-density conflict [16] seems to be the manifestation of technical problems, emerging from the density perspective, which can be settled using the trajectory (CTRW) perspective.
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