Helical Equilibria

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Contents

1 Introduction

Helical structures can be found nearly anywhere : Helices arise in nanosprings, vines, DNA, etc. These structures held great interest for scientists, some part of it for the equilibrium state.

Therefore, after a physical introduction, we will study the helical equilibria, here in the absence of body forces. We will particulary study the strains and how to caracterize them, mathematically first, then more physically. Finally, we will discuss the C++ implementation of our computations which leads to Mr. De Souza's program.

2 Physical origin

We will introduce here the physical elements leading to our problem.

2.1 Kinematics

A rod is defined by a centerline $\mathbf{r}(s)$ and two orthonormal vectors $\{\mathbf{d_1}, \mathbf{d_2}\}(s)$, with $s \in [-L, L]$ the arc length.
This gives an orthonormal framing $\{\mathbf{d_1}, \mathbf{d_2}, \mathbf{d_3}\}(s)$, and the condition with the constraint

$$\mathbf{r}' = \mathbf{d_3}$$

for an **unextensible and unshearable rod**.
We get the relations

$$\mathbf{d_i}' = \mathbf{u} \times \mathbf{d_i}$$

The components of $\mathbf{u}$ represents the **strains** on the rod.
On the other hand, we can consider the Frenet frame $\{\boldsymbol{\nu}, \boldsymbol{\beta}, \boldsymbol{\tau}\}$.
One can obtain the **Frenet-Serret equations** :

$$\begin{array}{rcl} \boldsymbol{\tau}' & = & \kappa \boldsymbol{\nu} \\ \boldsymbol{\beta}' & = & -\kappa \boldsymbol{\tau} + \tau \boldsymbol{\beta} \\ \boldsymbol{\nu}' & = & -\tau \boldsymbol{\nu} \end{array} \tag{2.1}$$

Since $\boldsymbol{\tau}$ is the tangent, we have $\boldsymbol{\tau} = \mathbf{d_3}$, and then $(\mathbf{d_1}, \mathbf{d_2})$ is a rotation of $(\boldsymbol{\nu}, \boldsymbol{\beta})$ through an angle ϕ.
One can obtain the following relations (ref.4):

$$\left\{ \begin{array}{rcl} u_1 & = & \boldsymbol{\kappa} \sin \phi \\ u_2 & = & \boldsymbol{\kappa} \cos \phi \\ u_3 & = & \boldsymbol{\tau} + \phi' \end{array} \right.$$

If the curvature κ and the torsion τ are constants, we obtain an helix which can be given in arc-length parametric form :

$$\mathbf{r}(s) = (r \cos(\eta_1 s), r \sin(\eta_1 s), p \eta_1 s)$$

in a basis, with r the radius and $2\pi p$ the pitch of the helix.
The relations are given by :

$$r = \frac{\kappa}{\kappa^2 + \tau^2},\ p = \frac{\tau}{\kappa^2 + \tau^2},\ \eta_1 = \frac{1}{\sqrt{r^2 + p^2}}$$

2.2 Mechanics

One can represent the stresses across the rod at $\mathbf{r}(s)$ as resultant force $\mathbf{n}(s)$ and moment $\mathbf{m}(s)$.

We will study the equilibria without any body forces, so the equilibrium relations gives

$$\begin{aligned} \mathbf{n}' &= \mathbf{0} \\ \mathbf{m}' + \mathbf{r}' \times \mathbf{n} &= \mathbf{0} \end{aligned} \tag{2.2}$$

Since we assume that the rod is hyperelastic, there exists a convex, coercive strain-energy density function WMoreover, we assume the rod is uniform, so we have no explicit dependance on s.We will particulary study the case where we can approximate W by a quadratic function of the strains.
Then W can be written as

$$W(u - \hat{u}) = \frac{1}{2}(u - \hat{u}) \cdot K(u - \hat{u}) \tag{2.3}$$

Where K is a symmetric positive definite which is assumed of the form

$$K = \begin{pmatrix} K_1 & 0 & K_{13} \\ 0 & K_2 & K_{23} \\ K_{13} & K_{23} & K_3 \end{pmatrix}$$

With the condition $K_1 \leq K_2$, and $\hat{\mathbf{u}}$ is the strain at the minimum energy.
Consequently, the expression of W is :

$$\begin{aligned} W = \frac{1}{2} & \left(K_1(u_1 - \hat{u}_1)^2 + K_2(u_2 - \hat{u}_2)^2 + K_3(u_3 - \hat{u}_3)^2 \right) \\ & + \left(K_{13}(u_1 - \hat{u}_1) + K_{23}(u_2 - \hat{u}_2) \right)(u_3 - \hat{u}_3) \end{aligned}$$

We have

$$m = \nabla W(u - \hat{u}) = K(u - \hat{u})$$

Then

$$\nabla W(u - \hat{u}) = \begin{pmatrix} K_1(u_1 - \hat{u}_1) + K_{13}(u_3 - \hat{u}_3) \\ K_2(u_2 - \hat{u}_2) + K_{23}(u_3 - \hat{u}_3) \\ K_{13}(u_1 - \hat{u}_1) + K_{23}(u_2 - \hat{u}_2) + K_3(u_3 - \hat{u}_3) \end{pmatrix}$$

2.3 Equilibrium

The equilibrium points minimizes W under the constraints

$$\frac{1}{2}\mathbf{u} \cdot \mathbf{u} = \frac{1}{2}\eta_1^2, \; u_3 = \eta_1 \eta_2$$

We can notice that these two strains are actually just a one-dimension strain, a circle

$$u_1^2 + u_2^2 = \eta_1^2(1 - \eta_2^2)$$

that has for tangent

$$\xi = (u_2, -u_1, 0)$$

Thus, minimizing W over the circle can be done by finding the solutions of

$$\xi \cdot \nabla W = 0$$

So we have the equation

$$u_2(K_1(u_1 - \hat{u}_1) + K_{13}(u_3 - \hat{u}_3)) - u_1(K_2(u_2 - \hat{u}_2) + K_{23}(u_3 - \hat{u}_3)) = 0$$

Which can be reorganized as

$$\begin{aligned} u_1 u_2(K_2 - K_1) + u_3(u_1 K_{23} - u_2 K_{13}) \\ -(K_2\hat{u}_2 + K_{23}\hat{u}_3)u_1 + (K_1\hat{u}_1 + K_{13}\hat{u}_3)u_2 = 0 \end{aligned} \tag{2.4}$$

This is the equation we are going to work with.

3 Hyperboloids

We can rewrite equation 2.4 a bit more simply :

$$u_1 u_2 - (a + bu_3)u_2 - (c + du_3)u_1 = 0 \tag{3.1}$$

or

$$(u_1 - (a + bu_3))(u_2 - (c + du_3) = (a + bu_3)(c + du_3) \tag{3.2}$$

where

$$a = -\frac{K_1\hat{u}_1 + K_{13}\hat{u}_3}{K_2 - K_1},\ c = \frac{K_2\hat{u}_2 + K_{23}\hat{u}_3}{K_2 - K_1}$$

$$b = \frac{K_{13}}{K_2 - K_1},\ d = -\frac{K_{23}}{K_2 - K_1}$$

These formulas imply that $K_1 \neq K_2$. We will consider that case later.

3.1 Standard Case

Before interesting ourselves to the general case, let's review the standard tools which will be used later on.

The standard Hyperboloid can be represented as :

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = 1$$

Where $x,\ y,\ z \in \mathbb{R} \setminus \{0\}$.

3.1.1 Usual Parametrization

A way of efficiently parametrize a mesh for our Standard Hyperboloid is :

$$\left\{ \begin{array}{rcl} x & = & A\sqrt{1+r^2}\cos\theta \\ y & = & B\sqrt{1+r^2}\sin\theta \\ z & = & C\,r \end{array} \right. \tag{3.3}$$

With $(\theta, r) \in [0, 2\pi] \times [-H, H]$, linearly variated, where $H > 0$ is the desired height display.

3.1.2 Rulings

We can notice that, passing $\frac{y^2}{B^2}$ on the other side of the equation, we have two identities which can be factorized as such :

$$\left(\frac{x}{A} - \frac{z}{C}\right)\left(\frac{x}{A} + \frac{z}{C}\right) = \left(1 - \frac{y}{B}\right)\left(1 + \frac{y}{B}\right)$$

So then we have two systems of two equations :

$$\left\{ \begin{array}{rcl} \frac{x}{A} - \frac{z}{C} & = & k\left(1 - \frac{y}{B}\right) \\ \frac{x}{A} + \frac{z}{C} & = & \frac{1}{k}\left(1 + \frac{y}{B}\right) \end{array} \right. \tag{3.4}$$

$$\left\{ \begin{array}{rcl} \frac{x}{A} - \frac{z}{C} & = & h\left(1 + \frac{y}{B}\right) \\ \frac{x}{A} + \frac{z}{C} & = & \frac{1}{h}\left(1 - \frac{y}{B}\right) \end{array} \right. \tag{3.5}$$

For every h, $k \in \mathbb{R}$.
Each of these two systems define a line in the space along the Hyperboloid. So by moving h and k we can cover the entire Hyperboloid with lines.

3.1.3 Explicit rulings

We can arrange these equations to obtain a parametrization in function of h and k.
The computations provide :

$$\left\{ \begin{array}{rcl} x & = & \frac{2-\frac{z}{C}\left(\frac{1}{k}-k\right)}{k+\frac{1}{k}}\,A \\ y & = & \frac{2\frac{z}{C}+k-\frac{1}{k}}{k+\frac{1}{k}}\,B \end{array} \right.$$

$$\left\{ \begin{array}{rcl} x & = & \frac{2+\frac{z}{C}\left(\frac{1}{h}-h\right)}{h+\frac{1}{h}}\,A \\ y & = & \frac{\frac{1}{h}-h-2\frac{z}{C}}{h+\frac{1}{h}}\,B \end{array} \right.$$

So we have two parametric equations of lines (with the parameter z) :

$$\left\{ \begin{array}{rcl} l_k(z) & = & \left(\frac{2-\frac{z}{C}\left(\frac{1}{k}-k\right)}{k+\frac{1}{k}}\,A, \frac{2\frac{z}{c}+k-\frac{1}{k}}{k+\frac{1}{k}}\,B, z\right) \\ l_h(z) & = & \left(\frac{2+\frac{z}{C}\left(\frac{1}{h}-h\right)}{h+\frac{1}{h}}\,A, \frac{\frac{1}{h}-h-2\frac{z}{C}}{h+\frac{1}{h}}\,B, z\right) \end{array} \right. \tag{3.6}$$

Note. We can express (x, y, z) in function of k and h directly with the formulas

$$x = A\frac{h-k}{h+k}, y = B\frac{hk+1}{h+k}, z = C\frac{hk-1}{h+k} \tag{3.7}$$

So we can plot the Hyperboloid using some k and h. However, we will lose the regular mesh we will obtain the other way, i.e the drawing with nice ellipses for $z = const$.

Once we have these equations, only one problem remains : we need to find good values for h and k so we have a regular figure, i.e. lines correctly positioned.

3.1.4 Regular k and h

We have now the equations of the lines. To efficiently nicely plot these lines, we will consider that we only display them within a z range of the form $[-H, H]$. Then, its is enough to compute the two extremities of the lines, i.e the points where $z = \pm H$.
In order to do that, we interest ourselves on these two positions for z, which corresponds for each to an ellipse :

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = r^2$$

with

$$(x, y, z) \in l_h, l_k$$

assuming

$$r := \sqrt{1 + \frac{z^2}{C^2}}$$

Since we search a regular repartition, we do search (x, y) in function of θ

$$\begin{cases} x &= A\, r\, \cos\theta \\ y &= B\, r\, \sin\theta \end{cases}$$

Which, with θ linearly partitioned, will give a nice set of $k(\theta)$ and $h(\theta)$.
We have to solve :

$$\begin{cases} r\, \cos\theta &= \frac{2-\frac{z}{C}\left(\frac{1}{k}-k\right)}{k+\frac{1}{k}} \\ r\, \sin\theta &= \frac{2\frac{z}{C}+k-\frac{1}{k}}{k+\frac{1}{k}} \end{cases}$$

$$\begin{cases} r\, \cos\theta &= \frac{2-\frac{z}{C}\left(\frac{1}{h}-h\right)}{h+\frac{1}{h}} \\ r\, \sin\theta &= \frac{\frac{1}{h}-h-2\frac{z}{C}}{h+\frac{1}{h}} \end{cases}$$

Let's concentrate on the $r\, \cos\theta$ expressions.
By developping the terms and multipling by k, h respectively we get a 2^{nd} degree polynomial for h and k :

$$\begin{cases} k^2(r\, \cos\theta - \frac{z}{C}) - 2k + (r\, \cos\theta + \frac{H}{C}) &= 0 \\ h^2(r\, \cos\theta + \frac{z}{C}) - 2h + (r\, \cos\theta - \frac{H}{C}) &= 0 \end{cases}$$

Both these equations have determinant $\sqrt{\Delta} = 2r \sin\theta$ and then we can choose k and h such that they correspond to the $r\, \sin\theta$ conditions :

$$\begin{cases} k(\theta) &= \frac{1+r\sin\theta}{r\cos\theta+z/C} \\ h(\theta) &= \frac{1+r\sin\theta}{r\cos\theta-z/C} \end{cases} \tag{3.8}$$

Then moving θ linearily :

$$\theta = \frac{2i\pi}{n},\ n \in \mathbb{N}, i \in \{0, ..., n-1\}$$

We have the regular rulings.

3.2 Transformation

A general quadratic function of $x \in R^3$

$$x \cdot Ax + b \cdot x + c = 1 \tag{3.9}$$

can be transformed into the canonical hyperboloid form via a transformation

$$y = Px + w$$

where P is orthogonal, provided only that the eigenvalues of A are two positive and one negative (or vice versa).
To work out this transformation, we need to compute the eigenvectors of A, which is theoretically possible, but in this case gives intractable expressions in terms of the original problem parameters.
There may be a simpler transformation, still of the form

$$y = Cx + v$$

but with C no longer orthogonal, but with much simpler coefficients that still transforms our hyperboloid into a standard form.
We will discuss this method in the following.

3.3 Parametrization

First consider the generic cases $bd \neq 0$ and $ac \neq 0$.

3.3.1 Transformations

Now that we have the rulings for the standard case, we would like to have the same kind of rulings for our Hyperboloid, which equation can be written as :

$$(u_1 - (a + bu_3))(u_2 - (c + du_3) = (a + bu_3)(c + du_3)$$

We now search a transformation which will put this equation into a diagonalized form.
Concretely, we search a invertible matrix C such that

$$x = Cu + w\ ,\ \Lambda(x) = 1$$

with

$$\Lambda(v_1, v_2, v_3) = \lambda_1 v_1^2 + \lambda_2 v_2^2 + \lambda_3 v_3^2$$

and precisely one lambda negative.

How to do it

We can easily see that, given :

$$\begin{cases} x &= u_1 - (a + bu_3) \\ y &= u_2 - (c + du_3) \end{cases}$$

We get the equation

$$(u_1 - x)(u_2 - y) = xy = \frac{1}{4}(x^2 - y^2)$$

On the other hand :

$$\begin{aligned} (u_1 - x)(u_2 - y) &= (a + bu_3)(c + du_3) = bdu_3^2 + (ad + bc)u_3 + ac \\ &= bd\left(\left(u_3 + \tfrac{bc+ad}{2bd}\right)^2 + ac - \tfrac{(bc+ad)^2}{4bd}\right) \end{aligned}$$

So we have the expression

$$\frac{1}{4}v_1^2 - \frac{1}{4}v_2^2 - bdv_3^2 + \frac{(bc+ad)^2}{4bd} - ac = Q_0(u) = 0$$

Providing

$$v = Cu + w, C = \begin{pmatrix} 1 & 1 & -d-b \\ 1 & -1 & d-b \\ 0 & 0 & 1 \end{pmatrix}$$

$$w = \begin{pmatrix} -c-a \\ c-a \\ \frac{bc+ad}{2bd} \end{pmatrix}, c_0 = ac - \frac{(bc+ad)^2}{4bd}$$

With

$$C^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & b \\ \frac{1}{2} & -\frac{1}{2} & d \\ 0 & 0 & 1 \end{pmatrix}$$

Thus we have the desired form!

$$\Lambda_0(Cu + w) - c_0 = 0$$

Which also can be written as

$$\Lambda(Cu + w) = 1$$

With

$$\Lambda = \Lambda_0 / c_0$$

i.e

$$\Lambda(a_1, a_2, a_3) = \frac{1}{4c_0}a_1^2 - \frac{1}{4c_0}a_2^2 - \frac{bd}{c_0}a_3^2 = 1$$

This is the equation of a standard Hyperboloid.

3.3.2 Considerations on P

We just found that

$$P = \begin{pmatrix} 1 & 1 & -d-b \\ 1 & -1 & d-b \\ 0 & 0 & 1 \end{pmatrix} \tag{3.10}$$

is the transformation between the original form and the standardized form.
What does this transformation corresponds to?
We can see that

$$\begin{pmatrix} 1 & 1 & -d-b \\ 1 & -1 & d-b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & -\cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -d \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

So we can decompose P in three transformations :

1. planar dilation in the (x, y) plane
2. rotation, still in the (x, y) plane
3. a shear parallel to the (x, y) plane

3.3.3 Conditions on the parameters

Observation

Note that

$$c_0 = ac - \frac{(bc+ad)^2}{4bd} = -\frac{(bc-ad)^2}{4bd}$$

Remark that the equation is a **one-sheeted Hyperboloid** if and only if **only one of the three components of Λ is negative**.
If we look at the product bd, we can see with our observation that c_0 and bd have opposite sign, which guarantees always to have a well-defined one-sheeted Hyperboloid :

- if $bd > 0$, then the only possibility is that λ_1 only would be negative, that we have with $c_0 < 0$.
- if $bd < 0$, then the only possibility is that λ_2 only would be negative, that we have with $c_0 > 0$.

The problem only remains with $c_0 = 0$ which leads to $bc - ad = 0$.
Note that our equation still can be written as :

$$\frac{1}{4}v_1^2 - \frac{1}{4}v_2^2 - bdv_3^2 = 0$$

Since the transformations we did are always valuable, because C is invertible $\forall b, d \in \mathbb{R}$ assuming $bd \neq 0$, we do not obtain here a one-sheeted Hyperboloid.

This is the **equation of a cone**, i.e a limit of a one-sheeted hyperboloid infinitely distended.
This remarks allows us to verify that the originial form, this one being an one-sheeted hyperboloid, is also one, since C preserves the signs of the eigenvalues.

3.4 Problematic situations

We have a good idea of the global form of the equation. Three situations now remain.

3.4.1 Case $ac = 0$, $bd \neq 0$

These two conditions provide a diagonal matrix

$$K = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}$$

Then we can see that

$$W(u - \hat{u}) = \frac{K_1}{2}(u_1 - \hat{u}_1)^2 + \frac{K_2}{2}(u_2 - \hat{u}_2)^2 + \frac{K_3}{2}(u_3 - \hat{u}_3)^2$$

Which leads to

$$\nabla W = (K_1(u_1 - \hat{u}_1), K_2(u_2 - \hat{u}_2), K_3(u_3 - \hat{u}_3))$$

and then the condition

$$\xi \cdot \nabla W = 0$$

becomes

$$u_1 u_2 (K_1 - K_2) + u_1 K_2 \hat{u}_2 - u_2 K_1 \hat{u}_1 = 0$$

Thus we have the equation of an Hyperbola defined for all $u_3 \in \mathbb{R}$, so what we get is a **cylindrical hyperbola**.

However, if $K_1 = K_2 = K$ we obtain :

$$u_1 K \hat{u}_2 = u_2 K \hat{u}_1$$

Which defines a **plane**.

3.4.2 Case $ac \neq 0$, $bd = 0$

This case could be more problematic because then w is badly defined, so we need to go back from the beginning.
But then we easily get :

$$(u_1 - a)(u_2 - c) = ac$$

this is once again a **cylindrical hyperbola**.

3.4.3 Case $ac = 0$, $bd = 0$

Here we have a, b, c, $d = 0$. This means that

$$u_1 u_2 = 0$$

And then the two planes O_{xz} and O_{yz} are the solutions.

3.4.4 Isotropic strain-energy function

We get the relations

$$K_1 - K_2 = K_{23} = K_{13} = 0$$

We have to go back to equation 2.4.
We get

$$K u_2 \hat{u}_1 = K u_1 \hat{u}_2 \tag{3.11}$$

Which defines a plane.

3.5 Conclusion

These analysis can be summarized as :

- $K_1 \neq K_2$
  - $b, d \neq 0$ ($\Longleftrightarrow K_{13}, K_{23} \neq 0$)
    - $a, c \neq 0$
      Standard Situation
    - $a, c = 0$
      Cylindrical Hyperbola
  - $b, d = 0$ ($\Longleftrightarrow K_{13}, K_{23} = 0$)
    - $a, c \neq 0$
      Cylindrical Hyperbola
    - $a, c = 0$
      Two planes
- Isotropic strain-energy function
  Plane

3.6 Rulings

Now that we have a nice description of the general case, let's see how we can apply it to the rulings with lines that we found earlier.
We will name :

- X Space the space where the Hyperboloid is in its standard form
- U Space the original space
- M Space the space of basis m.

3.6.1 X Space rulings

We have two possibilities that can occur : either $\lambda_1 < 0$ or $\lambda_2 < 0$. Let's assume for now that $\lambda_1 < 0$, and we will discuss how similar the other situation is later.
We then have the equation :

$$\frac{x_2^2}{\lambda_2} + \frac{x_3^2}{\lambda_3} = 1 + \frac{x_1^2}{\lambda_1}$$

This is the exact situation of the standard case.
Our previous computations provide :

$$\begin{cases} x_2 &= \frac{2 - \frac{x_1}{\sqrt{\lambda_1}}\left(\frac{1}{k} - k\right)}{k + \frac{1}{k}} \sqrt{\lambda_2} \\ x_3 &= \frac{2\frac{x_1}{\sqrt{\lambda_1}} + k - \frac{1}{k}}{k + \frac{1}{k}} \sqrt{\lambda_3} \end{cases}$$

$$\begin{cases} x_2 &= \frac{2+\frac{x_1}{\sqrt{\lambda_1}}\left(\frac{1}{h}-h\right)}{h+\frac{1}{h}}\sqrt{\lambda_2} \\ x_3 &= \frac{\frac{1}{h}-h-2\frac{x_1}{\sqrt{\lambda_1}}}{h+\frac{1}{h}}\sqrt{\lambda_3} \end{cases}$$

To obtain the other situation, we just need to switch the roles of x_1 and x_2, respectively λ_1 and λ_2.

Note that when these equations are used, it is simply needed to fix x_1 for two different values, say $-H$ and $+H$, so we get two points (x_1, x_2, x_3) which can be connected through a line, which will be in the Hyperboloid, since the variations $x_2(x_1)$ and $x_3(x_1)$ are linear.

3.6.2 *U* Space rulings

There, for practical reasons, we will still consider x_1 fixed in two values.
All we do now is applying the transformation. Since

$$x = Pu + w$$

we then have

$$u = P^{-1}(x - w) = \begin{pmatrix} \frac{x_1+x_2}{2} + a - \frac{bc+ad}{2d} \\ \frac{x_1-x_2}{2} + c - \frac{bc+ad}{2b} \\ x_3 - \frac{bc+ad}{2bd} \end{pmatrix}$$

This leads to

$$u = \begin{pmatrix} \frac{x_1 + \frac{2-\frac{x_1}{\sqrt{\lambda_1}}\left(\frac{1}{k}-k\right)}{k+\frac{1}{k}}\sqrt{\lambda_2}}{2} + a - \frac{bc+ad}{2d} \\ \frac{x_1 - \frac{2-\frac{x_1}{\sqrt{\lambda_1}}\left(\frac{1}{k}-k\right)}{k+\frac{1}{k}}\sqrt{\lambda_2}}{2} + c - \frac{bc+ad}{2b} \\ \frac{2\frac{x_1}{\sqrt{\lambda_1}}+k-\frac{1}{k}}{k+\frac{1}{k}}\sqrt{\lambda_3} - \frac{bc+ad}{2bd} \end{pmatrix}$$

with the h, and

$$u = \begin{pmatrix} \frac{1}{2}\left(x_1 + \frac{2+\frac{x_1}{\sqrt{\lambda_1}}\left(\frac{1}{h}-h\right)}{h+\frac{1}{h}}\sqrt{\lambda_2}\right) + a - \frac{bc+ad}{2d} \\ \frac{1}{2}\left(x_1 - \frac{2+\frac{x_1}{\sqrt{\lambda_1}}\left(\frac{1}{h}-h\right)}{h+\frac{1}{h}}\sqrt{\lambda_2}\right) + c - \frac{bc+ad}{2b} \\ \frac{\frac{1}{h}-h-2\frac{x_1}{\sqrt{\lambda_1}}}{h+\frac{1}{h}}\sqrt{\lambda_3} - \frac{bc+ad}{2bd} \end{pmatrix}$$

with k.

Once again we can invert the roles of x_1 and x_2, respectively λ_1 and λ_2. This transformation being still linear, the rulings also have the nice property that the line defined with two points will stay on the hyperboloid.

3.6.3 M Space rulings

The transformation is

$$m = K(u - \hat{u})$$

We apply this transformation to the equations in the previous case.

4 Informatic implementation in C++

Now that we have mathematical relations, let's see how we can add them to a program.

4.1 Tools

The C++ being a very basic language, we decided to use some libraries to gain some time on the displaying.
The two libraries are :

- Coin3D (OpenInventor free clone), http://www.coin3d.org/
- SoQt

Coin3D allows us to draw quite easily the standard figures, i.e. lines, cones, balls, and display them in an pre-made environment where the viewing system is already very advanced, one can rotate, zoom, translate.
SoQt comes from Qt, this tool gives window software, with menus, buttons.

4.2 Structure of the code

Since the C++ has the ability to produce classes, we used it a lot. This allows us to organize our work with "concrete" items such as an hyperboloid, an helix, etc.

4.3 Hyperboloid Parametrization

After the structuration, came the problem of implementation, particularly for the hyperboloid : we needed a mesh of points in order to draw the hyperboloid. An easy way to do this would have been to consider the points

$$(u_1, u_2, \frac{u_1 u_2 - a u_2 - c u_1}{b u_2 + d u_1})$$

which obviously are on the hyperboloid, and take some square mesh for (u_1, u_2) but then the drawing would have been extremely strange and unregular.
For these reasons, we wanted to find a similar parametrization to the standard situation (with r and θ).
Thus, with the transformation work, we get a nice Hyperboloid from which we get a better view of the situation.

5 Conclusion

We have studied the different possibilities of an helical equilibria.
We hope that the software we developped will help searchers in helical structures getting a better view of the situation. It was very formative for me, as a first project.
Although the mathematics used in this work are quite basic, we can come to something new : that is really powerful, and that's i discovered here.

References

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