Question 1

(i) Check if the parametrization $\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right)$ satisfies the equation of the curve:

$$\begin{aligned} x^3 + y^3 &= \left(\frac{3at}{1+t^3}\right)^3 + \left(\frac{3at^2}{1+t^3}\right)^3 \\ &= 3a\frac{9a^2t^3 + 9a^2t^6}{(1+t^3)^3} \\ &= 3a\frac{9a^2t^3(1+t^3)}{(1+t^3)^3} \\ &= 3a\frac{9a^2t^3}{(1+t^3)^2} \\ &= 3a\frac{3at}{(1+t^3)}\frac{3at^2}{(1+t^3)} \\ &= 3axy \end{aligned}$$

Note that α is continuous and positive for all $t \in J = (0, +\infty)$. It is also differentiable on J, with:

$$a'(t) = \left(\frac{3a}{1+t^3} - \frac{9at^3}{(1+t^3)^2}, \frac{6at}{1+t^3} - \frac{9at^4}{(1+t^3)^2}\right)$$

wgich is continuous on the whole J. Note also that for $t \in J$:

$$\begin{aligned} \alpha_x(t) &= 0 \\ 0 &= \frac{3a}{1+t^3} - \frac{9at^3}{(1+t^3)^2} \\ \frac{3t^3}{(1+t^3)^2} &= \frac{1}{1+t^3} \\ 3t^3 &= 1 + t^3 \\ 2t^3 &= 1 \\ t &= \sqrt[3]{\frac{1}{2}} \end{aligned}$$

and:

$$\begin{aligned} \alpha_y(t) &= 0 \\ 0 &= \frac{6at}{1+t^3} - \frac{9at^4}{(1+t^3)^2} \\ \frac{3t^3}{(1+t^3)^2} &= \frac{2}{1+t^3} \\ 3t^3 &= 2 + 2t^3 \\ t^3 &= 2 \\ t &= \sqrt[3]{2} \end{aligned}$$

so $\alpha'(t)$ is non-zero and both $\alpha_x(t)$ and $\alpha_y(t)$ are concave on the whole $J x < y \Leftrightarrow t < 1$. Aditionally note that for $t > 0$:

$$\begin{aligned} x = y &\Leftrightarrow \frac{3at}{1+t^3} = \frac{3at^2}{1+t^3} \\ &\Leftrightarrow 3at = 3at^2 \\ &\Leftrightarrow t = 1 \end{aligned}$$

In a similar way $x < y \Leftrightarrow t \in (0, 1)$ and $x > y \Leftrightarrow t \in (1, +\infty)$.

Because $y = tx$, is the relation between the coordinates of the curve and the the parateter t, then knowing that $x, y > 0$ we have that $alpha^{-1}(x, y) = \frac{y}{x}$. For any $x, y > 0$ the function $\frac{y}{x}$ is continuous. The above analysis shows as well that $\alpha(t)$ is bijective on J.

All the above shows that $\alpha(t)$ for $t \in J$ is a regular parametrization of $\partial\Omega$.

(ii) Consider functions $f(x, y) = x$ and $g(x, y) = -y$. Note that both are $C^1(\overline{\Omega})$ and our Ω is a regular domain, so by the Green's theorem:

$$\begin{aligned} \text{Area}_\Omega &= \int_\Omega \mathrm{d}x\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega 1 - (-1)\mathrm{d}x\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega \left\{\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right\} \mathrm{d}x\mathrm{d}x \\ &= \frac{1}{2}\int_{\overrightarrow{\partial\Omega}} f\mathrm{d}y + g\mathrm{d}x \\ &= \frac{1}{2}\int_{\overrightarrow{\partial\Omega}} x\mathrm{d}y - y\mathrm{d}x \\ &= \frac{1}{2}\int_{\overrightarrow{\partial\Omega}} x\mathrm{d}(tx) - tx\mathrm{d}x \\ &= \frac{1}{2}\int_{\overrightarrow{\partial\Omega}} x(x\mathrm{d}t + t\mathrm{d}x) - tx\mathrm{d}x \\ &= \frac{1}{2}\int_{\overrightarrow{\partial\Omega}} x^2\mathrm{d}t \end{aligned}$$

Using the found parametrization and the relation above, we have:

$$\begin{aligned} \text{Area}_\Omega &= \frac{1}{2}\int_{\overrightarrow{\partial\Omega}} x^2\mathrm{d}t \\ &= \frac{1}{2}\int_0^{+\infty} \frac{9a^2t^2}{(1+t^3)^2}\mathrm{d}t \\ &= \frac{3a^2}{2}\int_0^{+\infty} \frac{3t^2}{(1+t^3)^2}\mathrm{d}t \\ &= \frac{3a^2}{2}\int_1^{+\infty} \frac{1}{u^2}\mathrm{d}u \\ &= \frac{3a^2}{2}\lim_{s\to+\infty} -\frac{1}{u}\Big|_1^s \\ &= \frac{3a^2}{2} \end{aligned}$$

Question 2

(i) Let $f = (0, 0, g(x, y))$. First verify that the necessary condition is satisfied:

$$\nabla \cdot f = \frac{\partial}{\partial x}0 + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}g(x, y) = 0$$

Now let us find a vector potential $\varphi(x, y, z)$ of f s.t. $\varphi_3 \equiv 0$:

$$\begin{cases} -\frac{\partial}{\partial z}\varphi_2(x,y,z) & = & 0 & (1) \\ \frac{\partial}{\partial z}\varphi_1(x,y,z) & = & 0 & (2) \\ (\frac{\partial}{\partial x}\varphi_2 - \frac{\partial}{\partial y}\varphi_1)(x,y,z) & = & g(x,y) & (3) \end{cases}$$

$$(1) \Rightarrow \varphi_2(x, y, z) = k(x, y)$$
$$(2) \Rightarrow \varphi_1(x, y, z) = h(x, y)$$
$$(3) \Rightarrow (\frac{\partial}{\partial x}k(x, y) - \frac{\partial}{\partial y}h(x, y))(x, y, z) = g(x, y)$$

So if $h(x, y) \equiv 0$ and $k(x, y) = \int_0^x g(\tau, y)\mathrm{d}\tau$, we get a vector potential:

$$\varphi = (0, \int_0^x g(\tau, y)\mathrm{d}\tau, 0)$$

(ii) We first close the surface with a disk $D := \{(x, y, 0) : x^2 + y^2 \leq 1\}$ and set unit normals to $N(x, y, 0) := (0, 0, -1)$. By the previous exercise, there exists a vector potential $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with $f = \nabla \wedge \varphi$. We deduce from Stokes theorem, that the flux through the closed surface $(S \cup D)$ vanishes and therefore

$$0 = \int_{S \cup D} \langle f, N \rangle \,\mathrm{d}\sigma = \int_S \langle f, N \rangle \,\mathrm{d}\sigma + \int_D \langle f, N \rangle \,\mathrm{d}\sigma.$$

So we caluclate:

$$\begin{aligned} \int_S \langle f, N \rangle \,\mathrm{d}\sigma &= -\int_D \langle f, N \rangle \,\mathrm{d}\sigma \\ &= -\int_0^1 \int_0^{2\pi} -\sqrt{r^2 \sin^2 \Theta + r^2 \cos^2 \Theta}\, r \,\mathrm{d}\Theta\mathrm{d}r \\ &= \int_0^1 \int_0^{2\pi} r^2 \,\mathrm{d}\Theta\mathrm{d}r \\ &= \frac{2\pi}{3}. \end{aligned}$$

Question 3

(i) Posons $\mathbf{x} = (r\cos\theta, r\sin\theta, z)$.

Alors $h_r = \left|\frac{\partial \mathbf{x}}{\partial r}\right| = 1$, $h_\theta = \left|\frac{\partial \mathbf{x}}{\partial \theta}\right| = r$, $h_z = \left|\frac{\partial \mathbf{x}}{\partial z}\right| = 1$ et

$$\begin{aligned} \mathbf{e}_r &= \frac{1}{h_r}\frac{\partial \mathbf{x}}{\partial r} = \cos\theta\,\mathbf{e}_1 + \sin\theta\mathbf{e}_2, \\ \mathbf{e}_\theta &= \frac{1}{h_\theta}\frac{\partial \mathbf{x}}{\partial \theta} = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2, \\ \mathbf{e}_z &= \mathbf{e}_3. \end{aligned}$$

$$\Delta u(x,y) = \nabla \cdot \nabla u(x,y). \tag{1}$$

(ii) Le domaine Ω n'est pas le produit de deux intervalles, donc on peut pas utiliser la méthode de séparation des variables directement. Mais on peut essayer de chercher u sous la forme

$$u(x,y) = v(r,\theta)$$

où $x = r\cos\theta$, $y = r\sin\theta$, avec $1 \le r \le 2$ et $0 \le \theta \le \frac{\pi}{2}$.

Comme

$$\Delta u(x,y) = \partial_r^2 v(r,\theta) + \frac{1}{r}\partial_r v(r,\theta) + \frac{1}{r^2}\partial_\theta^2 v(r,\theta),$$

l'équation

$$\begin{cases} \Delta u(x,y) = 0 \quad \text{pour } \ (x,y) \in \Omega, \\ u(x,0) = 0, \\ u_x(0,y) = 0, \\ u(x,y) = 0 \quad \text{pour } x^2+y^2 = 1, \\ u(x,y) = 4y \quad \text{pour } x^2+y^2 = 4. \end{cases}$$

devient

$$r^2\partial_r^2 v(r,\theta) + r\partial_r v(r,\theta) + \partial_\theta^2 v(r,\theta) = 0 \quad \text{pour } 1 \le r \le 2, \quad 0 \le \theta \le \frac{\pi}{2}, \tag{2}$$

$$v(r,0) = 0 \quad \text{et } v_\theta\left(r,\frac{\pi}{2}\right) = 0, \tag{3}$$

$$v(1,\theta) = 0, \tag{4}$$

$$v(2,\theta) = 8\sin\theta. \tag{5}$$

Cherchons des solutions de la forme

$$v(r,\theta) = f(r)g(\theta).$$

Alors

$$r^2 f''(r)g(\theta) + rf'(r)g(\theta) + f(r)g''(\theta) = 0,$$

et donc

$$\frac{r^2 f''(r) + rf'(r)}{f(r)} = -\frac{g''(\theta)}{g(\theta)} = \lambda, \tag{6}$$

où λ est une constante.

La fonction g doit être une solution du problème aux limites

$$g''(\theta) = -\lambda g(\theta) \quad \text{pour } 0 \le \theta \le \frac{\pi}{2}, \tag{7}$$

$$g(0) = 0 \text{ et } g'\left(\frac{\pi}{2}\right) = 0. \tag{8}$$

Si $\lambda < 0$, la solution de (7) est $g(\theta) = Ae^{-\sqrt{-\lambda}\,\theta} + Be^{\sqrt{-\lambda}\,\theta}$. Mais de (8) on obtient, que $A = B = 0$.

Si $\lambda = 0$, $g(\theta) = A + B\theta$. Aussi (8) implique que $A = B = 0$.

Si $\lambda > 0$, $g(\theta) = A \sin \sqrt{\lambda}\, \theta + B \cos \sqrt{\lambda}\, \theta$. Cette fois (8) implique que $B = 0$, $A \in \mathbb{R}$ et $\lambda = n^2$, $n = 2k+1$, $k \in \mathbb{N}$.

Alors une fonction propre du problème (6) est

$$g_n = A_n \sin n\, \theta, \quad n = 2k+1, \quad k \in \mathbb{N}$$

et $\{\sigma = \lambda_n : n = 2k+1, k \in \mathbb{N}\}$ est le spectre avec $\lambda_n = n^2$.

Pour $\lambda = \lambda_n$ la fonction f doit satisfaire l'équation differentielle

$$r^2 f''(r) + r f'(r) = n^2 f(r) \quad \text{pour } 1 \leq r \leq 2. \tag{9}$$

Comme $n > 0$, la solution est

$$f_n(r) = P r^n + Q r^{-n}, \quad P, Q \in \mathbb{R}.$$

Pour que $v = f(r) g(\theta)$ vérifie (4), on doit avoir $f_n(1) = 0$, alors $Q = -P$.

Donc on obtient

$$v(r, \theta) = (P r^n - P r^{-n}) \sin n\, \theta, \quad P \in \mathbb{R}.$$

La solution générale de (2) et (3) est

$$v(r, \theta) = \sum_{n=1}^{m} P_n (r^n - r^{-n}) \sin n\, \theta.$$

Pour satisfaire aussi (5), on a que

$$\sum_{n=1}^{m} P_n (2^n - 2^{-n}) \sin n\, \theta = 8 \sin \theta.$$

Alors, $m = 1$, $P_1 = \frac{16}{3}$, et la solution du problème posé est

$$v(r, \theta) = 8(r - r^{-1}) \sin\, \theta = \frac{16}{3}(1 - r^{-2}) r \sin\, \theta$$

et

$$u(x, y) = \frac{16}{3} \left(1 - \frac{1}{x^2 + y^2} \right) y.$$