1. a) Compute

$$\boldsymbol{\alpha}' = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right),$$
$$\Rightarrow \quad ||\boldsymbol{\alpha}'|| = \sqrt{\frac{a^2+b^2}{c^2}} = 1. \tag{1}$$

b) Since $||\boldsymbol{\alpha}'|| = 1$ (from a.), we have $\mathbf{t} = \boldsymbol{\alpha}'$ and therefore

$$\mathbf{t}' = \kappa\mathbf{n} = \boldsymbol{\alpha}'' = \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right),$$
$$\Rightarrow \quad \kappa = ||\mathbf{t}'|| = \frac{a}{c^2}, \qquad \text{and,} \qquad \mathbf{n} = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right). \tag{2}$$

So, to compute the torsion, if we recall that $\tau(s) = \boldsymbol{b}\cdot\boldsymbol{n}'$, we have, by the previous,

$$\begin{aligned}
\tau(s) &= (\boldsymbol{t}\wedge\boldsymbol{n})\cdot\boldsymbol{n}' \\
&= \begin{vmatrix} -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \\ \frac{1}{c}\sin\frac{s}{c} & -\frac{1}{c}\cos\frac{s}{c} & 0 \end{vmatrix} \\
&= \frac{b}{c}\frac{1}{c}(\cos^2\tfrac{s}{c} + \sin^2\tfrac{s}{c}) \\
&= \frac{b}{c^2}
\end{aligned}$$

c) We already have $\mathbf{t}$ and $\mathbf{n}$ from a. and b.. All we need to do is therefore to compute

$$\mathbf{b} = \mathbf{t}\wedge\mathbf{n} = \left(\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right)$$

d) The lines $\boldsymbol{\beta}$ containing the normal $\mathbf{n}(s)$ and passing through $\boldsymbol{\alpha}(s)$ can be parameterised by

$$\boldsymbol{\beta}(s,t) = \boldsymbol{\alpha}(s) + at\mathbf{n}(s) = \left(a(1-t)\cos\frac{s}{c}, a(1-t)\sin\frac{s}{c} + \frac{a}{c}, \frac{b}{s}s\right).$$

So $\forall s \in I$, we have that $\boldsymbol{\beta}(s,1)$ is on the z axis. Furthermore, $\mathbf{e}_z\cdot\mathbf{n} = 0$.

e) By direct computation: $\boldsymbol{\alpha}'\cdot\mathbf{e}_z = \frac{b}{c}$.

2. Define a new parameter $S = L - s$. Then $\frac{\mathrm{d}}{\mathrm{d}S} = -\frac{\mathrm{d}}{\mathrm{d}s}$. We note $\{\mathbf{N},\mathbf{B},\mathbf{T}\}$, the Frenet frame associated with the new parameterisation and κ_S and τ_S the associated curvature and torsion. We find $\mathbf{T} = \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}S} = -\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}s} = -\mathbf{t}$. Similarly $K\mathbf{N} = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}S} = -\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}S} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa\mathbf{n}$. Accordingly $K = \kappa$ and $\mathbf{N} = \mathbf{n}$ since by definition both $K > 0$ and $\kappa > 0$. Next, we compute $\mathbf{B} = \mathbf{T}\wedge\mathbf{N} = -\mathbf{t}\wedge\mathbf{n} = -\mathbf{b}$. Finally, $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}S} = \tau_S\mathbf{N} = \frac{\mathrm{d}b}{\mathrm{d}s} = \tau\mathbf{n}$ so that $\tau_S = \tau$. So a re-parameterisation of the curve where we choose the other tip as $S = 0$ flips the directions of the tangent and binormal but leaves the normal unchanged. It is crucial to note that the definition of the curvature and torsion does not depend on the orientation.

3. Recall that because the curve is parameterised by arc-length,

$$\boldsymbol{\alpha}'' = \mathbf{t}' = \kappa\,\mathbf{n}, \quad \mathbf{n}' = -\kappa\,\mathbf{t} + \tau\,\mathbf{b}, \quad \mathbf{b}' = -\tau\,\mathbf{n}. \tag{3}$$

One more derivative gives $\boldsymbol{\alpha}''' = \kappa'\mathbf{n} + \kappa\mathbf{n}'$ which we rearrange as

$$\mathbf{n}' = \frac{\boldsymbol{\alpha}''' - \kappa'\mathbf{n}}{\kappa} \tag{4}$$

We then take a derivative of $\mathbf{b} = \mathbf{t} \wedge \mathbf{n}$:

$$\mathbf{b}' = \mathbf{t} \wedge \mathbf{n}'. \tag{5}$$

Substituting (3) and (4) in (5), we get

$$-\tau\mathbf{n} = \boldsymbol{\alpha}' \wedge \frac{\boldsymbol{\alpha}''' - \kappa'\mathbf{n}}{\kappa}. \tag{6}$$

Finally, the formula comes from taking the scalar product of (6) with $\mathbf{n} = \mathbf{t}'/\kappa = \boldsymbol{\alpha}''/\kappa$:

$$\begin{aligned}
\tau &= \tau(\mathbf{n} \cdot \mathbf{n}) \\
&= -(\boldsymbol{\alpha}' \wedge \frac{\boldsymbol{\alpha}''' - \kappa'\mathbf{n}}{\kappa}) \cdot \mathbf{n} \\
&= -\tfrac{1}{\kappa}\left((\boldsymbol{\alpha}' \wedge \boldsymbol{\alpha}''') \cdot \frac{\boldsymbol{\alpha}''}{\kappa} - (\boldsymbol{\alpha}' \wedge \kappa'\mathbf{n}) \cdot \mathbf{n}\right) \\
&= \frac{(\boldsymbol{\alpha}'(s) \wedge \boldsymbol{\alpha}''(s)) \cdot \boldsymbol{\alpha}'''(s)}{\kappa^2} \\
&= \frac{(\boldsymbol{\alpha}'(s) \wedge \boldsymbol{\alpha}''(s)) \cdot \boldsymbol{\alpha}'''(s)}{||\boldsymbol{\alpha}''(s)||^2}
\end{aligned}$$

4. Remember that a curve $\boldsymbol{\beta} : I \to \mathbb{R}^3$ is regular iff $\boldsymbol{\beta}' > 0$ for all $s \in I$. The indicatrix $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\beta} = (\cos\theta(s), \sin\theta(s))\,.$$

Since $\mathbf{t}' = \kappa\,\mathbf{n} = (-\sin\theta(s), \cos\theta(s))\,\theta'$, we indeed find that $\kappa = |\frac{\mathrm{d}\theta}{\mathrm{d}s}|$. Since we assumed that $\kappa > 0$, we conclude that for all $s \in I$, we have

$$\kappa(s) \neq 0 \Rightarrow \theta'(s) \neq 0 \Rightarrow \boldsymbol{\beta}'(s) \neq 0,$$

and accordingly, $\boldsymbol{\beta}$ is regular.

A note: we can compute the determinant $|\boldsymbol{t}\ \boldsymbol{n}|$ as follows:

$$\begin{aligned}
|\mathbf{t}\ \mathbf{n}| &= \begin{vmatrix} \cos(\theta(s)) & -\sin(\theta(s))\theta'(s) \\ \sin\theta(s) & \cos(\theta(s))\theta'(s) \end{vmatrix} \\
&= \theta'(s)
\end{aligned}$$

Since $|\mathbf{t}\ \mathbf{n}| = \pm\,\kappa$, this quantity is called the *signed curvature* of $\boldsymbol{\alpha}$ and denoted k.

5. We note $\mathbf{n}$ and $\mathbf{t}$ the normal and tangents to α. Also we define $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ where $s(t) = \int^t ||\boldsymbol{\alpha}'(\sigma)||\mathrm{d}\sigma$.

a) A differentiation of the definition of $\boldsymbol{\beta}$ by t while remembering that for a planar curve $\mathbf{n}' = -v\,\kappa\,\mathbf{t}$, yields

$$\begin{aligned}
\boldsymbol{\beta}' &= \boldsymbol{\alpha}' + \frac{\mathbf{n}'}{\kappa} - \frac{\kappa'\,\mathbf{n}}{\kappa^2}, \\
&= v\mathbf{t} - v\mathbf{t} - \frac{\kappa'}{\kappa^2}\,\mathbf{n}, \qquad (7)
\end{aligned}$$

where $'$ denotes derivation w.r.t. t. Hence the tangent to $\boldsymbol{\beta}$ is indeed parallel to the normal $\mathbf{n}$ to α.

Finally, the straight lines $\boldsymbol{\gamma}$, tangents to $\boldsymbol{\beta}$ can be parameterised as

$$\boldsymbol{\gamma}(t,u) = \boldsymbol{\beta} + u\,\boldsymbol{\beta}' = \boldsymbol{\alpha} + \frac{\mathbf{n}}{\kappa} - u\,\frac{\kappa'}{\kappa^2}\,\mathbf{n}. \qquad (8)$$

Accordingly, for each t, the line $\boldsymbol{\gamma}(t,\cdot)$ intercepts $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t)$ when $u = \kappa(t)/\kappa'(t)$.

b) The normal line going through $\boldsymbol{\alpha}(t)$ can be described as the set of points $\mathbf{x}$ such that

$$\boldsymbol{\alpha}'(t)\cdot(\mathbf{x} - \boldsymbol{\alpha}(t)) = 0$$

Consequently the point of intersection of the normals going through t_1 and t_2 is the point $\mathbf{x}$ such that

$$\begin{cases} \boldsymbol{\alpha}'(t_1)\cdot(\mathbf{x} - \boldsymbol{\alpha}(t_1)) &= 0 \\ \boldsymbol{\alpha}'(t_2)\cdot(\mathbf{x} - \boldsymbol{\alpha}(t_2)) &= 0 \end{cases}$$

And in the particular case where $t_2 = t_1 + \epsilon$, the second line becomes, if we Taylor expand and use the first line,

$$\begin{aligned}
&\boldsymbol{\alpha}'(t_1+\epsilon)\cdot(\mathbf{x} - \boldsymbol{\alpha}(t_1+\epsilon)) \\
&= \boldsymbol{\alpha}'(t_1)\cdot(\mathbf{x} - \boldsymbol{\alpha}(t_1) - \epsilon\boldsymbol{\alpha}'(t_1)) + \epsilon\boldsymbol{\alpha}''(t_1)\cdot(\mathbf{x} - \boldsymbol{\alpha}(t_1)) + O(\epsilon^2) \\
&= -\epsilon\boldsymbol{\alpha}'(t_1)\cdot\boldsymbol{\alpha}'(t_1) + \epsilon\boldsymbol{\alpha}''(t_1)\cdot(\mathbf{x} - \boldsymbol{\alpha}(t_1)) + O(\epsilon^2).
\end{aligned}$$

Rearranging then, we wish to solve for $\mathbf{x}$ in

$$\begin{cases} \boldsymbol{\alpha}'(t_1)\cdot\mathbf{x} &= \boldsymbol{\alpha}'(t_1)\cdot\boldsymbol{\alpha}(t_1) \\ \epsilon\boldsymbol{\alpha}''(s)\cdot\mathbf{x} &= \epsilon\boldsymbol{\alpha}'(t_1)\cdot\boldsymbol{\alpha}'(t_1) + \epsilon\boldsymbol{\alpha}''(t_1)\cdot\boldsymbol{\alpha}(t_1) + O(\epsilon^2) \end{cases}$$

and, reparameterizing by arclength, this amounts to solving

$$\begin{cases} \mathbf{t}(s_1)\cdot\mathbf{x} &= \mathbf{t}(s_1)\cdot\boldsymbol{\alpha}(s_1) \\ \mathbf{n}(s_1)\cdot\mathbf{x} &= \dfrac{\epsilon + \epsilon\boldsymbol{\alpha}''(s_1)\cdot\boldsymbol{\alpha}(s_1)}{\epsilon||\boldsymbol{\alpha}''(s_1)||} + O(\epsilon) \\ &= \frac{1}{\kappa} + \mathbf{n}(s_1)\cdot\alpha(s_1) + O(\epsilon) \end{cases}$$

but since $\mathbf{t}(s_1)$ and $\mathbf{n}(s_1)$ form an orthonormal basis for $\mathbb{R}^2$, this means that we have

$$\mathbf{x} = \boldsymbol{\alpha}(s_1) + \frac{\boldsymbol{n}(s_1)}{\kappa(s_1)} + O(\epsilon)\boldsymbol{n}s_1$$

which, as ϵ tends to 0 and switching back to time parameterization, gives us that $\mathbf{x} = \boldsymbol{\alpha}(t_1) + \frac{\boldsymbol{n}(t_1)}{\kappa(t_1)}$, which is clearly a point on the trace of the evolute.

6. For a planar curve $\boldsymbol{\alpha}(s)$ parameterised by its arc-length, we can always define a function $\theta(s)$ such that

$$\boldsymbol{\alpha}'(s) = \Big(\cos\theta(s), \sin\theta(s)\Big).$$

Then, we have

$$\boldsymbol{\alpha}''(s) = \Big(-\sin\theta(s), \cos\theta(s)\Big)\theta'(s) = \kappa(s)\,\mathbf{n}.$$

Then for $\mathbf{b}$ to have the correct orientation, we must therefore have $\theta'(s) = \kappa(s)$. The result follows by direct integration.

7. a) The lines containing $\mathbf{n}$ and passing through $\boldsymbol{\alpha}$ are parallel to a fixed plane iff there exists a constant vector $\mathbf{x}$ (the normal vector to the plane) such that

$$\mathbf{n}(s)\cdot\mathbf{x} = 0, \tag{9}$$

for all s in I.
Similarly, the tangent lines of $\boldsymbol{\alpha}$ make a constant angle with a fixed direction iff there exists a constant vector $\mathbf{y}$ (the direction in question) such that the dot product

$$\boldsymbol{\alpha}'(s)\cdot\mathbf{y} = C, \tag{10}$$

where C is a constant.
Equation (10) and (9) are equivalent after choosing $\mathbf{y} = \mathbf{x}$.

b) This follows in a similar way to the previous by the following compution:

$$\begin{aligned} \boldsymbol{b}(s)\cdot\boldsymbol{x} \text{ is constant} \quad &\Leftrightarrow \quad 0 = \frac{d}{ds}(\boldsymbol{b}(s)\cdot\boldsymbol{x}) = \boldsymbol{b}'(s)\cdot\boldsymbol{x} = -\tau\boldsymbol{n}(s)\cdot\boldsymbol{x}\forall s\in I \\ &\Leftrightarrow \quad 0 = \boldsymbol{n}(s)\cdot\boldsymbol{x}\forall s\in I \end{aligned}$$

c) Suppose that κ/τ is constant. Notice that $\boldsymbol{t}' = \kappa\boldsymbol{n} = \tau\frac{\kappa}{\tau}\boldsymbol{n} = -\frac{\kappa}{\tau}\boldsymbol{b}'$ since $\boldsymbol{t}' = \kappa\boldsymbol{n}$ and $\boldsymbol{b}' = -\tau\boldsymbol{n}$, so $\boldsymbol{t}' + \frac{\kappa}{\tau}\boldsymbol{b}' = 0$. Since $\frac{\kappa}{\tau}$ is constant, $\boldsymbol{t}' + \frac{\kappa}{\tau}\boldsymbol{b}' = (\boldsymbol{t} + \frac{\kappa}{\tau}\boldsymbol{b})'$, which means that $\boldsymbol{t} + \frac{\kappa}{\tau}\boldsymbol{b} = \boldsymbol{x}$ is a constant vector. Further, we then have $\boldsymbol{t}\cdot\boldsymbol{x} = \boldsymbol{t}\cdot(\boldsymbol{t} + \frac{\kappa}{\tau}\boldsymbol{b}) = ||\boldsymbol{t}|| = 1$ which is constant, so the angle between $\boldsymbol{t}$ and $\boldsymbol{x}$ is constant.

Conversely, if $\boldsymbol{\alpha}$ is a helix, there is a vector $\boldsymbol{x}$ such that $\boldsymbol{n}\cdot\boldsymbol{x} = 0$, so $0 = (\boldsymbol{n}\cdot\boldsymbol{x})' = \boldsymbol{n}'\cdot\boldsymbol{x} = \tau\boldsymbol{b}\cdot\boldsymbol{x} - \kappa\boldsymbol{t}\cdot\boldsymbol{x}$ and thus $\frac{\kappa}{\tau} = \frac{\boldsymbol{b}\cdot\boldsymbol{x}}{\boldsymbol{t}\cdot\boldsymbol{x}}$ must also be constant, since as we've shown before the products $\boldsymbol{b}\cdot\boldsymbol{x}$ and $\boldsymbol{t}\cdot\boldsymbol{x}$ are constant when $\boldsymbol{\alpha}$ is a helix.

d) We can compute

$$\begin{aligned} \boldsymbol{t}(s) &= \boldsymbol{\alpha}'(s) \\ &= (\tfrac{a}{c}\cos\theta(s), \tfrac{a}{c}\sin\theta(s), \tfrac{b}{c})^T \\ \boldsymbol{n}(s) &= \frac{\boldsymbol{\alpha}''(s)}{||\boldsymbol{\alpha}''(s)||} \\ &= \frac{(-\frac{a}{c}\sin(\theta(s))\theta'(s), \frac{a}{c}\cos(\theta(s))\theta'(s), \frac{b}{c})^T}{|\frac{a}{c}\theta'(s)|} \\ &= (-\sin(\theta(s)), \cos(\theta(s)), 0)^T \end{aligned}$$

and

$$\boldsymbol{b}(s) = \boldsymbol{t}(s)\wedge\boldsymbol{n}(s) = (-\frac{b}{c}\cos(\theta(s)), -\frac{b}{c}\sin(\theta(s)), \frac{a}{c})^T.$$

Since $\boldsymbol{t}(s) \cdot \boldsymbol{e}_z = \frac{b}{c}$ is constant, $\boldsymbol{\alpha}$ is a helix and also by the computation in c),

$$\frac{\kappa}{\tau} = \frac{\boldsymbol{b} \cdot \boldsymbol{e}_z}{\boldsymbol{t} \cdot \boldsymbol{e}_z} = \frac{a/c}{b/c} = \frac{a}{b}$$

8. We have the following equivalences:

$$\begin{aligned}
\boldsymbol{\alpha} \text{ lies on a sphere} \quad &\Leftrightarrow \quad \exists \text{ a constant vector } \boldsymbol{x} \text{ such that } ||\boldsymbol{\alpha}(s) - \boldsymbol{x}|| \text{ is constant} \\
&\Leftrightarrow \quad \exists \boldsymbol{x} : ||\boldsymbol{\alpha}(s) - \boldsymbol{x}||^2 = (\boldsymbol{\alpha}(s) - \boldsymbol{x}) \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{x}) \text{ is constant} \\
&\Leftrightarrow \quad \exists \boldsymbol{x} : 2\boldsymbol{\alpha}'(s) \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{x}) = 0 \\
&\Leftrightarrow \quad \exists \boldsymbol{x} : \boldsymbol{t}(s) \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{x}) = 0 \\
&\Leftrightarrow \quad \exists \boldsymbol{x} : \boldsymbol{\alpha}(s) - \boldsymbol{x} = \beta \boldsymbol{n} + \gamma \boldsymbol{b} \text{ for some } \beta, \gamma \in \mathbb{R}
\end{aligned}$$

Now, differentiating once, $\boldsymbol{t}(s) \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{x}) = 0$ implies $\boldsymbol{t}'(s) \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{x}) + \boldsymbol{t} \cdot \boldsymbol{t} = \kappa(s)\boldsymbol{n}(s) \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{x}) + 1 = 0$; replacing $\boldsymbol{\alpha}(s) - \boldsymbol{x}$ by $\beta\boldsymbol{n} + \gamma\boldsymbol{b}$ we see that $\kappa(s)\beta + 1 = 0$, i.e. $\beta = \frac{-1}{\kappa(s)}$ is the only possible value for β.

Further, $\kappa(s)\boldsymbol{n}(s) \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{x}) + 1$ implies, differentiating once more, that

$$\begin{aligned}
&\kappa(s)\boldsymbol{n}(s) \cdot \boldsymbol{t}(s) + (\boldsymbol{\alpha}(s) - \boldsymbol{x}) \cdot (\kappa'(s)\boldsymbol{n}(s) + \kappa(s)\boldsymbol{n}'(s)) \\
= \; &(\boldsymbol{\alpha}(s) - \boldsymbol{x}) \cdot (\kappa'(s)\boldsymbol{n}(s) + \kappa(s)^2\boldsymbol{t} + \kappa(s)\tau(s)\boldsymbol{b}(s))
\end{aligned}$$

which, replacing $\boldsymbol{\alpha}(s) - \boldsymbol{x}$ by $-\frac{1}{\kappa(s)}\boldsymbol{n}(s) + \gamma\boldsymbol{b}$, gives us

$$-\frac{\kappa'(s)}{\kappa(s)} + \kappa(s)\tau(s)\gamma = 0$$

so, since $\tau(s) \neq 0$, $\gamma = \dfrac{\kappa'(s)}{\kappa(s)^2\tau(s)} = \left(\dfrac{1}{\kappa(s)}\right)' \dfrac{1}{\tau(s)}$ is the only possible value for γ.

Thus, we have

$$\exists \boldsymbol{x} : \boldsymbol{\alpha}(s) - \boldsymbol{x} = \beta\boldsymbol{n} + \gamma\boldsymbol{b} \text{ for some } \beta, \gamma \in \mathbb{R} \quad \Leftrightarrow \quad \exists \boldsymbol{x} : \boldsymbol{\alpha}(s) - \boldsymbol{x} = -R\boldsymbol{n}(s) - R'T\boldsymbol{(}s)$$

Clearly then, $R^2 + (R')^2T^2 = ||\boldsymbol{\alpha}(s) - \boldsymbol{x}||^2$ is constant, and also $\boldsymbol{x} = \boldsymbol{\alpha}(s) + R\boldsymbol{n}(s) + R'T\boldsymbol{b}$.

Conversely, if we define $\boldsymbol{x}(s) = \boldsymbol{\alpha}(s) + R\boldsymbol{n} + R'T\boldsymbol{b}$ and we suppose $R^2 + (R')^2T^2$ is constant, then we can show that $\boldsymbol{x}(s)$ is constant. Indeed,

$$\begin{aligned}
\boldsymbol{x}'(s) &= \boldsymbol{t} + R'\boldsymbol{n} + R(\tau\boldsymbol{b} - \kappa\boldsymbol{t}) + R''T\boldsymbol{b} + R'T'\boldsymbol{b} + R'T(-\tau\boldsymbol{n}) \\
&= R\tau\boldsymbol{b} + R''T\boldsymbol{b} + R'T'\boldsymbol{b} \\
&= (R\tau + R''T + R'T')\boldsymbol{b} \\
&= \frac{\frac{d}{ds}(R^2 + (R')^2T^2)}{2R'T}\boldsymbol{b} \\
&= 0.
\end{aligned}$$

Thus,

$$\exists \boldsymbol{x} : \boldsymbol{\alpha} - \boldsymbol{x} = -R\boldsymbol{n} - R'T\boldsymbol{b} \Leftrightarrow R^2 + (R')^2T^2 \text{ is constant.}$$

and puting it all together, $\boldsymbol{\alpha}$ lies on a sphere iff $R^2 + (R')^2T^2$ is constant.

9. a) $\boldsymbol{\alpha}$ is at the very least continuous, since $\lim_{t\to 0} e^{-\frac{1}{t^2}} = 0$ $\boldsymbol{\alpha}$ is differentiable since $\boldsymbol{e}_x \cdot \boldsymbol{\alpha}(t)$ is smooth, and

$$\lim_{t\to 0^+} \frac{\boldsymbol{e}_y \cdot \boldsymbol{\alpha}(t)}{t} = \lim_{t\to 0^+} \frac{e^{-\frac{1}{t^2}}}{t} = \lim_{s\to\infty} se^{-s^2} = 0$$

so $\frac{d}{dt}\boldsymbol{e}_y \cdot \boldsymbol{\alpha}(t) = 0$ and $\frac{d}{dt}\boldsymbol{e}_z \cdot \boldsymbol{\alpha}(t) = 0$ are defined. Finally, since $\frac{d}{dt}e^{-\frac{1}{t^2}} = \frac{2}{t^3}e^{-\frac{1}{t^2}}$ tends to 0 as $t \to 0$, $\boldsymbol{\alpha}(s)$ is continuously differentiable.

b) Since $\boldsymbol{\alpha}(t) \cdot \boldsymbol{e}_z = t$, $\boldsymbol{e}_z \cdot \boldsymbol{\alpha}'(t) = 1$, so $||\boldsymbol{\alpha}'(t)|| \geq 1 \forall t$ and thus $\boldsymbol{\alpha}$ must be regular for all t. Further, using our computations in a), we have

$$\boldsymbol{\alpha}'(t) = \begin{cases} (1, \frac{2}{t^3}e^{-\frac{1}{t^2}}, 0) & \text{when } t < 0 \\ (1, 0, 0) & \text{when } t = 0 \\ (1, 0, \frac{2}{t^3}e^{-\frac{1}{t^2}}) & \text{when } t > 0 \end{cases}$$

Now, we need to figure out how to compute $\boldsymbol{\kappa}(t)$ when our curve is not parameterized by arclength. We have

$$\boldsymbol{t}(t) = \frac{\boldsymbol{\alpha}'(t)}{||\boldsymbol{\alpha}'(t)||} = \frac{\boldsymbol{\alpha}'(t)}{\frac{ds}{dt}},$$

and thus

$$\boldsymbol{\alpha}'(t) = \frac{ds}{dt}\boldsymbol{t}(t)$$

which in turn implies

$$\boldsymbol{\alpha}''(t) = \frac{d^2s}{dt^2}\boldsymbol{t}(t) + \left(\frac{ds}{dt}\right)^2 \kappa(t)\boldsymbol{n}(t)$$

(since $\frac{d}{dt}\boldsymbol{t}(t) = \kappa(t)\boldsymbol{n}(t)\frac{ds}{dt}$). So then,

$$\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t) = \left(\frac{ds}{dt}\right)^3 \kappa(t)\boldsymbol{b}(t)$$

and

$$||\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)|| = \left(\frac{ds}{dt}\right)^3 \kappa(t)$$

i.e.

$$\kappa(t) = \frac{||\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)||}{\left(\frac{ds}{dt}\right)^3} = \frac{||\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)||}{||\boldsymbol{\alpha}'(t)||^3}$$

thus, in this particular case,

$$\kappa(t) = \begin{cases} (0, 0, \frac{|-\frac{6}{t^4}+\frac{4}{t^6}|e^{-\frac{1}{t^2}}}{(1+\frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac{3}{2}}}) & \text{when } t < 0 \\ 0 & \text{when } t = 0 \\ (0, 0, \frac{|-\frac{6}{t^4}+\frac{4}{t^6}|e^{-\frac{1}{t^2}}}{(1+\frac{4}{t^6}e^{-\frac{2}{t^2}})^{\frac{3}{2}}}) & \text{when } t > 0 \end{cases}$$

which is zero only when $t = 0$ or $-\frac{6}{t^4} + \frac{4}{t^6} = 0$ i.e. $t = \pm\sqrt{\frac{2}{3}}$.

c) As we computed previously, $\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t) = ||\boldsymbol{\alpha}''(t)||^3\kappa(t)\boldsymbol{b}$, where $\boldsymbol{b}$ is (by definition) normal to the osculating plane. Thus, since $||\boldsymbol{\alpha}'(t)||$ and $\kappa(t)$ are nonzero in some set N 0 where N is a neighborhood of 0, and since

$$\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t) = \begin{cases} (0,0,(-\frac{6}{t^4}+\frac{4}{t^6})e^{-\frac{1}{t^2}}) & \text{when } t < 0 \\ (0,0,0) & \text{when } t = 0 \\ (0,(-\frac{6}{t^4}+\frac{4}{t^6})e^{-\frac{1}{t^2}},0) & \text{when } t < 0 \end{cases}$$

the osculating plane as $t \to 0^-$ is $z = 0$ and the osculating plane as $t \to 0^+$ is $y = 0$.

d) Now, as we did previously with κ, we have to figure out a way to compute τ when $\boldsymbol{\alpha}(t)$ is not parameterized by arclength. Recall from exercise 3 that

$$\tau(s) = \frac{|\boldsymbol{\alpha}'(s)\boldsymbol{\alpha}''(s)\boldsymbol{\alpha}'''(s)|}{\boldsymbol{\alpha}''(s)\cdot\boldsymbol{\alpha}''(s)}$$

Now, in terms of t, we have

$$\begin{aligned} \boldsymbol{\alpha}'(t) &= \boldsymbol{\alpha}'(s)\tfrac{ds}{dt} \\ \boldsymbol{\alpha}''(t) &= \boldsymbol{\alpha}''(s)(\tfrac{ds}{dt})^2 + \boldsymbol{\alpha}'(s)\tfrac{d^2s}{dt^2} \\ \boldsymbol{\alpha}'''(t) &= \boldsymbol{\alpha}'''(s)(\tfrac{ds}{dt})^3 + 3\boldsymbol{\alpha}''(s)\tfrac{ds}{dt}\tfrac{d^2s}{dt^2} + \boldsymbol{\alpha}'(s)\tfrac{d^3s}{dt^3} \end{aligned}$$

so, we then have

$$\begin{aligned} |\boldsymbol{\alpha}'(t)\boldsymbol{\alpha}''(t)\boldsymbol{\alpha}'''(t)| &= |\boldsymbol{\alpha}'(s)\boldsymbol{\alpha}''(s)\boldsymbol{\alpha}'''(s)|(\tfrac{ds}{dt})^6 \\ &= |\boldsymbol{\alpha}'(s)\boldsymbol{\alpha}''(s)\boldsymbol{\alpha}'''(s)|(\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t))^3 \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\alpha}''(s) &= \frac{\boldsymbol{\alpha}''(t) - \boldsymbol{\alpha}'(s)\frac{d^2s}{dt^2}}{(\frac{ds}{dt})^2} \\ &= \frac{\boldsymbol{\alpha}''(t) - \frac{\boldsymbol{\alpha}'(t)}{||\boldsymbol{\alpha}'(t)||}\frac{d^2s}{dt^2}}{\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t)} \end{aligned}$$

which means, since

$$\begin{aligned} \tfrac{d^2s}{dt^2} &= \tfrac{d}{dt}||\boldsymbol{\alpha}'(t)|| \\ &= \frac{\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}''(t)}{||\boldsymbol{\alpha}'(t)||} \end{aligned}$$

that

$$\begin{aligned} \boldsymbol{\alpha}''(s)\cdot\boldsymbol{\alpha}''(s) &= \frac{1}{(\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t))^2}\left(\boldsymbol{\alpha}''(t)\cdot\boldsymbol{\alpha}''(t) - 2\frac{(\boldsymbol{\alpha}''(t)\cdot\boldsymbol{\alpha}'(t))^2}{\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t)} + \frac{\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t)}{||\boldsymbol{\alpha}'(t)||^2}\frac{(\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}''(t))}{\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t)}\right) \\ &= \frac{\boldsymbol{\alpha}''(t)\cdot\boldsymbol{\alpha}''(t)}{(\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t))^2} - \frac{(\boldsymbol{\alpha}''(t)\cdot\boldsymbol{\alpha}'(t))^2}{(\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t))^3} \end{aligned}$$

thus,

$$\begin{aligned} \tau(t) &= \frac{|\boldsymbol{\alpha}'(t)\boldsymbol{\alpha}''(t)\boldsymbol{\alpha}'''(t)|}{(\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}')^2}\frac{1}{\frac{\boldsymbol{\alpha}''(t)\cdot\boldsymbol{\alpha}''(t)}{(\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t))^2} - \frac{(\boldsymbol{\alpha}''(t)\cdot\boldsymbol{\alpha}'(t))^2}{(\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t))^3}} \\ &= \frac{|\boldsymbol{\alpha}'(t)\boldsymbol{\alpha}''(t)\boldsymbol{\alpha}'''(t)|}{(\boldsymbol{\alpha}''(t)\cdot\boldsymbol{\alpha}''(t))(\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}'(t)) - (\boldsymbol{\alpha}''(t)\cdot\boldsymbol{\alpha}'(t))^2} \end{aligned}$$

Now, clearly, in this particular case, $\boldsymbol{\alpha}'''(t)$ and $\boldsymbol{\alpha}''(t)$ are linearly dependant when $t \neq 0$ and thus, in this case $\tau(t) = 0$ when $t \neq 0$, so $\lim_{t\to 0}\tau(t) = 0$.