

1. Composition of Darboux vectors

(a) We can compute

$$\begin{aligned} \mathbf{U}^\times &= D^T D' \\ &= (dQ)^T (dQ)' \\ &= Q^T d^T (d' Q + dQ') \\ &= Q^T \mathbf{u}^\times Q + Q^T Q' \end{aligned}$$

(b) By applying the result of question 1 of the week 2's exercise session, we find

$$(Q^T \mathbf{u})^\times = Q^T \mathbf{u}^\times Q, \tag{1}$$

the substitution of which in result a) of this question yields

$$\mathbf{U}^\times = (Q^T \mathbf{u})^\times + Q^T Q' = (Q^T \mathbf{u})^\times + \mathbf{p}^\times.$$

Hence

$$\begin{aligned} D\mathbf{U} &= D(Q^T \mathbf{u}) + D\mathbf{p}, \\ &= d\mathbf{u} + D\mathbf{p}. \end{aligned}$$

and thus

$$\mathbf{U} = \mathbf{u} + \mathbf{D}_i \mathbf{p}_i.$$

(c) If  $\mathbf{D}_3 = \mathbf{d}_3$ , we can write

$$Q = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and hence} \quad Q^T Q' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \varphi'.$$

Accordingly,  $\mathbf{p} = (0 \ 0 \ \varphi')^T$  and  $\mathbf{U} = \mathbf{u} + \varphi' \mathbf{d}_3 = \mathbf{u} + \varphi' \mathbf{D}_3$

(d) We have

$$\mathbf{U} = \tau \mathbf{t} + \kappa \mathbf{b}$$

so

$$\mathbf{u} = (\tau - \varphi') \mathbf{t} + \kappa \mathbf{b}$$

2. Factorisation of curves in  $SE(3)$

For the convenience of our discussion, we write  $\mathcal{U}(t) = \begin{pmatrix} \mathbf{u}^\times & \mathbf{a} \\ 0 & 0 \end{pmatrix}$  and  $\mathcal{V}(t) = \begin{pmatrix} \mathbf{v}^\times & \mathbf{b} \\ 0 & 0 \end{pmatrix}$ . We then have

$$\begin{aligned} \mathcal{Z}' &= (\mathcal{X}\mathcal{Y})' \\ &= \mathcal{X}'\mathcal{Y} + \mathcal{X}\mathcal{Y}' \\ &= \mathcal{X}\mathcal{U}\mathcal{Y} + \mathcal{X}\mathcal{Y}\mathcal{V} \\ &= \mathcal{Z}\mathcal{Y}^{-1}\mathcal{X}^{-1}\mathcal{X}\mathcal{U}\mathcal{Y} + \mathcal{Z}\mathcal{Y}^{-1}\mathcal{Y}\mathcal{V} \\ &= \mathcal{Z}(\mathcal{Y}^{-1}\mathcal{U}\mathcal{Y} + \mathcal{V}) \\ &= \mathcal{Z} \left( \begin{pmatrix} Y^T & -Y^T \mathbf{y} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u}^\times & \mathbf{a} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y & \mathbf{y} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \mathbf{v}^\times & \mathbf{b} \\ 0 & 0 \end{pmatrix} \right) \\ &= \mathcal{Z} \begin{pmatrix} Y^T \mathbf{u}^\times Y + \mathbf{v}^\times & Y^T \mathbf{u}^\times \mathbf{y} + Y^T \mathbf{a} + \mathbf{b} \\ 0 & 0 \end{pmatrix} \end{aligned}$$

As such, we have  $\mathbf{w}^\times = (Y^T \mathbf{u})^\times + \mathbf{v}^\times$  and so  $\mathbf{w} = Y^T \mathbf{u} + \mathbf{v}$ . Also,  $\mathbf{c} = Y^T(\mathbf{u}^\times \mathbf{y} + \mathbf{a}) + \mathbf{b}$ .

### 3. Offset of a curve in $\mathbb{R}^3$

(a) Let us write  $\mathbf{u}$  for the darboux triple of  $X(t)$  (i.e. the  $\mathbf{u}$  such that  $X' = Xu^\times$ ). We have

$$\begin{aligned}\mathbf{z}' &= \mathbf{x}' + \epsilon \mathbf{d}'_1 \\ &= \|\mathbf{x}'\| \mathbf{d}_3 + \epsilon(\mathbf{u}_3 \mathbf{d}_2 - \mathbf{u}_2 \mathbf{d}_3) \\ &= (\|\mathbf{x}'\| - \epsilon \mathbf{u}_2) \mathbf{d}_3 + \epsilon \mathbf{u}_3 \mathbf{d}_2\end{aligned}$$

$\|\mathbf{z}'\| = 0$  then only occurs if both components of  $\mathbf{z}'$  vanish, which in turn can only happen if  $\|\mathbf{x}'\| - \epsilon \mathbf{u}_2 = 0$ . Also note that if  $\mathbf{u}_3(t) \neq 0 \forall t$ , then  $\|\mathbf{z}'\|$  never vanishes as long as  $\mathbf{x}(t)$  is regular. The sufficient condition on  $\epsilon$  that is relevant for this course is

$$\epsilon < \min_t \frac{\|\mathbf{x}'(t)\|}{|\mathbf{u}_2(t)|}$$

(b) Since we want  $\mathbf{D}_1 = \mathbf{d}_1$ , we want

$$\mathbf{D} = X \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}$$

for some function  $\varphi(t)$ . In order for the framing to be adapted, we also need to make sure  $\mathbf{D}_3$  can be parallel to  $\mathbf{z}'$ , so we need to show that  $\mathbf{z}' \cdot \mathbf{d}_1 = 0$ . But we computed that  $\mathbf{z}' = (\|\mathbf{x}'\| - \epsilon \mathbf{u}_2) \mathbf{d}_3 + \epsilon \mathbf{u}_3 \mathbf{d}_2$  and therefore this is clear.

(c) We have

$$\mathcal{Z} = \begin{pmatrix} D & \mathbf{z} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} XY & X\mathbf{y} + \mathbf{x} \\ 0 & 1 \end{pmatrix}$$

hence,  $D = XY$  and  $\mathbf{z} = X\mathbf{y} + \mathbf{x} = \mathbf{x} + \epsilon \mathbf{d}_1$ . Now,  $D = XY$  implies  $Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}$

and  $\mathbf{v} = \begin{pmatrix} \varphi' \\ 0 \\ 0 \end{pmatrix}$  since  $\mathbf{v}^\times = Y'Y^T$ . Also,  $X\mathbf{y} + \mathbf{x} = \mathbf{x} + \epsilon \mathbf{d}_1$  gives us that  $X\mathbf{y} = \epsilon \mathbf{d}_1$  and

since  $X = [\mathbf{d}_1 \ \mathbf{d}_2 \ \mathbf{d}_3]$ , we must have that  $\mathbf{y} = \begin{pmatrix} \epsilon \\ 0 \\ 0 \end{pmatrix}$ .