Differential Geometry of Framed Curves	VERSION 1
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SESSION 3: SOLUTIONS	T. LESSINNES

1. **Composition of Darboux vectors**

   (a) We can compute
   $$\begin{aligned}\mathsf{U}^\times &= D^T D' \\ &= (d\,Q)^T (d\,Q)' \\ &= Q^T d^T (d'\,Q + d\,Q') \\ &= Q^T \mathsf{u}^\times Q + Q^T Q'\end{aligned}$$

   (b) By applying the result of question 10 of the week 1's exercise session, we find
   $$(Q^T \mathsf{u})^\times = Q^T\,\mathsf{u}^\times\,Q, \tag{1}$$
   the substitution of which in result a) of this question yields
   $$\mathsf{U}^\times = (Q^T\mathsf{u})^\times + Q^T Q' = (Q^T\mathsf{u})^\times + \mathsf{p}^\times.$$
   Hence
   $$\begin{aligned}D\,\mathsf{U} &= D\,(Q^T\mathsf{u}) + D\,\mathsf{p}, \\ &= d\,\mathsf{u} + D\,\mathsf{p}.\end{aligned}$$
   and thus
   $$\mathbf{U} = \mathbf{u} + \mathbf{D}_i \mathsf{p}_i.$$

   (c) If $\mathbf{D}_3 = \mathbf{d}_3$, we can write
   $$Q = \begin{pmatrix}\cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1\end{pmatrix}, \quad \text{and hence} \quad Q^T Q' = \begin{pmatrix}0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}\varphi'.$$
   Accordingly, $\mathsf{p} = (0\ 0\ \varphi')^T$ and $\mathbf{U} = \mathbf{u} + \varphi'\mathbf{d}_3 = \mathbf{u} + \varphi'\mathbf{D}_3$

   (d) From **1**f., we have
   $$\mathbf{U} = \tau\mathbf{t} + \kappa\mathbf{b}$$
   so
   $$\mathbf{u} = (\tau - \varphi')\mathbf{t} + \kappa\mathbf{b}$$

2. **Factorisation of curves in** $SE(3)$

   For the convenience of our discussion, we write $\mathcal{U}(s) = \begin{pmatrix}\mathsf{u}^\times & \mathsf{a} \\ 0 & 0\end{pmatrix}$ and $\mathcal{V}(s) = \begin{pmatrix}\mathsf{v}^\times & \mathsf{b} \\ 0 & 0\end{pmatrix}$. We then have
   $$\begin{aligned}\mathcal{Z}' &= (\mathcal{X}\mathcal{Y})' \\ &= \mathcal{X}'\mathcal{Y} + \mathcal{X}\mathcal{Y}' \\ &= \mathcal{X}\mathcal{U}\mathcal{Y} + \mathcal{X}\mathcal{Y}\mathcal{V} \\ &= \mathcal{Z}\mathcal{Y}^{-1}\mathcal{X}^{-1}\mathcal{X}\mathcal{U}\mathcal{Y} + \mathcal{Z}\mathcal{Y}^{-1}\mathcal{Y}\mathcal{V} \\ &= \mathcal{Z}(\mathcal{Y}^{-1}\mathcal{U}\mathcal{Y} + \mathcal{V}) \\ &= \mathcal{Z}\left(\begin{pmatrix}Y^T & -Y^T\mathbf{y} \\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathsf{u}^\times & \mathsf{a} \\ 0 & 0\end{pmatrix}\begin{pmatrix}Y & \mathbf{y} \\ 0 & 1\end{pmatrix} + \begin{pmatrix}\mathsf{v}^\times & \mathsf{b} \\ 0 & 0\end{pmatrix}\right) \\ &= \mathcal{Z}\begin{pmatrix}Y^T\mathsf{u}^\times Y + \mathsf{v}^\times & Y^T\mathsf{u}^\times\mathbf{y} + Y^T\mathsf{a} + \mathsf{b} \\ 0 & 0\end{pmatrix}\end{aligned}$$
   As such, we have $\mathsf{w}^\times = (Y^T\mathsf{u})^\times + \mathsf{v}^\times$ and so $\mathsf{w} = Y^T\mathsf{u} + \mathsf{v}$. Also, $\mathsf{c} = Y^T(\mathsf{u}^\times\mathbf{y} + \mathsf{a}) + \mathsf{b}$.

3. **Offset of a curve in $\mathbb{R}^3$**

(a) Let us write u for the darboux triple of $X(s)$ (i.e. the u such that $X' = X\mathsf{u}^\times$). We have

$$\begin{aligned} \mathbf{z}' &= \mathbf{x}' + \epsilon \mathbf{d}_1' \\ &= ||\mathbf{x}'||\mathbf{d}_3 + \epsilon(\mathsf{u}_3\mathbf{d}_2 - \mathsf{u}_2\mathbf{d}_3) \\ &= (||\mathbf{x}'|| - \epsilon\mathsf{u}_2)\mathbf{d}_3 + \epsilon\mathsf{u}_3\mathbf{d}_2 \end{aligned}$$

$||\mathbf{z}'|| = 0$ then only occurs if both components of $\mathbf{z}'$ vanish, which in turn can only happen if $||\mathbf{x}'|| - \epsilon\mathsf{u}_2 = 0$. Also note that if $\mathsf{u}_3(s) \neq 0 \ \forall s$, then $||\mathbf{z}'||$ never vanishes as long as $\mathbf{x}(s)$ is regular. The sufficient condition on ϵ that is relevant for this course is

$$\epsilon < \min_s \frac{||\mathbf{x}'(s)||}{|\mathsf{u}_2(s)|}$$

(b) Since we want $\mathbf{D}_1 = \mathbf{d}_1$, we want

$$\mathbf{D} = X \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

for some function $\varphi(s)$. In order for the framing to be adapted, we also need to make sure $\mathbf{D}_3$ can be parallel to $\mathbf{z}'$, so we need to show that $\mathbf{z}' \cdot \mathbf{d}_1 = 0$. But we computed that $\mathbf{z}' = (||\mathbf{x}'|| - \epsilon\mathsf{u}_2)\mathbf{d}_3 + \epsilon\mathsf{u}_3\mathbf{d}_2$ and therefore this is clear.

(c) We have from **1**. that if $\mathcal{Z} = \mathcal{X}\mathcal{Y}$, that

$$\mathcal{Z} = \begin{pmatrix} D & \mathbf{z} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} XY & X\mathbf{y} + \mathbf{x} \\ 0 & 1 \end{pmatrix}$$

hence, $D = XY$ and $\mathbf{z} = X\mathbf{y} + \mathbf{x} = \mathbf{x} + \epsilon\mathbf{d}_1$. Now, $D = XY$ implies $Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$ and $\mathsf{v} = \begin{pmatrix} \varphi' \\ 0 \\ 0 \end{pmatrix}$ since $\mathsf{v}^\times = Y'Y^T$. Also, $X\mathbf{y} + \mathbf{x} = \mathbf{x}x + \epsilon\mathbf{d}_1$ gives us that $X\mathbf{y} = \epsilon\mathbf{d}_1$ and since $X = [\mathbf{d}_1 \ \mathbf{d}_2 \ \mathbf{d}_3]$, we must have that $\mathbf{y} = \begin{pmatrix} \epsilon \\ 0 \\ 0 \end{pmatrix}$.