Before we start, we do the following computation.

Let $\boldsymbol{\alpha}(t)$ be a $\mathcal{C}^1(I, \mathbb{R}^3 \setminus \{0\})$ function and $\boldsymbol{e} = \boldsymbol{\alpha}/||\boldsymbol{\alpha}||$. We then have

$$\frac{d\boldsymbol{e}}{dt} = \frac{d}{dt}\frac{\boldsymbol{\alpha}}{||\boldsymbol{\alpha}||} = \frac{||\boldsymbol{\alpha}||\,\boldsymbol{\alpha}' - \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}'}{||\boldsymbol{\alpha}||}\,\boldsymbol{\alpha}}{||\boldsymbol{\alpha}||^2} = \frac{1}{||\boldsymbol{\alpha}||}\left(\boldsymbol{\alpha}' - (\boldsymbol{e}\cdot\boldsymbol{\alpha}')\,\boldsymbol{e}\right) = \boldsymbol{e} \times \left(\frac{\boldsymbol{\alpha}'}{||\boldsymbol{\alpha}||} \times \boldsymbol{e}\right). \tag{1}$$

1 Link as a signed area

Since $\mathbf{e}(\sigma, s) = \frac{\mathbf{y}(\sigma)-\mathbf{x}(s)}{||\mathbf{y}(\sigma)-\mathbf{x}(s)||}$, by the computation at the beginning of this solutions sheet we have

$$\mathbf{e}_\sigma = \frac{1}{||\mathbf{y}-\mathbf{x}||}(\mathbf{y}_\sigma - (\mathbf{y}_\sigma\cdot\mathbf{e})\mathbf{e})$$

and

$$\mathbf{e}_s = \frac{-1}{||\mathbf{y}-\mathbf{x}||}(\mathbf{x}_s - (\mathbf{x}_s\cdot\mathbf{e})\mathbf{e})$$

this means that

$$\begin{aligned}
\mathbf{e}\cdot(\mathbf{e}_s \times \mathbf{e}_\sigma) &= \mathbf{e}\cdot\left(\frac{-1}{||\mathbf{y}-\mathbf{x}||}(\mathbf{x}_s - (\mathbf{x}_s\cdot\mathbf{e})\mathbf{e}) \quad\times\quad \frac{1}{||\mathbf{y}-\mathbf{x}||}(\mathbf{y}_\sigma - (\mathbf{y}_\sigma\cdot\mathbf{e})\mathbf{e})\right) \\
&= \mathbf{e}\cdot\frac{-1}{||\mathbf{y}-\mathbf{x}||^2}\left(\mathbf{x}_s\times\mathbf{y}_\sigma - (\mathbf{y}_\sigma\cdot\mathbf{e})\mathbf{x}_s\times\mathbf{e} - (\mathbf{x}_s\cdot\mathbf{e})\mathbf{e}\times\mathbf{y}_\sigma\right) \\
&= \frac{1}{||\mathbf{y}-\mathbf{x}||^2}\;\mathbf{e}\cdot(\mathbf{y}_\sigma\times\mathbf{x}_s) = \frac{(\mathbf{y}(\sigma)-\mathbf{x}(s))\cdot(\mathbf{y}'(\sigma)\times\mathbf{x}'(s))}{||\mathbf{y}(\sigma)-\mathbf{x}(s)||^3}.
\end{aligned}$$

Now, given a survace $\mathbf{r}(x, y)$ with normal vector $\mathbf{n}_\mathbf{r}(x, y)$, the signed survace area of a patch S of $\mathbf{r}$ is given by

$$\Sigma = \iint_S \mathbf{n}_\mathbf{r}\cdot(\mathbf{r}_x \times \mathbf{y}_y)\;dx\,dy.$$

In this case, since $\mathbf{e}$ spans (a portion of) the unit sphere, we have that $\mathbf{n}_\mathbf{e} = \mathbf{e}$, and the statement follows.

2 Homotopy invariance

Under the variations

$$\mathbf{x}(s;\varepsilon) = \mathbf{x}(s) + \varepsilon\,\delta\mathbf{x}(s) \qquad\text{and}\qquad \mathbf{y}(\sigma;\varepsilon) = \mathbf{y}(\sigma) + \varepsilon\,\delta\mathbf{y}(\sigma),$$

we have

$$\mathbf{e}(s,\sigma;\varepsilon) = \frac{\mathbf{y}(\sigma)-\mathbf{x}(s)+\varepsilon\,(\delta\mathbf{y}(\sigma)-\delta\mathbf{x}(s))}{||\mathbf{y}(\sigma)-\mathbf{x}(s)+\varepsilon\,(\delta\mathbf{y}(\sigma)-\delta\mathbf{x}(s))||},$$

and therefore by (1), we have

$$\frac{d\mathbf{e}}{d\varepsilon} = \frac{\mathbf{e}(s,\sigma;\varepsilon)\times\left((\delta\mathbf{y}(\sigma)-\delta\mathbf{x}(s))\times\mathbf{e}(s,\sigma;\varepsilon)\right)}{||\mathbf{y}(\sigma)-\mathbf{x}(s)+\varepsilon\,(\delta\mathbf{y}(\sigma)-\delta\mathbf{x}(s))||}.$$

Then for $\varepsilon = 0$,

$$\delta\mathbf{e} = \frac{1}{||\mathbf{y}-\mathbf{x}||}\Big(\mathbf{e}\times\big((\delta\mathbf{y}-\delta\mathbf{x})\times\mathbf{e}\big)\Big).$$

The variation δLk is then

$$\begin{aligned}
4\pi\,\delta\mathrm{Lk} &= \delta\int_0^{l_1}\int_0^{l_2}\mathbf{e}\cdot(\mathbf{e}_\sigma\times\mathbf{e}_s)\;d\sigma\,ds = \int_0^{l_1}\int_0^{l_2}\delta\big(\mathbf{e}\cdot(\mathbf{e}_\sigma\times\mathbf{e}_s)\big)\;d\sigma\,ds \\
&= \int_0^{l_1}\int_0^{l_2}\delta\mathbf{e}\cdot(\mathbf{e}_\sigma\times\mathbf{e}_s)+\mathbf{e}\cdot(\delta\mathbf{e}_\sigma\times\mathbf{e}_s)+\mathbf{e}\cdot(\mathbf{e}_\sigma\times\delta\mathbf{e}_s)\;d\sigma\,ds.
\end{aligned}$$

Now, we have that

$$\begin{aligned}
\int_0^{l_2}\mathbf{e}\cdot(\delta\mathbf{e}_\sigma\times\mathbf{e}_s)\;d\sigma &= \int_0^{l_2}\delta\mathbf{e}_\sigma\cdot(\mathbf{e}_s\times\mathbf{e})\;d\sigma \\
&= \underbrace{\Big[\delta\mathbf{e}\cdot(\mathbf{e}_s\times\mathbf{e})\Big]_0^{l_2}}_{=0\text{ since all these are periodic}} - \int_0^{l_2}\delta\mathbf{e}\cdot\frac{d}{d\sigma}(\mathbf{e}_s\times\mathbf{e})\;d\sigma \\
&= -\int_0^{l_2}\delta\mathbf{e}\cdot(\mathbf{e}_{s\sigma}\times\mathbf{e})+\delta\mathbf{e}\cdot(\mathbf{e}_s\times\mathbf{e}_\sigma)\;d\sigma.
\end{aligned}$$

Similarly,

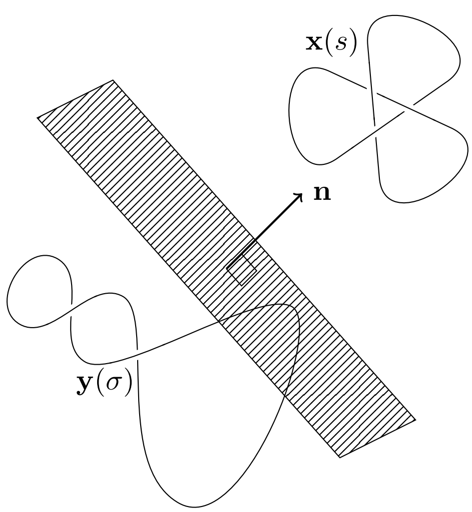
$$\int_0^{l_1}\mathbf{e}\cdot(\mathbf{e}_\sigma\times\delta\mathbf{e}_s)\;ds = -\int_0^{l_1}\delta\mathbf{e}\cdot(\mathbf{e}_s\times\mathbf{e}_\sigma)+\delta\mathbf{e}\cdot(\mathbf{e}\times\mathbf{e}_{\sigma s})\;d\sigma \tag{2}$$

Hence,

$$\delta\mathrm{Lk} = \frac{1}{4\pi}\iint\delta\big(\mathbf{e}\cdot(\mathbf{e}_\sigma\times\mathbf{e}_s)\big)\;d\sigma\,ds = \frac{1}{4\pi}\;3\iint\delta\mathbf{e}\cdot(\mathbf{e}_\sigma\times\mathbf{e}_s)\;d\sigma\,ds = 0,$$

where the last equality results from $\mathbf{e}_s$, $\mathbf{e}_\sigma$, and $\delta\mathbf{e}$ being coplanar (all are parallel to $\mathbf{e}$).

3 Unknotted curves and the Whitehead link



Let $\mathbf{n}$ be a unit vector normal to the plane separating $\mathbf{x}(s)$ and $\mathbf{y}(\sigma)$. For any real number α, define a closed curve $Tr(\alpha; C_1)$ parametrised as $\alpha\mathbf{n}+\mathbf{x}(s)$ which is just the curve C_1 translated. Since $\mathbf{x}(s)$ and $\mathbf{y}(\sigma)$ are closed and disjoint, there exist positive constants k_1, K_1 such that $k_1 \le ||\mathbf{x}(s)-\mathbf{y}(\sigma)|| \le K_1$ for any s, σ. Also, upon parametrising $\boldsymbol{x}$ and $\boldsymbol{y}$ by arc-length, we have $||\mathbf{y}'(\sigma)\times\mathbf{x}'(s)|| \le 1$ since $\mathbf{y}'(\sigma)$ and $\mathbf{x}'(s)$ are both of norm 1.

Then,

$$\begin{aligned}
&\left|\frac{(\mathbf{y}(\sigma)-\mathbf{x}(s)-\alpha\mathbf{n})\cdot(\mathbf{y}'(\sigma)\times\mathbf{x}'(s))}{||\mathbf{y}(\sigma)-\mathbf{x}(s)-\alpha\mathbf{n}||^3}\right| \\
=\ & \frac{1}{\alpha^2}\frac{\left|\left(\frac{1}{\alpha}(\mathbf{y}-\mathbf{x})-\mathbf{n}\right)\cdot(\mathbf{y}'\times\mathbf{x}')\right|}{||\frac{1}{\alpha}(\mathbf{y}-\mathbf{x})-\mathbf{n}||^3} \\
\leq\ & \frac{1}{\alpha^2}\frac{\left(\frac{1}{\alpha}||\mathbf{y}-\mathbf{x}||+||\mathbf{n}||\right)||\mathbf{y}'\times\mathbf{x}'||}{\left|\frac{1}{\alpha}||\mathbf{y}-\mathbf{x}||-||\mathbf{n}||\right|^3} \\
\leq\ & \frac{1}{\alpha^2}\frac{\frac{1}{\alpha}K_1+1}{\left|\frac{1}{\alpha}k_1-1\right|^3}
\end{aligned}$$

and as such we can compute

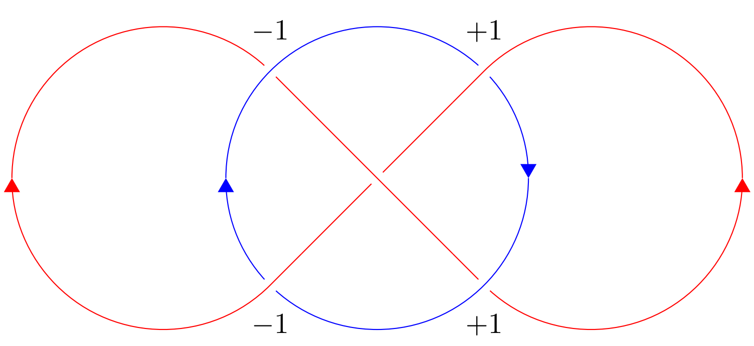
$$\begin{aligned}
|\mathrm{Lk}(Tr(\alpha;C_1),C_2)| &= \frac{1}{4\pi}\left|\int_0^{l_1}\int_0^{l_2}\frac{(\mathbf{y}(\sigma)-\mathbf{x}(s)-\alpha\mathbf{n})\cdot(\mathbf{y}'(\sigma)\times\mathbf{x}'(s))}{||\mathbf{y}(\sigma)-\mathbf{x}(s)-\alpha\mathbf{n}||^3}\,d\sigma\,ds\right| \\
&\leq \frac{1}{4\pi}\int_0^{l_1}\int_0^{l_2}\left|\frac{(\mathbf{y}(\sigma)-\mathbf{x}(s)-\alpha\mathbf{n})\cdot(\mathbf{y}'(\sigma)\times\mathbf{x}'(s))}{||\mathbf{y}(\sigma)-\mathbf{x}(s)-\alpha\mathbf{n}||^3}\right|\,d\sigma\,ds \\
&\leq \frac{1}{4\pi}\int_0^{l_1}\int_0^{l_2}\frac{1}{\alpha^2}\frac{(\frac{1}{\alpha}K_1+||\mathbf{n}||)}{\left|\frac{1}{\alpha}k_1-||\mathbf{n}||\right|^3}\,d\sigma\,ds \\
&= \frac{1}{4\pi}\,\frac{1}{\alpha^2}\frac{(\frac{1}{\alpha}K_1+1)}{\left|\frac{1}{\alpha}k_1-1\right|^3}l_1l_2.
\end{aligned}$$

At the same time, by exercise 2, we have that $\mathrm{Lk}(Tr(\alpha;C_1),C_2)=\mathrm{Lk}(C_1,C_2)$ for any α which means that we have that

$$|\mathrm{Lk}(C_1,C_2)|=|\mathrm{Lk}(Tr(\alpha;C_1),C_2)|\leq\frac{1}{4\pi}\frac{1}{\alpha^2}\frac{(\frac{1}{\alpha}K_1+1)}{\left|\frac{1}{\alpha}k_1-1\right|^3}l_1l_2$$

and thus, since that expression tends to 0 when α tends to infinity, $\mathrm{Lk}(C_1,C_2)=0$.

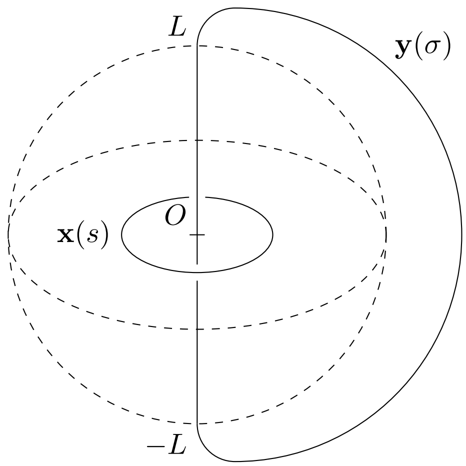
Now, we also want to compute the linking number of the Whitehead link using the method of counting signed crossings. The signs of the various crossings can be seen to be



and thus, since $\frac{1}{2}(1-1+1-1)=0$, we've computed that $\mathrm{Lk}=0$.

Another way to conclude that the link integral vanishes is to notice that if we reverse the orientation of one of C_1 or C_2 we get the original turned around a half turn (which is a transformation which can be accomplished by homotopy). As such, $\mathrm{Lk}=-\mathrm{Lk}$ and the conclusion follows.

4 The Hopf link



Let us first estimate the part of the link integral involving the part of $\mathbf{y}$ outside the radius L sphere. We can admit that the length of that part of the curve is equal to kL for some constant k. Also there exists some constant c such that $||\mathbf{y}-\mathbf{x}|| > c\,L$. We then compute

$$\begin{aligned} & \left|\int \frac{(\mathbf{y}-\mathbf{x})\cdot(\mathbf{y}'\times\mathbf{x}')}{||\mathbf{y}-\mathbf{x}||^3}\,d\sigma\right| \\ \leq & \int \left|\frac{(\mathbf{y}-\mathbf{x})\cdot(\mathbf{y}'\times\mathbf{x}')}{||\mathbf{y}-\mathbf{x}||^3}\right| d\sigma \\ \leq & \int \frac{||\mathbf{y}-\mathbf{x}||\,||\mathbf{y}'\times\mathbf{x}'||}{||\mathbf{y}-\mathbf{x}||^3}\,d\sigma \\ \leq & \int \frac{1}{(c\,L)^2}\,d\sigma \\ = & \frac{k}{c^2\,L} \end{aligned}$$

which is arbitrarily small as $L \to \infty$.

For the part of the integral containing the straight line part of $\mathbf{y}(\sigma)$, we have tha arc-length parameterisations of $\mathbf{x} = (\cos s\ \ \sin s\ 0)^T$ and $\mathbf{y} = (0\ 0\ \sigma)^T$ so that $\boldsymbol{y}'(\sigma)\times\boldsymbol{x}'(s) = (-\cos s\ \ -\sin s\ 0)^T$, $(\mathbf{y}-\mathbf{x})\cdot(\mathbf{y}'\times\mathbf{x}') = 1$, and $||\mathbf{y}-\mathbf{x}|| = \sqrt{\sigma^2+1}$.

Hence,

$$\mathrm{Lk} = \frac{1}{4\pi}\int_0^{2\pi}\int_{-L}^{L}\frac{1}{(1+\sigma^2)^{\frac{3}{2}}}\,d\sigma\,ds + O(1/L) = \frac{2\pi}{4\pi}\int_{-L}^{L}\frac{1}{(1+\sigma^2)^{\frac{3}{2}}}\,d\sigma + O(1/L)$$

taking $\sigma = \sinh(z)$ so that $d\sigma = \cosh(z)\,dz$, we have that

$$\mathrm{Lk} = \frac{1}{2}\int_{\mathrm{arcsinh}(-L)}^{\mathrm{arcsinh}(L)}\frac{1}{\cosh^2(z)}\,dz + O(1/L) = \tanh(\mathrm{arcsinh}(L)) \to 1 \text{ as } L \to \infty$$

Finally, since the link integral does not change under deformations where $\mathbf{x}$ intersect itself, the second link can be deformed to a hopf link with the $\mathbf{x}$ curve covered twice. The domain of integration along $\mathbf{x}$ can therefore be split into two parts each of which are the same as the link integral of the Hopf link. Accordingly, we have $\mathrm{Lk} = 2$.