Solution 1.

$$\begin{aligned}
\mathbf{r}(s_0+\epsilon,\sigma_0+\nu) &= \mathbf{r}(s_0,\sigma_0)+\epsilon\,\mathbf{r}_s(s_0,\sigma_0)+\nu\,\mathbf{r}_\sigma(s_0,\sigma_0)+\frac{\epsilon^2}{2}\mathbf{r}_{ss}(s_0,\sigma_0)+\epsilon\nu\,\mathbf{r}_{s\sigma}(s_0,\sigma_0)\\
&\quad+\frac{\nu^2}{2}\mathbf{r}_{\sigma\sigma}(s_0,\sigma_0)+O(\epsilon^3)+O(\nu^3)+O(\epsilon\nu^2)+O(\nu\epsilon^2),\\
&= \mathbf{r}-\epsilon\,\mathbf{t}^{[\mathbf{x}]}+\nu\,\mathbf{t}^{[\mathbf{y}]}-\frac{\epsilon^2}{2}\kappa^{[x]}\mathbf{n}^{[\mathbf{x}]}+\frac{\nu^2}{2}\kappa^{[y]}\mathbf{n}^{[\mathbf{y}]}\Big|_{(s_0,\sigma_0)}.
\end{aligned}$$

Solution 2. In general, we have

$$\begin{aligned}
\mathbf{r}_s\wedge\mathbf{r}_\sigma &= -\mathbf{x}'(s)\wedge\mathbf{y}'(\sigma)\\
&= -(\mathbf{t}^{[\mathbf{x}]}+\epsilon\kappa^{[x]}\mathbf{n}^{[\mathbf{x}]}+O(\epsilon^2))\wedge(\mathbf{t}^{[\mathbf{y}]}+\nu\kappa^{[y]}\mathbf{n}^{[\mathbf{y}]}+O(\nu^2))\\
&= \mathbf{t}^{[\mathbf{y}]}\wedge\mathbf{t}^{[\mathbf{x}]}+\epsilon\kappa^{[x]}\mathbf{t}^{[\mathbf{y}]}\wedge\mathbf{n}^{[\mathbf{x}]}+\nu\kappa^{[y]}\mathbf{n}^{[\mathbf{y}]}\wedge\mathbf{t}^{[\mathbf{x}]}+\text{h.o.t.},
\end{aligned}\tag{1}$$

where h.o.t. stands for higher order terms.

At a pair of points (s_0,σ_0) where the tangents align, the first term in (1) vanishes and accordingly, the vector $\mathbf{n}$ can not be defined as in the exercise sheet. To show that no definition of $\mathbf{n}$ at (s_0,σ_0) will provide a smooth function $\mathbf{n}$ in an open neighbourhood of (s_0,σ_0), we show that $\lim_{(s,\sigma)\to(s_0,\sigma_0)}\mathbf{n}(s,\sigma)$ depends on the path of approach. Locally, we specify this path by choosing a number $\phi\in[0,\pi)\setminus\{\pi/2\}$ and taking $\nu=\frac{\kappa^{[x]}}{\kappa^{[y]}}\tan\phi\,\epsilon$. Substituting this for ν in (1) yields

$$\begin{aligned}
\mathbf{r}_s\wedge\mathbf{r}_\sigma &= \epsilon\kappa^{[x]}\,(\mathbf{t}^{[\mathbf{y}]}\wedge\mathbf{n}^{[\mathbf{x}]}+\tan\phi\,\mathbf{n}^{[\mathbf{y}]}\wedge\mathbf{t}^{[\mathbf{x}]})+\text{h.o.t.},\\
&= \epsilon\kappa^{[x]}\,(\mathbf{t}^{[\mathbf{y}]}\wedge(\mathbf{n}^{[\mathbf{x}]}\pm\tan\phi\,\mathbf{n}^{[\mathbf{y}]}))+\text{h.o.t.}. \qquad (\because\ \mathbf{t}^{[\mathbf{x}]}=\pm\mathbf{t}^{[\mathbf{y}]})
\end{aligned}$$

So we must distinguish between two cases: either $\mathbf{n}^{[\mathbf{x}]}=\pm\mathbf{n}^{[\mathbf{y}]}$ or not. In the first (rare) case, we find $\mathbf{r}_s\wedge\mathbf{r}_\sigma=\pm\epsilon\kappa^{[x]}\,(1\pm\tan\phi)\mathbf{b}_y$ where the two $\pm$ lead to four possibilities depending on the geometry of the curves but not on the path of approach. In this case, $\mathbf{n}$ can actually be regularised by picking $\mathbf{n}(s_0,\sigma_0)=\pm\mathbf{b}_y$.

In the second case, by far the most common, the two curves do not have parallel osculating planes at (s_0,σ_0): $\mathbf{n}^{[\mathbf{x}]}\wedge\mathbf{n}^{[\mathbf{y}]}\neq 0$. In that case the direction of the vector $\mathbf{r}_s\wedge\mathbf{r}_\sigma$ depends on ϕ. Accordingly, it is impossible to make a choice for $\mathbf{n}(s_0,\sigma_0)$ that would lead to a smooth limit for all ϕ.

Solution 3. *Ad absurdum* Let $\mathbf{N}=\mathbf{r}_s\wedge\mathbf{r}_\sigma$. And assume there exist a positive number $a>0$ and a regular curve

$$\gamma: t\in(-a,a)\to\big(s(t),\sigma(t)\big),\tag{2}$$

such that both $\gamma(0)=(s_0,\sigma_0)$ and $\mathbf{N}(\gamma(t))=\mathbf{0}$ for all $t\in(-a,a)$. Then compute

$$\begin{aligned}
\mathbf{N}(\gamma(t))=0 \quad &\Rightarrow \quad \frac{\partial\mathbf{N}}{\partial s}\frac{ds}{dt}+\frac{\partial\mathbf{N}}{\partial\sigma}\frac{d\sigma}{dt}=0,\\
&\Rightarrow \quad (\mathbf{y}_\sigma\wedge\mathbf{x}_{ss})\,\dot{s}+(\mathbf{y}_{\sigma\sigma}\wedge\mathbf{x}_s)\,\dot{\sigma}=0.
\end{aligned}\tag{3}$$

A scalar product of (3) by $\mathbf{y}_{\sigma\sigma}$ yields

$$\dot{s}=0 \quad\text{or}\quad (\mathbf{x}_{ss}\wedge\mathbf{y}_{\sigma\sigma})\cdot\mathbf{y}_\sigma=0.\tag{4}$$

But since the curves $\mathbf{x}$ and $\mathbf{y}$ are regular, we have $\mathbf{x}_s \neq \mathbf{0}$ and $\mathbf{y}_\sigma \neq \mathbf{0}$. Accordingly $\mathbf{N}(\gamma) = 0$ implies that every point in the image of γ respects the hypothesis of question 2: the tangent to $\mathbf{x}$ is parallel to that of $\mathbf{y}$ at $\gamma(t)$. However, since $\mathbf{x}_s$ and $\mathbf{y}_\sigma$ are unit vectors, we know that $\mathbf{x}_{ss}$ (resp. $\mathbf{y}_{\sigma\sigma}$) is perpendicular to $\mathbf{x}_s$ (resp. $\mathbf{y}_\sigma$). But since $\mathbf{x}_s = \pm\,\mathbf{y}_\sigma$, we also have that $\mathbf{x}_{ss}$ is perpendicular to $\mathbf{y}_\sigma$. Accordingly, the RHS proposition in the alternative (4) can not occur at (s_0, σ_0) since $\mathbf{x}_{ss} \wedge \mathbf{y}_{\sigma\sigma}$ both can not vanish and must be parallel to $\mathbf{y}_\sigma$. We conclude that the left hand side must occur:

$$\dot{s}(\gamma(0)) = 0. \tag{5}$$

Similarly, the scalar product of (3) with $\mathbf{x}_{ss}$ implies that

$$\dot{\sigma}(\gamma(0)) = 0. \tag{6}$$

Together (5,6) imply that $\dot{\gamma}(0) = 0$ so that the curve γ can not be regular: a contradiction.

Solution 4. Assume that f vanishes for some (s_0, σ_0) (not necessarily meeting the hypothesis of question 2). The implicit function theorem guarantees that if

$$f_s(s_0, \sigma_0) \neq 0, \qquad \text{or} \qquad f_\sigma(s_0, \sigma_0) \neq 0, \tag{7}$$

then there exists a unique open curve γ through (s_0, σ_0) such that $f(\gamma(t)) = 0$.

To conclude we prove that (7) is true. *Ad absurdum*, if (7) is false then at (s_0, σ_0), we have

$$\begin{cases} f = 0, \\ f_s = 0, \\ f_\sigma = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y}_\sigma \wedge \mathbf{x}_s) = 0, \\ (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y}_\sigma \wedge \mathbf{x}_{ss}) = 0, \\ (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y}_{\sigma\sigma} \wedge \mathbf{x}_s) = 0. \end{cases} \quad \Leftrightarrow \quad (\mathbf{y} - \mathbf{x}),\ \mathbf{y}_\sigma,\ \mathbf{y}_{\sigma\sigma},\ \mathbf{x}_s \text{ and } \mathbf{x}_{ss} \text{ are coplanar.} \tag{8}$$

In particular, the last statement implies that $\mathbf{x}$ and $\mathbf{y}$ share an osculating plane at (s_0, σ_0): a contradiction.

Solution 5. If (s, σ_0) is such that $\mathbf{x}_s$ is parallel to $\mathbf{y}_\sigma$, then $(\mathbf{r}_s \wedge \mathbf{r}_\sigma)|_{(s_0,\sigma_0)} = \mathbf{0}$ and in particular $f(s_0, \sigma_0) = 0$. Then question 4 implies that such a curve exists.