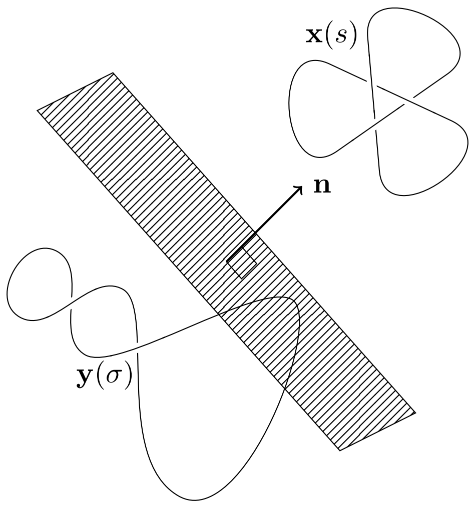
1 Unknotted curves and the Whitehead link



Let $\mathbf{n}$ be a unit vector normal to the plane separating $\mathbf{x}(s)$ and $\mathbf{y}(\sigma)$. For any real number α, define a closed curve $Tr(\alpha; C_1)$ parametrised as $\alpha\mathbf{n} + \mathbf{x}(s)$ which is just the curve C_1 translated. Since $\mathbf{x}(s)$ and $\mathbf{y}(\sigma)$ are closed and disjoint, there exist positive constants k_1, K_1 such that $k_1 \leq ||\mathbf{x}(s) - \mathbf{y}(\sigma)|| \leq K_1$ for any s, σ. Also, upon parametrising $\boldsymbol{x}$ and $\boldsymbol{y}$ by arc-length, we have $||\mathbf{y}'(\sigma) \times \mathbf{x}'(s)|| \leq 1$ since $\mathbf{y}'(\sigma)$ and $\mathbf{x}'(s)$ are both of norm 1.

Then,

$$\begin{aligned}
&\left|\frac{(\mathbf{y}(\sigma) - \mathbf{x}(s) - \alpha\mathbf{n}) \cdot (\mathbf{y}'(\sigma) \times \mathbf{x}'(s))}{||\mathbf{y}(\sigma) - \mathbf{x}(s) - \alpha\mathbf{n}||^3}\right| \\
=\ & \frac{1}{\alpha^2} \frac{\left|\left(\frac{1}{\alpha}(\mathbf{y} - \mathbf{x}) - \mathbf{n}\right) \cdot (\mathbf{y}' \times \mathbf{x}')\right|}{||\frac{1}{\alpha}(\mathbf{y} - \mathbf{x}) - \mathbf{n}||^3} \\
\leq\ & \frac{1}{\alpha^2} \frac{\left(\frac{1}{\alpha}||\mathbf{y} - \mathbf{x}|| + ||\mathbf{n}||\right) ||\mathbf{y}' \times \mathbf{x}'||}{\left|\frac{1}{\alpha}||\mathbf{y} - \mathbf{x}|| - ||\mathbf{n}||\right|^3} \\
\leq\ & \frac{1}{\alpha^2} \frac{\frac{1}{\alpha}K_1 + 1}{\left|\frac{1}{\alpha}k_1 - 1\right|^3}
\end{aligned}$$

and as such we can compute

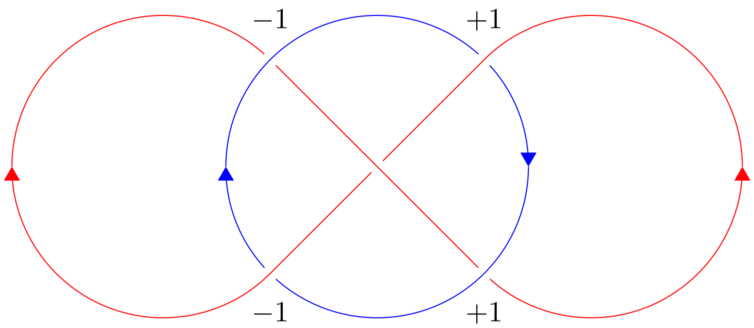
$$\begin{aligned}
|\mathrm{Lk}(Tr(\alpha; C_1), C_2)| &= \frac{1}{4\pi}\left|\int_0^{l_1}\int_0^{l_2} \frac{(\mathbf{y}(\sigma) - \mathbf{x}(s) - \alpha\mathbf{n}) \cdot (\mathbf{y}'(\sigma) \times \mathbf{x}'(s))}{||\mathbf{y}(\sigma) - \mathbf{x}(s) - \alpha\mathbf{n}||^3}\, d\sigma\, ds\right| \\
&\leq \frac{1}{4\pi}\int_0^{l_1}\int_0^{l_2} \left|\frac{(\mathbf{y}(\sigma) - \mathbf{x}(s) - \alpha\mathbf{n}) \cdot (\mathbf{y}'(\sigma) \times \mathbf{x}'(s))}{||\mathbf{y}(\sigma) - \mathbf{x}(s) - \alpha\mathbf{n}||^3}\right| d\sigma\, ds \\
&\leq \frac{1}{4\pi}\int_0^{l_1}\int_0^{l_2} \frac{1}{\alpha^2}\frac{(\frac{1}{\alpha}K_1 + ||\mathbf{n}||)}{\left|\frac{1}{\alpha}k_1 - ||\mathbf{n}||\right|^3}\, d\sigma\, ds \\
&= \frac{1}{4\pi}\frac{1}{\alpha^2}\frac{(\frac{1}{\alpha}K_1 + 1)}{\left|\frac{1}{\alpha}k_1 - 1\right|^3} l_1 l_2.
\end{aligned}$$

At the same time, by exercise 2, we have that $\mathrm{Lk}(Tr(\alpha; C_1), C_2) = \mathrm{Lk}(C_1, C_2)$ for any α which means that we have that

$$|\mathrm{Lk}(C_1, C_2)| = |\mathrm{Lk}(Tr(\alpha; C_1), C_2)| \leq \frac{1}{4\pi}\frac{1}{\alpha^2}\frac{(\frac{1}{\alpha}K_1 + 1)}{\left|\frac{1}{\alpha}k_1 - 1\right|^3} l_1 l_2$$

and thus, since that expression tends to 0 when α tends to infinity, $\mathrm{Lk}(C_1, C_2) = 0$.

Now, we also want to compute the linking number of the Whitehead link using the method of counting signed crossings. The signs of the various crossings can be seen to be



and thus, since $\frac{1}{2}(1-1+1-1)=0$, we've computed that $\mathrm{Lk}=0$.

Another way to conclude that the link integral vanishes is to notice that if we reverse the orientation of one of C_1 or C_2 we get the original turned around a half turn (which is a transformation which can be accomplished by homotopy). As such, $\mathrm{Lk}=-\mathrm{Lk}$ and the conclusion follows.

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Solution 1.

$$\begin{aligned}
\mathbf{r}(s_0+\epsilon,\sigma_0+\nu) &= \mathbf{r}(s_0,\sigma_0)+\epsilon\,\mathbf{r}_s(s_0,\sigma_0)+\nu\,\mathbf{r}_\sigma(s_0,\sigma_0)+\frac{\epsilon^2}{2}\mathbf{r}_{ss}(s_0,\sigma_0)+\epsilon\nu\,\mathbf{r}_{s\sigma}(s_0,\sigma_0)\\
&\quad+\frac{\nu^2}{2}\mathbf{r}_{\sigma\sigma}(s_0,\sigma_0)+O(\epsilon^3)+O(\nu^3)+O(\epsilon\nu^2)+O(\nu\epsilon^2),\\
&= \mathbf{r}-\epsilon\,\mathbf{t}^{[\mathbf{x}]}+\nu\,\mathbf{t}^{[\mathbf{y}]}-\frac{\epsilon^2}{2}\kappa^{[x]}\mathbf{n}^{[\mathbf{x}]}+\frac{\nu^2}{2}\kappa^{[y]}\mathbf{n}^{[\mathbf{y}]}\Big|_{(s_0,\sigma_0)}.
\end{aligned}$$

Solution 2. In general, we have

$$\begin{aligned}
\mathbf{r}_s\wedge\mathbf{r}_\sigma &= -\mathbf{x}'(s)\wedge\mathbf{y}'(\sigma)\\
&= -(\mathbf{t}^{[\mathbf{x}]}+\epsilon\kappa^{[x]}\mathbf{n}^{[\mathbf{x}]}+O(\epsilon^2))\wedge(\mathbf{t}^{[\mathbf{y}]}+\nu\kappa^{[y]}\mathbf{n}^{[\mathbf{y}]}+O(\nu^2))\\
&= \mathbf{t}^{[\mathbf{y}]}\wedge\mathbf{t}^{[\mathbf{x}]}+\epsilon\kappa^{[x]}\mathbf{t}^{[\mathbf{y}]}\wedge\mathbf{n}^{[\mathbf{x}]}+\nu\kappa^{[y]}\mathbf{n}^{[\mathbf{y}]}\wedge\mathbf{t}^{[\mathbf{x}]}+\text{h.o.t.}, \qquad (1)
\end{aligned}$$

where h.o.t. stands for higher order terms.

At a pair of points (s_0,σ_0) where the tangents align, the first term in (1) vanishes and accordingly, the vector $\mathbf{n}$ can not be defined as in the exercise sheet. To show that no definition of $\mathbf{n}$ at (s_0,σ_0) will provide a smooth function $\mathbf{n}$ in an open neighbourhood of (s_0,σ_0), we show that $\lim_{(s,\sigma)\to(s_0,\sigma_0)}\mathbf{n}(s,\sigma)$ depends on the path of approach. Locally, we specify this path by choosing a number $\phi\in[0,\pi)\setminus\{\pi/2\}$ and taking $\nu=\frac{\kappa^{[x]}}{\kappa^{[y]}}\tan\phi\,\epsilon$. Substituting this for ν in (1) yields

$$\begin{aligned}
\mathbf{r}_s\wedge\mathbf{r}_\sigma &= \epsilon\kappa^{[x]}\,(\mathbf{t}^{[\mathbf{y}]}\wedge\mathbf{n}^{[\mathbf{x}]}+\tan\phi\;\mathbf{n}^{[\mathbf{y}]}\wedge\mathbf{t}^{[\mathbf{x}]})+\text{h.o.t.},\\
&= \epsilon\kappa^{[x]}\,(\mathbf{t}^{[\mathbf{y}]}\wedge(\mathbf{n}^{[\mathbf{x}]}\pm\tan\phi\,\mathbf{n}^{[\mathbf{y}]}))+\text{h.o.t.}. \qquad (\because\mathbf{t}^{[\mathbf{x}]}=\pm\mathbf{t}^{[\mathbf{y}]})
\end{aligned}$$

So we must distinguish between two cases: either $\mathbf{n}^{[\mathbf{x}]}=\pm\mathbf{n}^{[\mathbf{y}]}$ or not. In the first (rare) case, we find $\mathbf{r}_s\wedge\mathbf{r}_\sigma=\pm\epsilon\kappa^{[x]}\,(1\pm\tan\phi)\mathbf{b}_y$ where the two $\pm$ lead to four possibilities depending on the geometry of the curves but not on the path of approach. In this case, $\mathbf{n}$ can actually be regularised by picking $\mathbf{n}(s_0,\sigma_0)=\pm\mathbf{b}_y$.

In the second case, by far the most common, the two curves do not have parallel osculating planes at (s_0,σ_0): $\mathbf{n}^{[\mathbf{x}]}\wedge\mathbf{n}^{[\mathbf{y}]}\neq0$. In that case the direction of the vector $\mathbf{r}_s\wedge\mathbf{r}_\sigma$ depends on ϕ. Accordingly, it is impossible to make a choice for $\mathbf{n}(s_0,\sigma_0)$ that would lead to a smooth limit for all ϕ.

Solution 3. *Ad absurdum* Let $\mathbf{N} = \mathbf{r}_s \wedge \mathbf{r}_\sigma$. And assume there exist a positive number $a > 0$ and a regular curve

$$\gamma : t \in (-a, a) \to \big(s(t), \sigma(t)\big), \tag{2}$$

such that both $\gamma(0) = (s_0, \sigma_0)$ and $\mathbf{N}(\gamma(t)) = \mathbf{0}$ for all $t \in (-a, a)$. Then compute

$$\begin{aligned} \mathbf{N}(\gamma(t)) = 0 \quad &\Rightarrow \quad \frac{\partial \mathbf{N}}{\partial s}\frac{ds}{dt} + \frac{\partial \mathbf{N}}{\partial \sigma}\frac{d\sigma}{dt} = 0, \\ &\Rightarrow \quad (\mathbf{y}_\sigma \wedge \mathbf{x}_{ss})\,\dot{s} + (\mathbf{y}_{\sigma\sigma} \wedge \mathbf{x}_s)\,\dot{\sigma} = 0. \end{aligned} \tag{3}$$

A scalar product of (3) by $\mathbf{y}_{\sigma\sigma}$ yields

$$\dot{s} = 0 \quad \text{or} \quad (\mathbf{x}_{ss} \wedge \mathbf{y}_{\sigma\sigma}) \cdot \mathbf{y}_\sigma = 0. \tag{4}$$

But since the curves $\mathbf{x}$ and $\mathbf{y}$ are regular, we have $\mathbf{x}_s \neq \mathbf{0}$ and $\mathbf{y}_\sigma \neq \mathbf{0}$. Accordingly $\mathbf{N}(\gamma) = 0$ implies that every point in the image of γ respects the hypothesis of question 2: the tangent to $\mathbf{x}$ is parallel to that of $\mathbf{y}$ at $\gamma(t)$. However, since $\mathbf{x}_s$ and $\mathbf{y}_\sigma$ are unit vectors, we know that $\mathbf{x}_{ss}$ (resp. $\mathbf{y}_{\sigma\sigma}$) is perpendicular to $\mathbf{x}_s$ (resp. $\mathbf{y}_\sigma$). But since $\mathbf{x}_s = \pm\,\mathbf{y}_\sigma$, we also have that $\mathbf{x}_{ss}$ is perpendicular to $\mathbf{y}_\sigma$. Accordingly, the RHS proposition in the alternative (4) can not occur at (s_0, σ_0) since $\mathbf{x}_{ss} \wedge \mathbf{y}_{\sigma\sigma}$ both can not vanish and must be parallel to $\mathbf{y}_\sigma$. We conclude that the left hand side must occur:

$$\dot{s}(\gamma(0)) = 0. \tag{5}$$

Similarly, the scalar product of (3) with $\mathbf{x}_{ss}$ implies that

$$\dot{\sigma}(\gamma(0)) = 0. \tag{6}$$

Together (5,6) imply that $\dot{\gamma}(0) = 0$ so that the curve γ can not be regular: a contradiction.

Solution 4. Assume that f vanishes for some (s_0, σ_0) (not necessarily meeting the hypothesis of question 2). The implicit function theorem guarantees that if

$$f_s(s_0, \sigma_0) \neq 0, \qquad \text{or} \qquad f_\sigma(s_0, \sigma_0) \neq 0, \tag{7}$$

then there exists a unique open curve γ through (s_0, σ_0) such that $f(\gamma(t)) = 0$.

To conclude we prove that (7) is true. *Ad absurdum*, if (7) is false then at (s_0, σ_0), we have

$$\begin{cases} f = 0, \\ f_s = 0, \\ f_\sigma = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y}_\sigma \wedge \mathbf{x}_s) = 0, \\ (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y}_\sigma \wedge \mathbf{x}_{ss}) = 0, \\ (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y}_{\sigma\sigma} \wedge \mathbf{x}_s) = 0. \end{cases} \quad \Leftrightarrow \quad (\mathbf{y} - \mathbf{x}),\ \mathbf{y}_\sigma,\ \mathbf{y}_{\sigma\sigma},\ \mathbf{x}_s \text{ and } \mathbf{x}_{ss} \text{ are coplanar.} \tag{8}$$

In particular, the last statement implies that $\mathbf{x}$ and $\mathbf{y}$ share an osculating plane at (s_0, σ_0): a contradiction.

Solution 5. If (s, σ_0) is such that $\mathbf{x}_s$ is parallel to $\mathbf{y}_\sigma$, then $(\mathbf{r}_s \wedge \mathbf{r}_\sigma)|_{(s_0,\sigma_0)} = \mathbf{0}$ and in particular $f(s_0, \sigma_0) = 0$. Then question 4 implies that such a curve exists.