Solution 1 Both factors change sign in the numerator while the denominator is left unchanged.

Solution 2

- Under a translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^3$:

$$(\mathbf{x}(\sigma) + \mathbf{a}) - (\mathbf{x}(s) + \mathbf{a}) = \mathbf{x}(\sigma) - \mathbf{x}(s)$$

$$(\mathbf{x}(\sigma) + \mathbf{a})' \wedge (\mathbf{x}(s) + \mathbf{a})' = \mathbf{x}'(\sigma) \wedge \mathbf{x}'(s)$$

- Under a rotation $\mathbf{x} \mapsto R\mathbf{x}$ for some $R \in \mathrm{SO}(3)$:

$$R\mathbf{x}(\sigma) - R\mathbf{x}(s) = R(\mathbf{x}(\sigma) - \mathbf{x}(s)).$$

$$\begin{aligned}(R\mathbf{x}(\sigma))' \wedge (R\mathbf{x}(s))' &= (R\mathbf{x}'(\sigma))^{\times}(R\mathbf{x}'(s)) \\ &= |R| R^{-T} (\mathbf{x}'(\sigma))^{\times} R^{-1} (R\mathbf{x}'(s)) \\ &= R\,(\mathbf{x}'(\sigma) \wedge \mathbf{x}'(s)). \end{aligned} \quad (1)$$

Accordingly, under rigid rotation, the denominator is left unchanged while the numerator becomes

$$\begin{aligned}(R\,\mathbf{x}(\sigma) - R\,\mathbf{x}(s)) \cdot (R\mathbf{x}(\sigma))' \wedge (R\mathbf{x}(s))' &= \Big[R\,(\mathbf{x}(\sigma) - \mathbf{x}(s))\Big] \cdot \Big[R\,(\mathbf{x}'(\sigma) \wedge \mathbf{x}'(s))\Big], \\ &= (\mathbf{x}(\sigma) - \mathbf{x}(s)) \cdot R^T\, R\,(\mathbf{x}'(\sigma) \wedge \mathbf{x}'(s)), \\ &= (\mathbf{x}(\sigma) - \mathbf{x}(s)) \cdot (\mathbf{x}'(\sigma) \wedge \mathbf{x}'(s)). \end{aligned} \quad (2)$$

- Under a dilation $\mathbf{x} \mapsto \lambda\mathbf{x}$, we get the factor λ^3 both in the nominator and the denominator, so Wr is invariant under dilations as well.

Note that if $R \in O(3) \setminus SO(3)$, then $|R| = -1$ and a minus sign would appear in (1). Accordingly, the Writhe flips sign under isometries of determinant -1.

Solution 3 For a planar curve, the three vectors $(\mathbf{x}(\sigma) - \mathbf{x}(s))$, $\mathbf{x}'(\sigma)$ and $\mathbf{x}'(s)$ are coplanar. Their scalar triple product vanishes and $I_{\mathrm{Wr}} = 0$.

We can obtain the same result making use of question 2 by noticing that the curve is left unchanged by reflexion through its plane (an isometry of determinant -1) while the Writhe must change sign.

Solution 4 Given $\mathbf{e} \cdot (\mathbf{e}_s \wedge \mathbf{e}_\sigma) = 0$, we have

$$(\mathbf{x}(s) - \mathbf{x}(\sigma)) \cdot (\mathbf{x}'(s) \wedge \mathbf{x}'(\sigma)) = 0,$$

which is equivalent to

$$\mathbf{x}'(s) \cdot [\mathbf{x}'(\sigma) \wedge (\mathbf{x}(s) - \mathbf{x}(\sigma))] = 0 \quad (3)$$

Differentiate (3) with respect to s to get

$$\mathbf{x}''(s) \cdot [\mathbf{x}'(\sigma) \wedge (\mathbf{x}(s) - \mathbf{x}(\sigma))] = 0. \quad (4)$$

But $\mathbf{x}' = v\mathbf{t}$ with $v = ||\mathbf{x}'|| = \frac{\mathrm{d}S}{\mathrm{d}s}$ where S is the arc-length along $\mathbf{x}$, so we have

$$\mathbf{x}'' = v'\mathbf{t} + v\mathbf{t}' = v'\mathbf{t} + v^2\kappa\mathbf{n}. \tag{5}$$

Eqs. (3-5) imply that $\mathbf{x}'(\sigma) \wedge (\mathbf{x}(s) - \mathbf{x}(\sigma))$ is perpendicular to the $(\mathbf{t}, \mathbf{n})$-plane so it must be along $\mathbf{b}(s)$:

$$\mathbf{b}(s) = \pm \frac{\mathbf{x}'(\sigma) \wedge (\mathbf{x}(s) - \mathbf{x}(\sigma))}{||\mathbf{x}'(\sigma) \wedge (\mathbf{x}(s) - \mathbf{x}(\sigma))||}. \tag{6}$$

Differentiate (6) with respect to s to get

$$\mathbf{b}'(s) = \pm \frac{\mathbf{x}'(\sigma) \wedge \mathbf{x}'(s)}{||\mathbf{x}'(\sigma) \wedge (\mathbf{x}(s) - \mathbf{x}(\sigma))||} - \mathbf{w},$$

where $\mathbf{w}$ is parallel to $\mathbf{b}$.

Finally $\frac{d\mathbf{b}}{ds} = \frac{d\mathbf{b}}{dS}\frac{dS}{ds} = -v\tau\mathbf{n}$ so that

$$\begin{aligned} -v\,\kappa\,\tau = \mathbf{b}'(s) \cdot \kappa\mathbf{n}(s) &= \mathbf{b}'(s) \cdot \frac{\mathbf{x}''(s) - \frac{v'(s)}{v(s)}\mathbf{x}'(s)}{v^2} \\ &= \frac{\mathbf{x}'(\sigma) \wedge \mathbf{x}'(s)}{||\mathbf{x}'(\sigma) \wedge (\mathbf{x}(s) - \mathbf{x}(\sigma))||} \cdot \frac{\mathbf{x}''(s)}{v^2} \\ &= \frac{(\mathbf{x}'(s) \wedge \mathbf{x}''(s)) \cdot \mathbf{x}'(\sigma)}{||\mathbf{x}'(\sigma) \wedge (\mathbf{x}(s) - \mathbf{x}(\sigma))||v^2}. \end{aligned}$$

But $(\mathbf{x}'(s) \wedge \mathbf{x}''(s))$ is perpendicular to both $\mathbf{t}$ and $\mathbf{n}$ and so it must be parallel to $\mathbf{x}'(\sigma) \wedge (\mathbf{x}(s) - \mathbf{x}(\sigma))$ and so it is perpendicular to $\mathbf{x}'(\sigma)$. Hence the torsion vanishes identically. The curve is planar.

Solution 5

Step 1: Consider the planar problem of finding a circle through two points A and X and tangent to a unit vector $\mathbf{t}$ at B. Let ℓ be the straight line through A and B, θ be the angle between ℓ and the line q subtended by $\mathbf{t}$, a be the distance between A and B, O the centre of the circle, p the perpendicular to AB through O, and P and Q the respective intersections of p with AB and q. The angle AOP has amplitude θ since the triangles POA and AQO are similar. Then in the right-angled triangle θ, we find

$$\sin\theta = \frac{a}{2\,R}, \tag{7}$$

where R is the radius of the circle.

Step 2: Modify the expression for writhe. We have already showed (technically, we did it for Lk but the proof is identical for Wr) that

$$Wr = \frac{1}{4\,\pi} \int\int \mathbf{e} \cdot (\mathbf{e}_s \times \mathbf{e}_\sigma)\; ds\, d\sigma, \tag{8}$$

where $e(s, \sigma) = \frac{\mathbf{x}(\sigma) - \mathbf{x}(s)}{||\mathbf{x}(\sigma) - \mathbf{x}(s)||}$. Then consider the fact that on the one hand

$$(\mathbf{e} \times \mathbf{e}_s) \times (\mathbf{e} \times \mathbf{e}_\sigma) = \Big(\mathbf{e} \cdot (\mathbf{e}_s \times \mathbf{e}_\sigma)\Big)\;\; \mathbf{e}, \tag{9}$$

and on the other hand, the fact that

$$\mathbf{e} \times \mathbf{e}_\sigma = \mathbf{e} \times \frac{\mathbf{x}'(\sigma)}{a(s,\sigma)} = \frac{\sin\theta_1}{a(s,\sigma)}\,\mathbf{n}_1 \overset{(7)}{=} \frac{\mathbf{n}_1}{d(s,\,\sigma)}, \quad \text{and} \quad \mathbf{e} \times \mathbf{e}_s = \frac{\mathbf{n}_2}{d(\sigma,\,s)}. \tag{10}$$

where θ_1 (resp. θ_2) is the angle between the straight lines subtended by $\mathbf{e}$ and by $\mathbf{e}_\sigma$ (resp. $\mathbf{e}_s$), $a(s,\,\sigma) = ||\mathbf{x}(\sigma) - \mathbf{x}(s)||$, and $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) is the unit vector along $\mathbf{e} \times \mathbf{e}_\sigma$ (resp. $\mathbf{e} \times \mathbf{e}_s$). Substituting (10) in (9) gives the result.

Solution 6

- $\mathbf{x}(s) = (r\cos s,\, r\sin s,\, rps), \quad \mathbf{x}'(s) = (-r\sin s,\, r\cos s,\, rp)$.
- $\mathbf{r}(s,\sigma) := \mathbf{x}(s) - \mathbf{x}(\sigma) = r(\cos s - \cos\sigma,\, \sin s - \sin\sigma,\, p(s-\sigma))$.
- $||\mathbf{r}|| = r\sqrt{4\sin^2\left(\frac{s-\sigma}{2}\right) + p^2(s-\sigma)^2}$.
- $$\begin{aligned} \mathbf{r}\cdot(\mathbf{r}_\sigma \wedge \mathbf{r}_s) &= r^3 \begin{vmatrix} \cos s - \cos\sigma & \sin s - \sin\sigma & p(s-\sigma) \\ -\sin s & \cos s & p \\ -\sin\sigma & \cos\sigma & p \end{vmatrix} \\ &= r^3 p \left[4\sin^2\left(\frac{s-\sigma}{2}\right) - (s-\sigma)\sin(s-\sigma)\right]. \end{aligned}$$
- $$\begin{aligned} I_{\mathrm{Wr}} &= \frac{\mathbf{r}\cdot(\mathbf{r}_\sigma \wedge \mathbf{r}_s)}{||\mathbf{r}||^3} \\ &= \frac{p}{2}\cdot\frac{\sin^2\left(\frac{s-\sigma}{2}\right) - \frac{s-\sigma}{2}\sin\left(\frac{s-\sigma}{2}\right)\cos\left(\frac{s-\sigma}{2}\right)}{\left[\sin^2\left(\frac{s-\sigma}{2}\right) + p^2\left(\frac{s-\sigma}{2}\right)^2\right]^{\frac{3}{2}}}. \end{aligned}$$

As indicated in the hint we compute

$$\left(\frac{w}{\sqrt{\sin^2 w + p^2 w^2}}\right)' = \frac{\sin^2 w - w\sin w\cos w}{(\sin^2 w + p^2 w^2)^{\frac{3}{2}}}. \tag{11}$$

Finally, compute

$$\begin{aligned}
\mathrm{Wr}(\mathbf{x}) &= \frac{1}{4\pi}\int_0^L d\sigma \int_0^L I_{\mathrm{Wr}}(s,\sigma)\,ds \\
&= \frac{1}{4\pi}\int_0^L d\sigma \int_{-\frac{\sigma}{2}}^{\frac{L-\sigma}{2}} \frac{p}{2}\,\frac{\sin^2 t - t\sin t\cos t}{(\sin^2 t + p^2t^2)^{\frac{3}{2}}}\,2\,dt \quad \text{where } t = \frac{s-\sigma}{2} \\
&= \frac{p}{4\pi}\lim_{\varepsilon\to 0^+}\left\{\int_0^L d\sigma\left[\int_{-\frac{\sigma}{2}}^{-\varepsilon}\frac{\sin^2 t - t\sin t\cos t}{(\sin^2 t + p^2t^2)^{\frac{3}{2}}}\,dt + \int_{\varepsilon}^{\frac{L-\sigma}{2}}\frac{\sin^2 t - t\sin t\cos t}{(\sin^2 t + p^2t^2)^{\frac{3}{2}}}\,dt\right]\right\} \\
&\overset{(11)}{=} \frac{p}{4\pi}\int_0^L d\sigma \lim_{\varepsilon\to 0^+}\left\{\left[\frac{t}{\sqrt{\sin^2 t + p^2t^2}}\right]_{-\frac{\sigma}{2}}^{-\varepsilon} + \left[\frac{t}{\sqrt{\sin^2 t + p^2t^2}}\right]_{\varepsilon}^{\frac{L-\sigma}{2}}\right\}, \qquad (12) \\
&= \frac{p}{4\pi}\left[\int_0^L \frac{(L-\sigma)/2}{\sqrt{\sin^2(L-\sigma)/2 + p^2\,(L-\sigma)^2/4}}\,d\sigma + \int_0^L \frac{\sigma/2}{\sqrt{\sin^2\sigma/2 + p^2\,\sigma^2/4}}\,d\sigma\right] \\
&\qquad -\frac{p\,L}{2\pi}\lim_{\varepsilon\to 0}\frac{\varepsilon}{\sqrt{\sin^2\varepsilon + p^2\,\varepsilon^2}}, \\
&= \frac{p}{4\pi}\left[\int_{L/2}^0 \frac{-2\,u\,du/2}{\sqrt{\sin^2 u + p^2\,u^2}} + \int_0^{L/2}\frac{2\,v\,dv}{\sqrt{\sin^2 v + p^2\,v^2}} - \frac{2L}{\sqrt{1+p^2}}\right], \\
&\qquad \text{with } u = \frac{L-\sigma}{2} \text{ and } v = \frac{\sigma}{2}, \\
&= \frac{p}{\pi}\int_0^{\frac{L}{2}}\frac{v\,dv}{\sqrt{\sin^2 v + p^2v^2}} - \frac{p\,L}{2\pi}\frac{1}{\sqrt{1+p^2}}. \qquad (13)
\end{aligned}$$

To study the asymptotic behaviour, we first take a derivative w.r.t. L which yields

$$\frac{\mathrm{dWr}}{\mathrm{d}L} = \frac{p}{2\pi}\left(\frac{L}{\sqrt{\sin^2 L + p^2L^2}} - \frac{1}{\sqrt{1+p^2}}\right), \tag{14}$$

$$= \frac{p}{2\pi}\left(\frac{1}{\sqrt{\frac{1}{L^2}\sin^2 L + p^2}} - \frac{1}{\sqrt{1+p^2}}\right) \xrightarrow[L\to\infty]{} \frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}}\right). \tag{15}$$

Hence for large L, we have the following asymptotic behaviour for Wr:

$$\mathrm{Wr}(L) \overset{L\to\infty}{\sim} \frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}}\right) L. \tag{16}$$

This asymptotic result can also be derived somewhat more rigorously by the following analysis: change variables according to $g = 1/L$ so that $L \to \infty$ becomes $g \to 0^+$. The differential equation (14) becomes singular:

$$-g^2\frac{\mathrm{dWr}}{\mathrm{d}g} = \frac{p}{2\pi}\left(\frac{1}{\sqrt{g^2\,\sin^2 1/g + p^2}} - \frac{1}{\sqrt{1+p^2}}\right). \tag{17}$$

We want to study the differential equation (17) as $g \to 0$. First note that $|p| \le \sqrt{g^2 \sin^2 1/g + p^2} \le \sqrt{g^2 + p^2}$. Accordingly

$$\frac{1}{|p|} \ge \frac{1}{\sqrt{g^2 \sin^2 1/g + p^2}} \ge \frac{1}{\sqrt{g^2+p^2}} = \frac{1}{|p|} - \frac{g^2}{2\,|p|^3} + O(g^4), \tag{18}$$

so that

$$\frac{1}{\sqrt{g^2 \sin^2 1/g + p^2}} = \frac{1}{|p|} + O(g^2). \tag{19}$$

Next we expand $\mathrm{Wr}(g)$ close to $g = 0$ as

$$\mathrm{Wr}(g) = w_{-1}\, g^{-1} + w_0 + w_1\, g + O(g^2) \Rightarrow \mathrm{Wr}'(g) = -\frac{w_{-1}}{g^2} + w_1 + O(g), \tag{20}$$

where w_{-1}, w_0 and w_1 are constant coefficients that we will try to find.

Substituting (19) and (20) in (17) yields

$$-w_{-1} + w_1\, g^2 + O(g^3) = \frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}} + O(g^2)\right). \tag{21}$$

Hence we find $w_{-1} = \frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}}\right)$ but note that this analysis does not prescribe w_0 or w_1. Substituting our result and $g = L^{-1}$ in (20) gives

$$\mathrm{Wr}(L) = \frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}}\right) L + w_0 + w_1\frac{1}{L} + O\left(\frac{1}{L^2}\right) \overset{L\to\infty}{\sim} \frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}}\right) L. \tag{22}$$

We see that we have gathered a little more information as to the behaviour of the solutions at higher order in $1/L$. To find w_0, we could write an expansion close to $L = 0$ and then enforce that both expansions must match smoothly at say $L = 1$. That sort of argument is useful to obtain approximate solutions to many non-linear problems and is well worth keeping in mind.