**Differential Geometry of Framed Curves**
Prof. John Maddocks
Session 9: Solutions — T. Lessinnes

# 1 Lk and Wr for curves on a sphere

**1.** The curve and a surface offset can be homotopied to opposite halves of their respective spheres. They are then separated by a plane and so their link vanishes.

**2.** See correction of session 6. The total twist, is an integral of the local twist and so it must also vanish.

**3.** Considering $Lk = Tw + Wr$ for the surface framing, and the results in 1. and 2., immediately yields $Wr(\mathbf{r}) = 0$. The important point is that Writhe only depends on the curve and not on the framing so that now that we have established this, we can use it for any other framings of curves on spheres.

The integrand of Writhe need not vanish and in general does not vanish as it is easy to find a pair of points $\mathbf{x}_1$ and $\mathbf{x}_2$ on the sphere and define vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ tangent to the sphere respectively at $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{x}_2 - \mathbf{x}_1$ are not coplanar.

**4.** The self-link of a curve is the link between it and an offset defined by its Serret-Frenet framing. For it to be well defined, the curve must have non-vanishing curvature. Consider an arc-length parameterisation $\mathbf{r}(s)$ of the regular curve $\mathbf{r}$. Because $\mathbf{r}$ is on a sphere, we have

$$\mathbf{r} \cdot \mathbf{r} = R^2 \Rightarrow \mathbf{r} \cdot \mathbf{r}' = 0 \Rightarrow \mathbf{r}' \cdot \mathbf{r}' + \mathbf{r} \cdot \mathbf{r}'' = 0 \Rightarrow \mathbf{r} \cdot \mathbf{r}'' = -1. \tag{1}$$

Therefore, $\mathbf{r}''$ can not vanish and neither can $\kappa$. Finally, substituting $\mathbf{r}'' = \kappa\,\mathbf{n}$ and $\mathbf{r}(s) = \mathbf{N}(s)/R$ (where $\mathbf{N}(s)$ is the normal to the sphere at $\mathbf{r}(s)$) in (1) gives $\mathbf{N} \cdot \mathbf{n} = -1/(R\kappa) < 0$ hence a Frenet offset is strictly inside the sphere, and we can use a similar homotopy to what was done in question 1 to show that self-link does vanish.

**5.** The total Frenet Tw vanishes by application of C-F-W and question 3. and 4.. This is actually a result first attributed to Fenchel (W. Fenchel, "Über einen Jacobischen Satz der Kurventheorie", Mathematics Jour., vol. 39 (1934), pp. 95–97.)

No, the local twist of the frenet frame need not vanish: any non-planar spherical curve serves as a counter-example.

**6.** We have seen that the surface framing is natural. For any smooth –not necessarily on a sphere– curve, the register angle between two natural framings is always constant (the proof of this comes by directly substituting the zero local twists of natural framings in the definition of the register angle). Then, the fact that the surface framing of a spherical curve is natural and closed implies that all natural framings of spherical curves are closed.

# 2 An open-ended problem

C-F-W says nothing about those three ribbons because they are not closed. A potential strategy to bridge the gap is to consider closures of them... see coming lecture.