Differential Geometry of Framed Curves
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SESSION 10: SOLUTIONS — T. LESSINNES

1 Framings of closed curves (revisiting material from last lecture)

1. Existence of a closed framing.

a) First, because the curve is regular, we can re-parameterise it by arc-length so that we are able to apply the results of session 2 (we keep the letter s as a parameter and simply assume that it is now arc-length). Then we pick an arbitrary smooth function, say $\mathsf{u}_3(s) = \cos s$ and pick a unit vector d_{10} perpendicular to $\mathbf{x}'(0)$. Then we build a framing $D(s) = \{\mathbf{d}_1(s),\, \mathbf{d}_2(s),\, \mathbf{d}_3(s)\}$ of $\mathbf{x}$ by solving the following initial value problem

$$\begin{aligned} \mathbf{d}_i'(s) &= \big(\mathsf{u}_3(s)\,\mathbf{d}_3(s) + \mathbf{x}' \times \mathbf{x}''\big) \times \mathbf{d}_i(s), \\ \mathbf{d}_1(0) &= \mathbf{d}_{10}, \\ \mathbf{d}_2(0) &= \mathbf{x}'(0) \times \mathbf{d}_{10}, \\ \mathbf{d}_3(0) &= \mathbf{x}'(0). \end{aligned} \tag{1}$$

See session 2 for a proof that the solution of (1) is indeed an adapted framing to $\mathbf{x}$ and that the third component of its Darboux vector is u_3. In particular, because $\mathbf{x}$ is smoothly closed, we have $\mathbf{d}_3(0) = \mathbf{d}_3(L)$. However, in general there exists an angle ϕ (that depends on the function u_3 and on the shape of the curve) such that

$$D(L) = D(0)\, R_{3,\phi}. \tag{2}$$

If ϕ is an integer multiply of $2\,\pi$, then we were lucky and our function u_3 provided a closed framing.

b) If not, we define a new function

$$\mathsf{q}_3(s) = \mathsf{u}_3(s) - \phi/L. \tag{3}$$

and define an associated framing $C(s) = \{\mathbf{c}_1(s),\, \mathbf{c}_2(s),\, \mathbf{c}_3(s)\}$ as the solution of the IVP

$$\begin{aligned} \mathbf{c}_i'(s) &= \big(\mathsf{q}_3(s)\,\mathbf{d}_3(s) + \mathbf{x}' \times \mathbf{x}''\big) \times \mathbf{c}_i(s), \\ \mathbf{c}_1(0) &= \mathbf{d}_{10}, \\ \mathbf{c}_2(0) &= \mathbf{x}'(0) \times \mathbf{d}_{10}, \\ \mathbf{c}_3(0) &= \mathbf{x}'(0). \end{aligned} \tag{4}$$

Note in particular that $C(0) = D(0)$.
The register angle between D and C is given by

$$\varphi(s) = \int_0^s \mathsf{q}_3(s') - \mathsf{u}_3(s')\; ds' = -\phi\, \frac{s}{L}. \tag{5}$$

In particular, we have

$$C(L) = D(L)\, R_{3,\,\varphi(L)} = D(0) = C(0), \tag{6}$$

so that C is a closed framing.

2. Writhe framings are always closed.

a) Let $W(s) = \{\mathbf{w}_1(s),\, \mathbf{w}_2(s),\, \mathbf{w}_3(s)\}$ be a Writhe framing with $\mathbf{w}_1(0) = \mathbf{d}_{10}$ as defined in the previous question. That is, the local twist of $W(s)$ is $\omega_3(s) = -\frac{1}{2}\int_0^L \mathrm{wr}(s,\sigma)\, d\sigma)$. Let $\varphi(s)$ be the register angle between W and the closed framing C from the previous question. We have

$$\varphi(s) = \int_0^s \mathsf{q}_3(s') - \omega_3(s')\; ds' \tag{7}$$

b) Accordingly, we have

$$\varphi(L) = 2\,\pi(Tw(C) + Wr(x)) = 2\,\pi Lk(\mathbf{x};\, C). \tag{8}$$

Hence

$$W(L) = C(L)\, R_{3,\,-\varphi_L} = C(L)\, R_{3,\,2\,k\,\pi} = C(L) = C(0) = W(0). \tag{9}$$

So that $W(s)$ is a closed framing.

3. Writhe framings have zero link. Now that we know that $W(s)$ is closed, we can apply the CFW theorem which readily gives $Lk = 0$.

4. $Wr(\mathbf{x}) = 0 \Rightarrow$ parallel transport framings are closed. Let $P(s)$ be a parallel transport framing. Then by definition, it has vanishing local twist. Then define $\psi(s)$ as the register angle between P and W, the Writhe framing defined in question 2. Then, we have

$$\psi(L) - \psi(0) = -\int_0^L \omega_3\; ds = 2\,\pi\, Wr(\mathbf{x}) = 0. \tag{10}$$

Hence $\psi(L) = \psi(0)$ and since W is closed so must P.

5. The same computation as in 4. shows that $\psi(L) - \psi(0) = Wr(\mathbf{x})$. So we have

$$P(L) = W(L)\, R_{3,\,\psi(L)} = W(0)\, R_{3,\,\psi(L)} = P(0)\, R_{3,\,-\psi(0)} R_{3,\,\psi(L)} = P(0)\, R_{3,2\,\pi\, Wr(\mathbf{x})}. \tag{11}$$

2 Writhe of a figure 8 curve

We start with proving equation (2) of the exercise sheet. We have

$$\mathbf{x}(s) = \begin{pmatrix} \cos s \\ \frac{\sin 2s}{2} \\ \nu\, \sin s \end{pmatrix}, \qquad \text{and therefore} \qquad \mathbf{x}'(s) = \begin{pmatrix} -\sin s \\ \cos 2s \\ \nu\, \cos s \end{pmatrix}. \tag{12}$$

We compute

$$\mathbf{x}(s) - \mathbf{x}(\sigma) = \begin{pmatrix} -2\, \sin \frac{s+\sigma}{2}\, \sin \frac{s-\sigma}{2} \\ \sin(s-\sigma)\, \cos(s+\sigma) \\ 2\,\nu\, \sin \frac{s-\sigma}{2}\, \cos \frac{s+\sigma}{2} \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}'(s) \times \mathbf{x}'(\sigma) = \begin{pmatrix} \nu\,(\cos 2s\, \cos\sigma - \cos s\, \cos 2\sigma) \\ \nu\, \sin(s-\sigma) \\ \sin\sigma\, \cos 2s - \sin s \cos 2\sigma \end{pmatrix}, \tag{13}$$

and

$$\begin{aligned} \cos 2s\, \cos\sigma - \cos s\, \cos 2\sigma &= (2\, \cos^2 s - 1)\, \cos\sigma - \cos s\, (2\, \cos^2 \sigma - 1) \\ &= (\cos s - \cos\sigma)\, (1 + 2\, \cos s\, \cos\sigma) \\ &= -2 \sin \tfrac{s+\sigma}{2}\, \sin \tfrac{s-\sigma}{2} \Big(1 + 2\, \cos s\, \cos\sigma\Big). \end{aligned} \tag{14}$$

$$\sin\sigma\, \cos 2s - \sin s \cos 2\sigma = 2\, \sin \tfrac{\sigma-s}{2}\, \cos \tfrac{\sigma+s}{2} \Big(1 + 2\, \sin\sigma\, \sin\sigma\Big). \tag{15}$$

Gathering, we find that

$$
\begin{aligned}
&\Big(\mathbf{x}(s)-\mathbf{x}(\sigma)\Big)\cdot\big(\mathbf{x}(s)\times\mathbf{x}(\sigma)\big)\\
&\quad=\nu\Big[4\,\sin^2\tfrac{s+\sigma}{2}\,\sin^2\tfrac{s-\sigma}{2}\,(1+2\,\cos s\,\cos\sigma)+\sin^2(s-\sigma)\,\cos(s+\sigma)\\
&\qquad\quad-4\,\sin^2\tfrac{s-\sigma}{2}\,\cos^2\tfrac{s+\sigma}{2}\,(1+2\,\sin s\,\sin\sigma)\Big],\\
&\quad=4\,\nu\,\sin^2\tfrac{s-\sigma}{2}\Big[\sin^2\tfrac{s+\sigma}{2}\,(1+2\,\cos s\,\cos\sigma)+\cos^2\tfrac{s-\sigma}{2}\,\cos(s+\sigma)-\cos^2\tfrac{s+\sigma}{2}\,(1+2\,\sin s\,\sin\sigma)\Big],\\
&\quad=4\,\nu\,\sin^2\tfrac{s-\sigma}{2}\Big[\cos^2\tfrac{s-\sigma}{2}\,\cos(s+\sigma)-\cos(s+\sigma)-\cos^2\tfrac{s+\sigma}{2}\big(\cos(s-\sigma)-\cos(s+\sigma)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\sin^2\tfrac{s+\sigma}{2}\big(\cos(s-\sigma)+\cos(s+\sigma)\big)\Big],\\
&\quad=4\,\nu\,\sin^2\tfrac{s-\sigma}{2}\,\cos(s+\sigma)\Big[1-\sin^2\tfrac{s-\sigma}{2}-1-\cos(s-\sigma)+1\Big],\\
&\quad=4\,\nu\,\sin^4\tfrac{s-\sigma}{2}\,\cos(s+\sigma).
\end{aligned}
\tag{16}
$$

We also have

$$
||\mathbf{x}(s)-\mathbf{x}(\sigma)||^2=4\,\sin^2\tfrac{s-\sigma}{2}\Big[\sin^2\tfrac{s+\sigma}{2}+\cos^2\tfrac{s-\sigma}{2}\,\cos^2(s+\sigma)+\nu^2\,\cos^2\tfrac{s+\sigma}{2}\Big].
\tag{17}
$$

Gathernig (16,17), we find

$$
\begin{aligned}
Wr(\mathbf{x})&=\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{4\,\nu\,\sin^4\frac{s-\sigma}{2}\,\cos(s+\sigma)}{8\,\left|\sin^3\frac{s-\sigma}{2}\right|\,\Big[\sin^2\frac{s+\sigma}{2}+\cos^2\frac{s-\sigma}{2}\,\cos^2(s+\sigma)+\nu^2\,\cos^2\frac{s+\sigma}{2}\Big]^{3/2}}\,ds\,d\sigma,\\
&=\frac{\nu}{4\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|\sin(\frac{\mu}{2}-\sigma)\right|\,\cos\mu}{2\Big[\sin^2\frac{\mu}{2}+\nu^2\,\cos^2\frac{\mu}{2}+\cos^2(\frac{\mu}{2}-\sigma)\,\cos^2\mu\Big]^{3/2}}\,d\mu\,d\sigma,\\
&=\frac{\nu}{4\pi}\int_{-\pi}^{\pi}\int_{0}^{\pi}\frac{\sin\tau\,\cos\mu}{\Big[\sin^2\frac{\mu}{2}+\nu^2\,\cos^2\frac{\mu}{2}+\cos^2\tau\,\cos^2\mu\Big]^{3/2}}\,d\tau\,d\mu,
\end{aligned}
\tag{18}
$$

where the second equality comes by changing variables in the inner integral $\mu=s+\sigma$ and shifting the domain of integration using the fact that the integrand is periodic; the third equality comes after permuting the order of integration and changing variables according to $\tau=\sigma-\mu/2$.

It turns out that we can explicitly integrate the inner integral of (18) using

$$
\int_0^{\pi}\frac{\sin\tau\;d\tau}{(A+B\,\cos^2\tau)^{3/2}}=\int_{-1}^{1}\frac{dx}{(A+B\,x^2)^{3/2}}=\frac{1}{A}\left[\frac{x}{\sqrt{A+B\,x^2}}\right]_{-1}^{1}=\frac{2}{A\sqrt{A+B}},
\tag{19}
$$

where we changed variables according to $x=\cos\tau$.

Substituting (19) in (18), we find

$$
Wr(\mathbf{x})=\frac{\nu}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\mu\;d\mu}{\Big(\sin^2\frac{\mu}{2}+\nu^2\,\cos^2\frac{\mu}{2}\Big)\,\sqrt{\sin^2\frac{\mu}{2}+\cos^2\mu+\nu^2\,\cos^2\frac{\mu}{2}}}.
\tag{20}
$$

And we get to (2) from the exercise sheet by noting that the integrand in (20) is even.

Next we want to study the limit of (20) as $\nu\to 0$. To this intend, we will bound the integral above and below and show that both bounds tend to 1 as $\nu\to 0^+$.

1. We find an upper bound to the function $g(\mu,\nu) = \frac{\cos\mu}{\sqrt{\nu^2\cos^2\frac{\mu}{2}+\sin^2\frac{\mu}{2}+\cos^2\mu}}$ on the domain $\mu \in (0,\pi/2)$. We note that for a given μ, g is a monotonously decreasing function of ν. Hence we have

$$g(\mu,\,\nu) \leq g(\mu,1) = \frac{\cos\mu}{\sqrt{1+\cos^2\mu}} \leq \frac{\cos\mu}{\sqrt{\cos^2\mu}} = 1. \tag{21}$$

Also, consider the function $h(\mu,\,a) = \frac{1}{\sqrt{a+\cos^2\mu}}$ with $a \in (0,1)$. We have

$$h(\mu,\,a) \geq \frac{1}{\sqrt{a+1}} \geq 1-a, \tag{22}$$

where the last inequality can be proven by squaring:

$$1 \geq (1-a)^2(1+a) \Leftrightarrow 1 \geq \left(1-a^2\right)(1-a) \Leftrightarrow 1+a \geq a^2,$$

which is manifestly true for $a \in (0,1)$. Since we have that $\nu^2\cos^2\frac{\mu}{2}+\sin^2\frac{\mu}{2} \in (0,1)$, this proves the inequality (4) of the question sheet.

2. Define $f(\mu,\,\nu) = \frac{\cos\mu\;d\mu}{\left(\sin^2\frac{\mu}{2}+\nu^2\cos^2\frac{\mu}{2}\right)\sqrt{\sin^2\frac{\mu}{2}+\cos^2\mu+\nu^2\cos^2\frac{\mu}{2}}}$. To make use of the inequalities found in **1.**, we need to split the domain of integration of (2) from the question sheet:

$$Wr(\mathbf{x};\,\nu) = \frac{\nu}{\pi}\int_0^{\frac{\pi}{2}} f(\mu,\,\nu)\;d\mu + \frac{\nu}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(\mu,\,\nu)\;d\mu.$$

Then we have

$$\lim_{\nu\to 0^+} Wr(\mathbf{x};\,\nu) = \lim_{\nu\to 0^+}\frac{\nu}{\pi}\int_0^{\frac{\pi}{2}} f(\mu,\,\nu)\;d\mu \quad + \quad 0. \tag{23}$$

We next we provide upper and lower bounds for the integral in (23) by noting that all factors in f are positive when $\mu \in (0,\frac{\pi}{2})$:

$$\int_0^{\frac{\pi}{2}} \frac{\cos\mu\;d\mu}{\sin^2\frac{\mu}{2}+\nu^2\cos^2\frac{\mu}{2}} - \int_0^{\frac{\pi}{2}}\cos\mu\;d\mu \leq \int_0^{\frac{\pi}{2}} f(\mu,\nu)\,d\mu \leq \int_0^{\frac{\pi}{2}} \frac{d\mu}{\sin^2\frac{\mu}{2}+\nu^2\cos^2\frac{\mu}{2}}, \tag{24}$$

where the first inequality comes from (4) in the question sheet and the second inequality comes from (3) in the question sheet. To conclude, we compute

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{d\mu}{\sin^2\frac{\mu}{2}+\nu^2\cos^2\frac{\mu}{2}} &= \frac{2}{\nu}\int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{\nu^2}\tan^2\frac{\mu}{2}}\,\frac{d\mu}{2\,\nu\cos^2\frac{\mu}{2}},\\ &= \frac{2}{\nu}\int_0^{1/\nu}\frac{1}{1+x^2}\;dx,\\ &= \frac{2}{\nu}\;\arctan\left[1/\nu\right],\end{aligned} \tag{25}$$

where $x = \frac{1}{\nu}\tan\mu/2$.

We also compute

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos\mu}{\sin^2\frac{\mu}{2}+\nu^2\cos^2\frac{\mu}{2}}\;d\mu &= 2\int_0^1 \frac{(1-y^2)\;dy}{(1+y^2)\,(\nu^2+y^2)},\\ &= -\frac{4}{1-\nu^2}\int_0^1\frac{dy}{1+y^2} + 2\,\frac{1+\nu^2}{1-\nu^2}\int_0^1\frac{dy}{\nu^2+y^2},\\ &= -\frac{\pi}{1-\nu^2} + 2\,\frac{1+\nu^2}{1-\nu^2}\,\frac{1}{\nu}\;\arctan[1/\nu],\end{aligned} \tag{26}$$

where the first equality comes by changing variables according to $y = \tan\mu/2$.

Substituting (24-26) in (23), we find

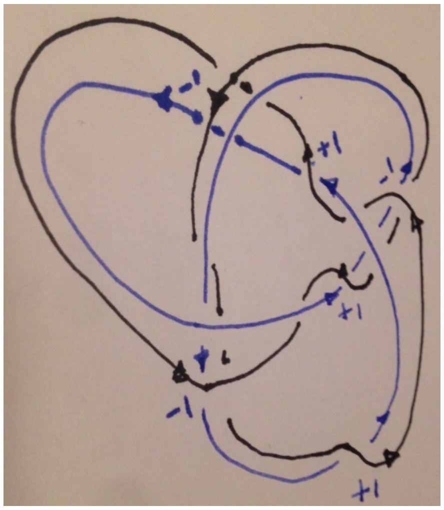
$$\begin{array}{lcccccc} \lim_{\nu\to 0^+}\left[\frac{-\nu}{1-\nu^2}+\frac{1+\nu^2}{1-\nu^2}\frac{2}{\pi}\arctan[1/\nu]-\frac{\nu}{\pi}\right] & \leq & \lim_{\nu\to 0^+} Wr(\mathbf{x};\,\nu) & \leq & \lim_{\nu\to 0^+}\frac{2}{\pi}\arctan[1/\nu], \\ \Leftrightarrow \qquad 1 & \leq & \lim_{\nu\to 0^+} Wr(\mathbf{x};\,\nu) & \leq & 1, \end{array}$$

which proves the property

3. The RHS of (2) of the question sheet is an odd function of ν so $\lim_{\nu\to 0^-} Wr = -1$.

3 Writhe Framing of the trefoil

The crossing count develops as follows



4 Writhe of an offset curve

1. Equip the curve $\mathbf{z}(s)$ with the adapted frame $Z(s) = \{\mathbf{c}_1(s),\,\mathbf{c}_2(s),\,\mathbf{c}_3(s)\}$ defined by

$$\mathbf{c}_1 = -\mathbf{d}_1, \qquad \mathbf{c}_3 = \frac{\mathbf{z}'}{||\mathbf{z}'||}, \quad \text{and} \quad \mathbf{c}_2 = \mathbf{c}_3 \times \mathbf{c}_1. \tag{27}$$

We also define $\mathbf{w}(s) = \mathsf{w}_i(s)\,\mathbf{c}_i(s)$ as the Darboux vector of $Z(s)$ and note that

$$\mathsf{w}_3 = \mathbf{c}_1' \cdot \mathbf{c}_2. \tag{28}$$

Taking a derivative of (6) from the question sheet, we find

$$\mathbf{z}' = ||\mathbf{z}'||\,\mathbf{c}_3 = ||\mathbf{x}'||\,\mathbf{d}_3 + \eta\,\mathbf{d}_1' \Rightarrow -\mathbf{c}_1' \overset{(27)}{=} \mathbf{d}_1' = \frac{1}{\eta}\Big(||\mathbf{z}'||\,\mathbf{c}_3 - ||\mathbf{x}'||\,\mathbf{d}_3\Big). \tag{29}$$

On the other hand, we also have

$$-\mathbf{c}_2 = -\mathbf{c}_3 \times \mathbf{c}_1 \overset{(27)}{=} \frac{\mathbf{z}'}{||\mathbf{z}'||} \times \mathbf{d}_1 \overset{(29)}{=} \left(\frac{||\mathbf{x}'||}{||\mathbf{z}'||}\,\mathbf{d}_3 + \eta\,\frac{\mathbf{d}_1'}{||\mathbf{z}'||}\right) \times \mathbf{d}_1. \tag{30}$$

Substituting (29,30) in (28), we find

$$
\begin{aligned}
\mathsf{w}_3 &= (-\mathbf{c}_1') \cdot (-\mathbf{c}_2) \overset{(29)}{=} \frac{1}{\eta}\left(||\mathbf{z}'||\,\mathbf{c}_3 - ||\mathbf{x}'||\,\mathbf{d}_3\right) \cdot (-\mathbf{c}_2), \\
&= -\frac{||\mathbf{x}'||}{\eta}\,\mathbf{d}_3 \cdot (-\mathbf{c}_2) \overset{(30)}{=} -\left(\frac{||\mathbf{x}'||}{\eta}\,\mathbf{d}_3\right) \cdot \left(\frac{||\mathbf{x}'||}{||\mathbf{z}'||}\,\mathbf{d}_3 \times \mathbf{d}_1 + \eta\,\frac{1}{||\mathbf{z}'||}(\mathbf{u} \times \mathbf{d}_1) \times \mathbf{d}_1\right), \\
&= \frac{||\mathbf{x}'||}{||\mathbf{z}'||}(\mathbf{u} \times \mathbf{d}_1) \cdot \mathbf{d}_2 = \frac{||\mathbf{x}'||}{||\mathbf{z}'||}\,\mathsf{u}_3.
\end{aligned}
\tag{31}
$$

2. We note that equation (6) from the question sheet can be re-written as

$$
\mathbf{x} = \mathbf{z} - \eta\,\mathbf{d}_1 = \mathbf{z} + \eta\,\mathbf{c}_1, \tag{32}
$$

where we can view $\mathbf{x}$ as an offset of $\mathbf{z}$ based on the adapted frame Z to $\mathbf{z}$. Next we apply $Lk = Tw + Wr$ to (6) from the question sheet and to (32) from this sheet. We find

$$
\begin{aligned}
Lk(\mathbf{x},\,\mathbf{z}) &= Wr(\mathbf{x}) + Tw(X) &&= Wr(\mathbf{x}) + \frac{1}{2\pi}\int_a^b \mathsf{u}_3(s)\,ds \\
\| \quad & && \\
Lk(\mathbf{z},\,\mathbf{x}) &= Wr(\mathbf{z}) + Tw(Z) &&= Wr(\mathbf{z}) + \frac{1}{2\pi}\int_a^b \mathsf{w}_3(s)\,ds.
\end{aligned}
\tag{33}
$$

Substituting (31) in (33) gives the result.