

1 Yet another variant of the Euler-Rodrigues formula

For any $\mathbf{v} \in \mathbb{R}^3$, we have

$$\mathbf{n}^\times \mathbf{n}^\times \mathbf{v} = \mathbf{n}^\times (\mathbf{n}^\times \mathbf{v}) = \mathbf{n} \times (\mathbf{n} \times \mathbf{v}) = (\mathbf{n} \cdot \mathbf{v})\mathbf{n} - \mathbf{v} = (\mathbf{n} \otimes \mathbf{n} - \text{Id}) \mathbf{v}. \quad (1)$$

Since (1) holds for all \mathbf{v} we have $\mathbf{n} \otimes \mathbf{n} = \text{Id} + \mathbf{n}^\times \mathbf{n}^\times$ which can be substituted in Eq. (2) from the question sheet.

2 Eulers-Rodrigues parameters

- (a) As discussed $q \in S^3$ is associated to a rotation of angle ϕ and axis \mathbf{n} . However, the rotation matrix $R(\phi, \mathbf{n}) \in SO(3)$ corresponding to a rotation of angle ϕ and axis along the unit vector \mathbf{n} is given by the Rodrigues formula:

$$\begin{aligned} R(\phi, \mathbf{n}) &= \cos \phi \text{Id} + (1 - \cos \phi) \mathbf{n} \otimes \mathbf{n} + \sin \phi \mathbf{n}^\times, \\ &= \left(\cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2} \right) \text{Id} + 2 \sin^2 \frac{\phi}{2} \mathbf{n} \otimes \mathbf{n} + 2 \cos \frac{\phi}{2} \sin \frac{\phi}{2} \mathbf{n}^\times, \\ &= (q_0^2 - \mathbf{q} \cdot \mathbf{q}) \text{Id} + 2 \mathbf{q} \otimes \mathbf{q} + 2q_0 \mathbf{q}^\times = Q(q). \end{aligned} \quad (2)$$

Expanding (2) gives

$$\begin{aligned} Q(q) &= \begin{pmatrix} q_0^2 - q_1^2 - q_2^2 - q_3^2 & 0 & 0 \\ 0 & q_0^2 - q_1^2 - q_2^2 - q_3^2 & 0 \\ 0 & 0 & q_0^2 - q_1^2 - q_2^2 - q_3^2 \end{pmatrix} \\ &+ 2 \begin{pmatrix} q_1^2 & q_1 q_2 & q_1 q_3 \\ q_2 q_1 & q_2^2 & q_2 q_3 \\ q_3 q_1 & q_3 q_2 & q_3^2 \end{pmatrix} + \begin{pmatrix} 0 & -2q_0 q_3 & 2q_0 q_2 \\ 2q_0 q_3 & 0 & -2q_0 q_1 \\ -2q_0 q_2 & 2q_0 q_1 & 0 \end{pmatrix}. \end{aligned} \quad (3)$$

- (b) Simply computing $Q(q)\mathbf{q}$ with $Q(q)$ as defined by (6) on the exercise sheet immediately shows $Q\mathbf{q} = \mathbf{q}$. However, note that it is even simpler to use the first equality in equation (5) of the exercise sheet. If $\mathbf{q} = 0$, the $q \in S^3$ implies $q_0 = \pm 1$ and $Q = \text{Id}$.
- (c) We have shown in question 3 of session 2 that

$$\text{tr}(Q) = 1 + 2 \cos \theta, \quad (4)$$

and also computing the Trace as the sum of the diagonal entries in (5) of the question sheet, we find

$$4q_4^2 - 1 = 1 + 2 \cos \theta \Leftrightarrow q_4 = \pm \cos \frac{\theta}{2}. \quad (5)$$

- (d) The fact that any $q \in S^3$ corresponds to a single $Q \in SO(3)$ is manifest from equation (5) of the exercise sheet.

Given a rotation matrix $R \in SO(3)$, define the vector $\mathbf{r} \in \mathbb{R}^3$ and the unit vector $\mathbf{n} \in S^2$ according to $\mathbf{r}^\times = R - R^T$ and $\mathbf{n} = \mathbf{r}/\|\mathbf{r}\|$. Also, define θ as the unique solution of $1 + 2 \cos \theta = \text{tr} R$ on $[0, \pi]$.

The angle θ and the vector \mathbf{n} completely specify R . That is, if R_1 and R_2 have same θ and same \mathbf{n} , then $R_1 = R_2$. That is because for any vector \mathbf{x} , we have $\mathbf{x} = (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + (\mathbf{n} \times \mathbf{x}) \times \mathbf{n}$. The vector $(\mathbf{n} \times \mathbf{x}) \times \mathbf{n}$ is perpendicular to \mathbf{n} and we showed in question 3.c of session 2 that if $\mathbf{v} \perp \mathbf{w}$, for \mathbf{w} along the axis of $Q \in SO(3)$, then $\mathbf{v} = \cos \theta \mathbf{v} + \sin \theta \mathbf{n} \times \mathbf{v}$ (in fact we almost did... make sure you can close the argument). So we have

$$\begin{aligned}
R\mathbf{x} &= R\left[(\mathbf{x} \cdot \mathbf{n})\mathbf{n} + (\mathbf{n} \times \mathbf{x}) \times \mathbf{n}\right], \\
&= (\mathbf{x} \cdot \mathbf{n})R\mathbf{n} + R[(\mathbf{n} \times \mathbf{x}) \times \mathbf{n}], \\
&= (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos \theta (\mathbf{n} \times \mathbf{x}) \times \mathbf{n} + \sin \theta \mathbf{n} \times ((\mathbf{n} \times \mathbf{x}) \times \mathbf{n}), \\
&= (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos \theta (\mathbf{n} \times \mathbf{x}) \times \mathbf{n} + \sin \theta (\mathbf{n} \times \mathbf{x}).
\end{aligned} \tag{6}$$

The point of this last equation is that the action of R on \mathbb{R}^3 is completely specified by \mathbf{n} and θ . Accordingly, if two matrices in $SO(3)$ have same \mathbf{n} and same θ , they must be equal.

Now we show that for any $R \in SO(3)$, $\exists q \in S^3 : Q(q) = R$. Given a matrix $R \in SO(3)$ we compute θ and \mathbf{n} and build the matrix $Q((\cos \theta/2, \sin \theta/2 n_1, \sin \theta/2 n_2, \sin \theta/2 n_3))$. By construction $\theta(R) = \theta(Q)$ and $\mathbf{n}(R) = \mathbf{n}(Q)$ and it must therefore be that $Q = R$.

Finally we prove the following

Proposition 1. *Let $p, q \in S^3$, then*

$$Q(p) = Q(q) \Leftrightarrow p = \pm q. \tag{7}$$

Proof.

$$Q(p) = Q(q) \Leftrightarrow \begin{pmatrix} p_1^2 - p_2^2 - p_3^2 + p_0^2 & 2(p_1p_2 - p_3p_0) & 2(p_1p_3 + p_2p_0) \\ 2(p_1p_2 + p_3p_0) & -p_1^2 + p_2^2 - p_3^2 + p_0^2 & 2(-p_1p_0 + p_2p_3) \\ 2(p_1p_3 - p_2p_0) & 2(p_1p_0 + p_2p_3) & -p_1^2 - p_2^2 + p_3^2 + p_0^2 \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} q_1^2 - q_2^2 - q_3^2 + q_0^2 & 2(q_1q_2 - q_3q_0) & 2(q_1q_3 + q_2q_0) \\ 2(q_1q_2 + q_3q_0) & -q_1^2 + q_2^2 - q_3^2 + q_0^2 & 2(-q_1q_0 + q_2q_3) \\ 2(q_1q_3 - q_2q_0) & 2(q_1q_0 + q_2q_3) & -q_1^2 - q_2^2 + q_3^2 + q_0^2 \end{pmatrix}. \tag{9}$$

The equalities between the diagonal entries imply that $q_i^2 = p_i^2$. Accordingly, there exist four number $s_0, s_1, s_2, s_3 \in \{-1, +1\}$ such that $p_i = s_i q_i$. Next the equalities between the off-diagonal terms are satisfied iff $s_0 = s_1 = s_2 = s_3$. \square

- (e) The result comes from computing the 10 possible scalar products between pairs among $\{q, B_1 q, B_2 q, B_3 q\}$.

3 Quaternion manipulation

1. We will use the four equations

$$ijk = -1, \tag{10}$$

$$ii = -1, \tag{11}$$

$$jj = -1, \tag{12}$$

$$kk = -1. \tag{13}$$

Multiplying (10) on the right by k and using (13) there in gives

$$ijk = -1 \Rightarrow ijkk = -k \Rightarrow ij = k. \tag{14}$$

We also find that

$$ijk = -1 \Rightarrow iijk = -i \stackrel{(11)}{\Rightarrow} jk = i. \quad (15)$$

Finally multiplying (10) on the left by j and on the right by kj gives

$$ijk = -1 \Rightarrow jijk kj = -jk j \stackrel{(13,15)}{\Rightarrow} -jijj = -ij \stackrel{(12)}{\Rightarrow} ji = -ij = -k. \quad (16)$$

Similar arguments allow to complete the ring: $ij = k$, $jk = i$, and $ki = j$ and any pair of non-identical elements in $\{i, j, k\}$ anti-commutes.

2. By direct computations

$$\begin{aligned} qp &= (q_0 + i q_1 + j q_2 + k q_3) (p_0 + i p_1 + j p_2 + k p_3), \\ &= q_0 p_0 + q_0 (i p_1 + j p_2 + k p_3) + p_0 (i q_1 + j q_2 + k q_3) + (i q_1 + j q_2 + k q_3) (i p_1 + j p_2 + k p_3), \\ &= q_0 p_0 + q_0 (i p_1 + j p_2 + k p_3) + p_0 (i q_1 + j q_2 + k q_3) \\ &\quad - q_1 p_1 + i q_1 (j p_2 + k p_3) - q_2 p_2 + j q_2 (i p_1 + k p_3) - q_3 p_3 + k q_3 (i p_1 + j p_2) \\ &= (q_0 p_0 - \mathbf{q} \cdot \mathbf{p}) + q_0 (i p_1 + j p_2 + k p_3) + p_0 (i q_1 + j q_2 + k q_3) \\ &\quad + k (q_1 p_2 - q_2 p_1) + j (q_3 p_1 - q_1 p_3) + i (q_2 p_3 - q_3 p_2), \end{aligned} \quad (17)$$

and the result comes from gathering scalar and vector parts.

3. We use equation (10) from the question sheet to show that

$$\begin{aligned} qr\bar{q} &= (q_0, \mathbf{q}) (0, \mathbf{r}) (q_0, -\mathbf{q}), \\ &= (-\mathbf{q} \cdot \mathbf{r}, q_0 \mathbf{r} + \mathbf{q} \times \mathbf{r}) (q_0, -\mathbf{q}), \\ &= \left(-q_0 (\mathbf{q} \cdot \mathbf{r}) + (q_0 \mathbf{r} + \mathbf{q} \times \mathbf{r}) \cdot \mathbf{q}, \right. \\ &\quad \left. (\mathbf{q} \cdot \mathbf{r}) \mathbf{q} + q_0 (q_0 \mathbf{r} + \mathbf{q} \times \mathbf{r}) - (q_0 \mathbf{r} + \mathbf{q} \times \mathbf{r}) \times \mathbf{q} \right) \\ &= \left(0, (\mathbf{q} \otimes \mathbf{q}) \mathbf{r} + q_0^2 \mathbf{r} + 2 q_0 \mathbf{q} \times \mathbf{r} + \mathbf{q} \times (\mathbf{q} \times \mathbf{r}) \right), \\ &= \left(0, (q_0^2 + \mathbf{q} \cdot \mathbf{q}) \mathbf{r} + 2 q_0 \mathbf{q} \times \mathbf{r} + 2 \mathbf{q} \times (\mathbf{q} \times \mathbf{r}) \right), \end{aligned} \quad (18)$$

where we used the fact that $\mathbf{q} \otimes \mathbf{q} = (\mathbf{q} \cdot \mathbf{q}) \text{Id} + \mathbf{q}^\times \mathbf{q}^\times$.

This is identical to the Rodrigues formula whenever $q\bar{q} = 1$. In particular, this shows that if we have a rotation of quaternion q (of norm 1), then it rotates a vector \mathbf{r} onto the vector part of $qr\bar{q}$ as discussed during the lecture.