

1 Quaternions of particular curves in $SO(3)$

1.1 Quaternion associated with the Frenet frame of a helix

The Frenet frame to α can be recovered from the reference frame $(\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3)$ by the following sequence of (right) rotations:

$$(\mathbf{n} \ \mathbf{b} \ \mathbf{t}) = (\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3) R_{3,s/c+\pi} R_{1,\phi}, \quad (1)$$

where the matrices $R_{i,\alpha}$ were defined in the lecture and where $\phi = \text{Arctan}(a/b)$. In particular,

$$\cos \phi/2 = \sqrt{\frac{c+b}{2c}}, \quad \text{and} \quad \sin \phi/2 = \sqrt{\frac{c-b}{2c}}. \quad (2)$$

Hence the quaternion associated to the rotation from the lab frame $(\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3)$ to the Frenet-Serret frame $(\mathbf{n} \ \mathbf{b} \ \mathbf{t})$ is

$$\begin{aligned} q(s) &= \left[\cos\left(\frac{s}{2c} + \frac{\pi}{2}\right) + k \sin\left(\frac{s}{2c} + \frac{\pi}{2}\right) \right] [\cos(\phi/2) + i \sin(\phi/2)], \\ &= \left[-\sin\left(\frac{s}{2c}\right) + k \cos\left(\frac{s}{2c}\right) \right] \left[\sqrt{\frac{c+b}{2c}} + i \sqrt{\frac{c-b}{2c}} \right], \\ &= -\sin \frac{s}{2c} \sqrt{\frac{c+b}{2c}} - i \sin \frac{s}{2c} \sqrt{\frac{c-b}{2c}} + j \cos \frac{s}{2c} \sqrt{\frac{c-b}{2c}} + k \cos \frac{s}{2c} \sqrt{\frac{c+b}{2c}}. \end{aligned} \quad (3)$$

1.2 Quaternion of the multiply covered circle

1. We simply take the quaternion of the previous question in the case $b = 0$ and a the radius of the circle:

$$q(s) = \cos \frac{s}{2a} \frac{j+k}{\sqrt{2}} - \sin \frac{s}{2a} \frac{1+i}{\sqrt{2}}. \quad (4)$$

The fact that the circle is covered n times is encoded in the domain of $s \in [0, 2\pi n]$. Also $(j+k)/\sqrt{2}$ and $(1+i)/\sqrt{2}$ are themselves unit quaternions so that the image of $q(s)$ is a great circle in S^3 . It is covered $n/2$ times: closed if n is even and open if n is odd.

2. The register angle ψ from the frame of the previous question to a frame with prescribed twist $\mathbf{u}_3(s)$ is

$$\psi(s) = \int_0^s \mathbf{u}_3(\sigma) d\sigma + \psi(0). \quad (5)$$

Accordingly the quaternion that we are looking for is

$$q(s) = \left(\cos \frac{s}{2a} \frac{j+k}{\sqrt{2}} - \sin \frac{s}{2a} \frac{1+i}{\sqrt{2}} \right) \left(\cos(\psi(s)/2) + k \sin(\psi(s)/2) \right), \quad (6)$$

$$= -\sin \frac{\psi(s) + s/a}{2} + i \sin \frac{\psi(s) - s/a}{2} + j \cos \frac{\psi(s) - s/a}{2} + k \cos \frac{\psi(s) + s/a}{2}. \quad (7)$$

2 Cayley transforms

2.1 A few general properties

1. *Ad absurdum*, if $(I + M)$ is not invertible, then there exists a non-trivial (i.e. $\neq \mathbf{0}$) vector \mathbf{v} such that $(I + M)\mathbf{v} = \mathbf{0}$. This implies that

$$M\mathbf{v} = -\mathbf{v} \Leftrightarrow (I + N)(I - N)^{-1}\mathbf{v} = -\mathbf{v}, \quad (8)$$

$$\Leftrightarrow (I - N)^{-1}(I + N)\mathbf{v} = -\mathbf{v}, \quad (9)$$

$$\Leftrightarrow (I + N)\mathbf{v} = -(I - N)\mathbf{v}, \quad (10)$$

$$\Leftrightarrow \mathbf{v} + N\mathbf{v} = -\mathbf{v} + N\mathbf{v}, \quad \Leftrightarrow \mathbf{v} = -\mathbf{v}, \quad (11)$$

a contradiction.

2. We simply have

$$M = (I + N)(I - N)^{-1} \Leftrightarrow M(I - N) = (I + N) \quad (12)$$

$$\Leftrightarrow M - I = (I + M)N \stackrel{2.1.1}{\Leftrightarrow} N = (I + M)^{-1}(M - I). \quad (13)$$

3. First note that if Q is a Cayley transform, then $(I + Q)$ is invertible and so Q can not have -1 as an eigenvalue. This in turn implies $Q \notin O(n) \setminus SO(n)$ so that if Q is orthogonal, then it must be that $Q \in SO(n)$. Then we have

$$Q^T Q = I \Leftrightarrow (I - S)^{-T}(I + S)^T(I + S)(I - S)^{-1} = I, \quad (14)$$

$$\Leftrightarrow (I + S)^T(I + S) = (I - S)^T(I - S), \quad (15)$$

$$\Leftrightarrow I + S^T + S + S^T S = I - S^T - S + S^T S, \quad (16)$$

$$\Leftrightarrow S^T = -S. \quad (17)$$

2.2 The case $n = 3$

1. We simply follow the proof highlighted during the lecture. To find Q using the fact that it is the Cayley transform of $S = \mathbf{u}^\times$, we first need to compute $(I - S)^{-1}$.

Let $P \in SO(3)$ be a right-handed orthonormal basis of \mathbb{R}^3 such that the third column of P is parallel to \mathbf{u} , then we have

$$P^{-1}SP = \begin{pmatrix} 0 & -u & 0 \\ u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (18)$$

where $u := \|\mathbf{u}\|$. The matrix $P^{-1}SP$ has characteristic polynomial $-\lambda(\lambda^2 + u^2)$. Applying the Cayley-Hamilton theorem then leads to

$$S^3 = -S u^2. \quad (19)$$

Next we look for three numbers α , β and γ such that

$$(I - S)(\alpha I + \beta S + \gamma S^2) = I, \quad (20)$$

$$\Leftrightarrow (\alpha - 1)I + (\beta - \alpha)S + (\gamma - \beta)S^2 - \gamma S^3 = 0, \quad (21)$$

$$\stackrel{(19)}{\Leftrightarrow} (\alpha - 1)I + (\beta - \alpha + \gamma u^2)S + (\gamma - \beta)S^2 = 0. \quad (22)$$

The condition (22) is satisfied by

$$\alpha = 1, \quad \beta = \gamma = \frac{1}{1+u^2}. \quad (23)$$

Before we conclude, note that for any vector $\mathbf{v} \in \mathbb{R}^3$, we have

$$S^2 \mathbf{v} = \mathbf{u}^\times (\mathbf{u}^\times \mathbf{v}) = \mathbf{u}^\times (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u} (\mathbf{u} \cdot \mathbf{v}) - u^2 \mathbf{v} = (\mathbf{u} \otimes \mathbf{u} - u^2 I) \mathbf{v}. \quad (24)$$

Finally, compute

$$Q = (I + S)(I - S)^{-1}, \quad (25)$$

$$= (I + S) \frac{1}{1+u^2} \left((1+u^2)I + S + S^2 \right), \quad (26)$$

$$= \frac{1}{1+u^2} \left((1+u^2)I + (2+u^2)S + 2S^2 + S^3 \right) \quad (27)$$

$$\stackrel{(19)}{=} \frac{1}{1+u^2} \left((1+u^2)I + 2S + 2S^2 \right), \quad (28)$$

$$\stackrel{(24)}{=} \frac{1}{1+u^2} \left((1-u^2)I + 2\mathbf{u}^\times + 2\mathbf{u} \otimes \mathbf{u} \right). \quad (29)$$

As for the second part of the question, the Rodrigues formula directly gives the matrix of a right-handed rotation of angle ϕ and axis \mathbf{n} :

$$R(\phi, \mathbf{n}) = \cos \phi I + (1 - \cos \phi) \mathbf{n} \otimes \mathbf{n} + \sin \phi \mathbf{n}^\times. \quad (30)$$

For $\phi = 2 \operatorname{Arctan}(u)$ with $u > 0$, we have $\tan \frac{\phi}{2} = u$. So that both

$$\cos \phi = \frac{1 - \tan^2 \frac{\phi}{2}}{1 + \tan^2 \frac{\phi}{2}} = \frac{1 - u^2}{1 + u^2}, \quad \text{and} \quad \sin \phi = \frac{2 \tan \frac{\phi}{2}}{1 + \tan^2 \frac{\phi}{2}} = \frac{2u}{1 + u^2}. \quad (31)$$

Substituting both (31) and $\mathbf{n} = \frac{1}{u}\mathbf{u}$ in (30) yields

$$R(2 \operatorname{Arctan}(u), \mathbf{u}/u) = \frac{1}{1+u^2} \left[(1-u^2)I + 2u^2 \frac{\mathbf{u}}{u} \otimes \frac{\mathbf{u}}{u} + 2u \left(\frac{\mathbf{u}}{u} \right)^\times \right] = Q. \quad (32)$$

2. We will use the result of 2.1.2 to find S . To this intend, we first compute $(I + Q)^{-1}$. The characteristic polynomial of a 3×3 matrix A is

$$-\lambda^3 + (\operatorname{tr} A)\lambda^2 - \frac{1}{2} ((\operatorname{tr} A)^2 - \operatorname{tr} A^2) \lambda + \det A. \quad (33)$$

Since $Q \in SO(3)$, the Cayley-Hamilton theorem gives

$$Q^3 - t Q^2 + t Q - I = 0, \quad (34)$$

where $t := \operatorname{Trace}(Q)$. Multiplying (34) by Q^T yields

$$Q^2 = t Q - t I + Q^T. \quad (35)$$

Next we try to find three numbers α, β and γ such that

$$(I + Q)(\alpha Q^T + \beta I + \gamma Q) = I. \quad (36)$$

To this intend compute

$$\begin{aligned} \text{Eq. (36)} &\Leftrightarrow \alpha Q^T + (\alpha + \beta - 1)I + (\gamma + \beta)Q + \gamma Q^2 = 0, \\ &\stackrel{(35)}{\Leftrightarrow} (\alpha + \gamma)Q^T + (\alpha + \beta - 1 - t\gamma)I + (\gamma(1+t) + \beta)Q = 0, \end{aligned} \quad (37)$$

$$\Leftrightarrow \begin{cases} \alpha + \gamma = 0, \\ \alpha + \beta - t\gamma = 1, \\ \beta + (1+t)\gamma = 0, \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{1}{2(1+t)}, \\ \beta = \frac{1+t}{2(1+t)}, \\ \gamma = \frac{-1}{2(1+t)}. \end{cases} \quad (38)$$

Accordingly, the final result in (38) contains the three numbers that we were looking for: we have shown that

$$(I + Q)^{-1} = \frac{1}{2(1+t)} (Q^T + (1+t)I - Q). \quad (39)$$

Now, making use of the result 2.1.2 of this sheet, we compute

$$S = (I + Q)^{-1} (Q - I) = \frac{1}{2(1+t)} (Q^T + (1+t)I - Q) (Q - I), \quad (40)$$

$$= \frac{1}{2(1+t)} (-Q^T - tI + (2+t)Q - Q^2), \quad (41)$$

$$\stackrel{(35)}{=} \frac{Q - Q^T}{1+t} \quad (42)$$

2.3 Composition of rotations

We have seen in class that if $Q \in SO(3)$, it has eigenvalues $\{1, e^{i\theta}, e^{-i\theta}\}$ for some $\theta \in (-\pi, \pi]$. If -1 is not an eigenvalue of Q (that is $-\pi < \theta < \pi$), then the Cayley vector \mathbf{u} of Q satisfies $u := \|\mathbf{u}\| = |\tan \theta/2|$. Also \mathbf{u} is in the eigenspace of the $+1$ eigenvalue of Q . On the other hand, the quaternion associated to Q is

$$q = (q_0, \mathbf{q}) = \pm(\cos \theta/2, \sin \theta/2 \mathbf{n}), \quad (43)$$

where \mathbf{n} is in the eigenspace of the eigenvalue $+1$ of Q . It follows that

$$\mathbf{u} = \mathbf{q}/q_0. \quad (44)$$

Now consider the two matrices Q_1 and Q_2 of the exercise. They have associated quaternions q and p and the composition of which reads (see last week's exercise session)

$$\begin{aligned} qp &= (q_0 p_0 - \mathbf{q} \cdot \mathbf{p}, q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p}) \\ &= q_0 p_0 \left(1 - \frac{\mathbf{q} \cdot \mathbf{p}}{q_0 p_0}, \frac{\mathbf{p}}{p_0} + \frac{\mathbf{q}}{q_0} + \frac{\mathbf{q} \times \mathbf{p}}{q_0 p_0} \right) \\ &= q_0 p_0 (1 - \mathbf{u}_1 \cdot \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_1 \times \mathbf{u}_2). \end{aligned} \quad (45)$$

Substituting (45) in (44) gives the Cayley vector of the composition as expected.

2.4 Cayley transforms in $SE(3)$

First, we prove that if \mathcal{S} is of the form

$$\mathcal{S} = \begin{pmatrix} \mathbf{u}^\times & \mathbf{v} \\ 0 & 0 \end{pmatrix}, \quad (46)$$

then its Cayley transform is in $SE(3)$. This is because

$$(\mathcal{I} + \mathcal{S})(\mathcal{I} - \mathcal{S})^{-1} = \begin{pmatrix} I + \mathbf{u}^\times & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I - \mathbf{u}^\times & -\mathbf{v} \\ 0 & 1 \end{pmatrix}^{-1}, \quad (47)$$

$$= \begin{pmatrix} I + \mathbf{u}^\times & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (I - \mathbf{u}^\times)^{-1} & (I - \mathbf{u}^\times)^{-1}\mathbf{v} \\ 0 & 1 \end{pmatrix} \quad (48)$$

$$= \begin{pmatrix} Q & Q\mathbf{v} + \mathbf{v} \\ 0 & 1 \end{pmatrix}, \quad (49)$$

where $Q := (I + \mathbf{u}^\times)(I - \mathbf{u}^\times)^{-1} \in SO(3)$ because it is the Cayley transform of a skew matrix. Accordingly, the RHS of (49) is in $SE(3)$.

Next, we show that if \mathcal{Q} is the Cayley transform of some matrix $S \in \mathbb{R}^{4 \times 4}$ and $\mathcal{Q} \in SE(3)$, then \mathcal{S} is of the form (46).

If $\mathcal{Q} \in SE(3)$ then there exist a matrix $Q \in SO(3)$ and a vector \mathbf{q} such that

$$\mathcal{Q} = \begin{pmatrix} Q & \mathbf{q} \\ 0 & 1 \end{pmatrix}. \quad (50)$$

Furthermore, because \mathcal{Q} is a Cayley transform, $\mathcal{Q} + \mathcal{I}$ must be invertible. This in turn implies that $Q + I$ is invertible. Then after defining $S = (Q + I)^{-1}(Q - I)$ a similar argument to that developed in 2.1.1 (make sure you can do it) shows that $I - S$ is invertible and a similar argument to 2.1.2 shows that Q is the Cayley transform of S . Finally, 2.1.3 ensures that S must be skew so there exists a vector \mathbf{u} such that $S = \mathbf{u}^\times$.

The result 2.1.2 implies

$$\begin{aligned} \mathcal{S} &= (\mathcal{Q} + \mathcal{I})^{-1}(\mathcal{Q} - \mathcal{I}) = \begin{pmatrix} Q + I & \mathbf{q} \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} Q - I & \mathbf{q} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (Q + I)^{-1} & -\frac{1}{2}(Q + I)^{-1}\mathbf{q} \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} Q - I & \mathbf{q} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{u}^\times & (Q + I)^{-1}\mathbf{q} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$