~~Ex 2~~ Writhe of an offset curve

Ex 10 / 2

$$\underset{\sim}{z} = \underset{\sim}{x} + \eta \underset{\sim}{d_2} \qquad (*)$$

Consider the adapted frame $Z(s) = \left(\underset{\sim}{-d_1}, \ \dfrac{-\underset{\sim}{z}'}{\|z'\|} \times \underset{\sim}{d_1}, \ \dfrac{\underset{\sim}{z}'}{\|z'\|} \right)$, (see session 3), with $\underset{\sim}{D_1} = -\underset{\sim}{d_1}$, $\underset{\sim}{D_2} = \dfrac{-\underset{\sim}{z}'}{\|z'\|} \times \underset{\sim}{d_1}$, $\underset{\sim}{D_3} = \dfrac{\underset{\sim}{z}'}{\|z'\|}$.

We already have $\underset{\sim}{u} = u_i \underset{\sim}{d_i}$ the ~~adapted~~ Darboux vector of the adapted frame $X(s)$ to $\underset{\sim}{x}(s)$. Let $\underset{\sim}{\omega} = \omega_i \underset{\sim}{D_i}$ be the ______ of $Z(s)$.

Then $\omega_3 = \underset{\sim}{D_1}' \cdot \underset{\sim}{D_2}$ (indeed $\underset{\sim}{D_1}' \cdot \underset{\sim}{D_2} = (\underset{\sim}{\omega} \times \underset{\sim}{D_1}) \cdot \underset{\sim}{D_2} = (\underset{\sim}{D_1} \times \underset{\sim}{D_2}) \cdot \underset{\sim}{\omega}$)

On ~~the other~~ one hand, taking a derivative of $(*)$ by s gives.

$$(\text{¥}) \quad \|z'\| \underset{\sim}{D_3} = \|x'\| \underset{\sim}{d_3} + \eta \underset{\sim}{d_1}' \implies \underset{\sim}{d_1}' = \frac{1}{\eta}\left(\|z'\| \underset{\sim}{D_3} - \|x'\| \underset{\sim}{d_3} \right)$$

$$\underset{\sim}{d_1}' = -\underset{\sim}{D_1}'$$

On the other hand,

$$(\text{¥}) \quad -\underset{\sim}{D_2} = +\frac{\underset{\sim}{z}'}{\|z'\|} \times \underset{\sim}{d_1} \overset{(\text{¥})}{=} \left(\frac{\|x'\|}{\|z'\|} d_3 + \eta \frac{d_1'}{\|z'\|} \right) \times d_1$$

Then, we have.

$$\omega_3 = \underset{\sim}{D_1}' \cdot D_2 = (-\underset{\sim}{D_1}') \cdot (-\underset{\sim}{D_2}) \overset{(\text{¥})}{=} \frac{1}{\eta}\left(\|z'\| \underset{\sim}{D_3} - \|x'\| \underset{\sim}{d_3} \right) \cdot (-D_2).$$

$$= \left(-\frac{\|x'\|}{\eta} \underset{\sim}{d_3} \right) \cdot (-\underset{\sim}{D_2}) \overset{(\text{¥})}{=} \left(-\frac{\|x'\|}{\eta} \underset{\sim}{d_3} \right) \cdot \left(\frac{\|x'\|}{\|z'\|} d_3 \times d_1 + \frac{\eta}{\|z'\|} (u \times d_1) \times d_1 \right).$$

$$= -\frac{\|x'\|}{\eta} \underset{\sim}{d_3} \cdot \left(\frac{\eta}{\|z'\|} (u_1 \underset{\sim}{d_1} - \underset{\sim}{u}) \right) = \frac{\|x'\|}{\|z'\|} u_3 \qquad (\bullet)$$

~~Finally~~ Next, note that $(*)$ can be written.

$$\underset{\sim}{x} = z - \eta \underset{\sim}{d_1} = z + \eta \underset{\sim}{D_1}. \qquad (**)$$

Then we apply $Lk = Tw + Wr$ on $(*)$ and on $(**)$.

$$L(\underset{\sim}{x}, \underset{\sim}{z}) = Lk(\underset{\sim}{z}, \underset{\sim}{x})$$

$$L(\underset{\sim}{x}, \underset{\sim}{z}) = Wr(\underset{\sim}{x}) + Tw(X) = Wr(\underset{\sim}{x}) + \frac{1}{2\pi}\int_a^b u_3(s)\, ds$$

$$Lk(\underset{\sim}{z}, \underset{\sim}{x}) = Wr(\underset{\sim}{z}) + Tw(Z) = Wr(\underset{\sim}{z}) + 1/2\pi \int_a^b \omega_3\, ds.$$

This last equation gives

$$Wr(z) = Wr(x) + \frac{1}{2\pi}\int_a^b u_3(s) - \omega_3(s)\, ds$$

$$\overset{(\circ)}{=} Wr(x) + \frac{1}{2\pi}\int_a^b u_3(s)\left(1 - \frac{\|x'\|}{\|z'\|}\right) ds.$$

Particular example

•1) $z(s)$ is an offset of $x(s) = \cos s\, e_1 + \sin s\, e_2$

Indeed, after defining $d_1 = \cos(\omega(s))\underbrace{(\cos s\, e_1 + \sin s\, e_2)}_{e_r} \mp \sin(\omega(s))\, e_3$,

we have

$$z(s) = x(s) + \eta\, d_1(s).$$

First we need to make sure that $d_1 \cdot x' = 0$.

It is best seen in cylindrical coordinates, defining

$$e_r = \cos\theta\, e_1 + \sin\theta\, e_2 \quad ; \quad e_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2.$$

And $\theta(s) = s$.

$x'(s) = e_\theta$ then $\begin{cases} d_1 = \cos\omega\, e_r + \sin\omega\, e_3 \\ x' = e_\theta \end{cases} \Rightarrow d_1 \cdot x' = 0.$

Also we have $\|x'(s)\| = 1$ so $d_3 = x' = e_\theta$

Next to apply (2), we need to compute

$\|z'\|$: First $z' = x' + \eta\, d_1'$

$$= d_3 + \eta\left(-\sin\omega\, \omega'\, e_r + \cos\omega\, e_\theta \mp \omega' \cos\omega\, e_3\right)$$

$$= -\eta \sin\omega\, \omega'\, e_r + (1+\eta\cos\omega)\, e_\theta \mp \eta\,\omega'\cos\omega\, e_3$$

Then

$$\|z'\| = \sqrt{\eta^2 \sin^2\omega\, \omega'^2 + (1+\eta\cos\omega)^2 + \eta^2\omega'^2\cos^2\omega}$$

$$= \sqrt{(1+\eta\cos\omega)^2 + \eta^2\omega'^2} \qquad (1)$$

Finally, we need the local twist u_3 of the frame (d_1, d_2, d_3):

~~We first~~ $u_3 = d_1' \cdot d_2$

But $d_2 = d_3 \times d_1 = e_\theta \times (\cos\omega\, e_r \mp \sin\omega\, e_3) = \mp\sin\omega\, e_r - \cos\omega\, e_3$

Then $u_3 = (-\omega' \sin\omega\, e_r + \cos\omega\, e_\theta \mp \omega' \cos\omega\, e_3) \cdot (\mp\sin\omega\, e_r - \cos\omega\, e_3)$

$$= \omega' \qquad (2)$$

Gathering (1) & (2) and substituting in (2) of the exercise sheet, we get

$$\boxed{Wr(z) = \frac{1}{2\pi}\int_0^{2\pi} \omega'(s)\left(1 - \frac{1}{\sqrt{(1+\eta\cos\omega)^2 + \eta^2\omega'^2}}\right) ds.} \qquad (\bullet 1)$$

0.2] . ~~The first term is~~ Let's compute $\omega(0)$ & $\omega(2\pi)$. Ex 10 / 3

The first term gives.

~~at s = 0~~

$$2\pi \,\mathrm{Ceil}\left(\cancel{\tfrac{k\cdot 0}{2\pi}} - 1/2\right) = 0.$$

The second term gives.

$$2\arctan\left[\sqrt{\frac{1+\eta}{1-\eta}}\tan(0)\right] = 0$$

the sum is.

$$0$$

at 2π

$$2\pi\,\mathrm{Ceil}\left((k-1/2)\right) = k2\pi.$$

$$2\arctan\left[\sqrt{\quad}\tan\left[4\pi\sqrt{\frac{1-\eta^2}{\eta}}\right]\right].$$

the sum is.

$$2k\pi + 2\arctan[\dots].$$

Accordingly, for the closure condition to hold, we need to choose η such that the arctan term vanishes. This can only happen if the argument of the arctan vanishes. We then ask that.

$$\tan\left[4\pi\sqrt{\frac{1-\eta^2}{\eta}}\frac{1}{\eta}\right] = 0 \qquad \cancel{\text{for some } p\in\mathbb{Z}}$$

$$\iff 4\pi\sqrt{1-\eta^2}\,\frac{1}{\eta} = p\pi \quad \text{for some } p\in\mathbb{Z}$$

$$\iff 1-\eta^2 = \frac{p^2}{16}\eta^2 \underset{\text{as } \eta\in(0,1)}{\iff} \eta = \frac{1}{\sqrt{1+p^2/16}}$$

In that case, $\frac{2}{\eta}\sqrt{1-\eta^2} = p/2$

However, we also need ω to be C^1. The term $2\pi\,\mathrm{Ceil}\left(\frac{ks-\pi}{2\pi}\right)$ jumps by 2π when $s = \frac{\pi}{k} + \frac{2\pi}{k}n$ for $n\in\mathbb{N}$. Hence, we also need the second term to jump by -2π.

It occurs when the argument of the tangent ~~jumps by~~ cross $\pi/2 + n\pi$.

So we need $\left.\frac{ps}{2}\right|_{s=\frac{\pi}{k}+\frac{2\pi}{k}n} = \frac{\pi}{2} + n\pi.$ $\forall n\in\mathbb{N}$

$$\iff \frac{p}{k}\left(\pi/2 + n\pi\right) = \pi/2 + n\pi$$

$$\implies p = k.$$

In conclusion, we choose

$$\boxed{\eta = \frac{1}{\sqrt{1+\frac{k^2}{16}}}} \qquad (0.2)$$

•3| We start by solving the differential equation. Ex10

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$$\omega' = \frac{4}{\eta}\left(1+\eta\cos\omega(s)\right) \quad \text{with IV} \quad \omega(0)=0.$$

$$\Leftrightarrow \frac{d\omega}{1+\eta\cos\omega} = \frac{4}{\eta}\,ds.$$

$$\Rightarrow \int_0^{\omega(s)} \frac{d\sigma}{1+\eta\cos\sigma} = \int_0^s \frac{4}{\eta}\,d\tau \qquad (*)$$

but
$$\frac{d\sigma}{1+\eta\cos\sigma} = \frac{d\sigma}{1+\eta(2\cos^2\sigma/2-1)} = \frac{d\sigma}{1-\eta+\eta 2\frac{1}{1+tg^2\sigma/2}}$$

$$= \frac{1+tg^2\sigma/2}{(1+\eta)+(1-\eta)tg^2\sigma/2}\,d\sigma$$

change of variable $tg\frac{\sigma}{2} = x \Rightarrow \left(1+tg^2\frac{\sigma}{2}\right)\frac{d\sigma}{2} = dx$

then
$$\frac{d\sigma}{1+\eta\cos\sigma} = \frac{2\,dx}{(1+\eta)+(1-\eta)x^2} = \frac{2\,dx}{1+\left(\sqrt{\frac{1-\eta}{1+\eta}}\,x\right)^2}\,\frac{1}{1+\eta} \qquad (\Psi)$$

Which can be substituted in (*) to get.

$$\frac{2}{1+\eta}\int_0^{x(s)} \frac{dx}{1+\left(\sqrt{\frac{1-\eta}{1+\eta}}\,x\right)^2} = \frac{4}{\eta}s = \frac{2}{1+\eta}\,\frac{\arctan\left(\sqrt{\frac{1-\eta}{1+\eta}}\,x(s)\right)}{\sqrt{\frac{1-\eta}{1+\eta}}}.$$

$$\Rightarrow x(s) = \sqrt{\frac{1+\eta}{1-\eta}}\,\tan 2\frac{\sqrt{1-\eta^2}}{\eta}s$$

$$= \tan\frac{\omega(s)}{2}$$

$$\Rightarrow \omega(s) = 2\left(z\pi + \text{Arctan}\left[\sqrt{\frac{1+\eta}{1-\eta}}\,\tan\left(2\frac{\sqrt{1-\eta^2}}{\eta}s\right)\right]\right) \quad \text{for } z\in\mathbb{Z}.$$

Hence the choice of ω made in the session does solve the IVP.

Accordingly,
$$(\omega'\eta)^2 = 16\cdot\left(1+\eta\cos\omega(s)\right)^2$$

$\frac{Ex\,90}{5}$

· Now, we go back to the formula that we found in (•1).

$$W_2(z) = \frac{1}{2\pi}\int_0^{2\pi} \omega'(s)\left(1 - \frac{1}{\sqrt{}}\right) ds$$

$$= \frac{\omega(2\pi)}{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \frac{\omega'(s)\, ds}{\sqrt{(1+\eta\cos\omega)^2 + 16(1+\eta\cos\omega)^2}}.$$

$$= k - \frac{1}{\sqrt{17}}\int_0^{2\pi} \frac{\omega'(s)\, ds}{1+\eta\cos\omega(s)} \quad \frac{1}{2\pi}$$

$$= k - \frac{1}{\sqrt{17}}\int_0^{k2\pi} \frac{d\omega}{1+\eta\cos\omega}\left(\frac{1}{2\pi}\right) = k - \frac{k}{\sqrt{17}}\int_0^{2\pi}\frac{d\omega}{1+\eta\cos\omega} \quad \left(\frac{1}{2\pi}\right)$$

$$= k - \frac{k}{\sqrt{17}}\frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2}\frac{d\omega}{1+\eta\cos\omega}$$

$$= k - \frac{k}{\sqrt{17}}\frac{1}{2\pi}\left(\int_{-\pi/2}^{\pi/2}\frac{d\omega}{1+\eta\cos\omega} + \int_{-\pi/2}^{\pi/2}\frac{d\sigma}{1-\eta\cos\sigma}\right) \quad \text{where } \sigma = \pi - \omega$$

$$\overset{(\psi)}{=} k - \frac{k}{\sqrt{17}}\frac{1}{2\pi}\left(\frac{1}{1+\eta}\int_{-1}^{1}\frac{dx}{1+\left(\sqrt{\frac{1-\eta}{1+\eta}}\,x\right)^2} + \frac{1}{1-\eta}\int_{-1}^{1}\frac{dy}{1+\left(\sqrt{\frac{1+\eta}{1-\eta}}\,y\right)^2}\right).$$

$$= k - \frac{k}{2\pi\sqrt{17}}2\left(\frac{1}{1+\eta}\,2\,\frac{\operatorname{Arctan}\left(\sqrt{\frac{1-\eta}{1+\eta}}\right)}{\sqrt{\frac{1-\eta}{1+\eta}}} + \frac{1}{1-\eta}\,2\,\frac{\operatorname{Arctan}\left(\sqrt{\frac{1+\eta}{1-\eta}}\right)}{\sqrt{\frac{1+\eta}{1-\eta}}}\right).$$

$$= k - \frac{k\,4}{2\pi\sqrt{17}}\frac{1}{\sqrt{1-\eta^2}}\underbrace{\left(\operatorname{Arctan}\left[\sqrt{\frac{1-\eta}{1+\eta}}\right] + \operatorname{Arctan}\left[\left(\sqrt{\frac{1-\eta}{1+\eta}}\right)^{-1}\right]\right)}_{\pi/2}.$$

$$= k - \frac{k}{\sqrt{17}}\frac{2\pi}{\sqrt{1-\eta^2}}\frac{1}{2\pi}$$

$$= k - \frac{k}{\sqrt{17}}\frac{\sqrt{16+k^2}}{k}$$

$$= k - \sqrt{\frac{16+k^2}{17}}$$

$$(\bullet 2) \Rightarrow 1-\eta^2 = 1 - \frac{1}{1+\frac{k^2}{16}} = \frac{k^2/16}{1+k^2/16}.$$

$$\Rightarrow \frac{1}{\sqrt{1-\eta^2}} = \frac{\sqrt{16+k^2}}{k}.$$

Generalisation.

Take $\underset{\sim}{x}_0 = \underset{\sim}{x}$ & $\underset{\sim}{x}_1 = \underset{\sim}{y}$ as defined in (1).

Also take $x_\lambda = \underset{\sim}{x} + \lambda \eta \underset{\sim}{d_1}$

then $x_\lambda' = \|x'\| d_3 + \lambda \eta \underset{\sim}{d_1}'$ & $x_1' = \|x'\| d_3 + \eta d_1'$.

$\Rightarrow x_\lambda' \cdot x_1' = \|x'\|^2 + \eta(1+\lambda)\, d_1' \cdot d_3 + \lambda^2 \eta^2 d_1'^2 > 0$ for η sufficiently small.

$\Rightarrow |\angle x_\lambda' \cdot x_1'| \le \pi/2 < \pi$.

In the formula of the theorem, change variable according to $s = (b-a)t + a$.

$\Rightarrow ds = (b-a)\, dt$.

to get.

$$Wr(x_1) - Wr(x_0) = Wr(y) - Wr(x)$$

$$= \frac{1}{2\pi}\int_a^b \frac{d_3(s) \times D_3(s)}{1 + d_3(s) \cdot D_3(s)} \cdot \frac{d}{ds}\big(d_3(s) + D_3(s)\big)\, ds \qquad (*)$$

The numerator in (*) can be written.

Recall

$y' = \|x'\| d_3 + \eta (u \times d_1)$.

$$\left(\frac{x'}{\|x'\|} \times \frac{y'}{\|y'\|}\right) \cdot \frac{d}{ds}\left(\frac{x'}{\|x'\|} + \frac{y'}{\|y'\|}\right)$$

$$= \left[d_3 \times \left(\frac{\|x'\|}{\|y'\|} d_3 + \frac{\eta}{\|y'\|}(u \times d_1)\right)\right] \cdot \left(d_3' + \frac{y''}{\|y''\|}\right)$$

Cause scalar prod with $(d_3 \times \cdot)$

$$= \frac{\eta}{\|y'\|}\left[d_3 \times (u \times d_1)\right] \cdot \left(d_3' + \frac{1}{\|y'\|}\left(\cancel{\|x'\|' d_3} + \|x'\|\, d_3' + \eta\, u' \times d_1 + \eta\, u \times (u \times d_1)\right)\right)$$

$$= \frac{\eta}{\|y'\|^2}(-u_3)\, d_1 \cdot \left((\|y'\| + \|x'\|)\, d_3' + \eta\, \cancel{u' \times d_1}^{\,0} + \eta\,(u_1 u - u^2 d_1)\right)$$

$$= -\frac{\eta u_3}{\|y'\|^2}\left((\|y'\| + \|x'\|)\, u_2 - \eta\,(u_2^2 + u_3^2)\right).$$

But $\|y'\|^2 = (\|x'\| d_3 + \eta (u \times d_1)) \cdot (\|x'\| d_3 + \eta (u \times d_1))$

$$= \|x'\|^2 + 2\|x'\|\eta(-u_2) + \eta^2 (u_2^2 + u_3^2)$$

So that $-\eta^2(u_2^2 + u_3^2) = \|x'\|^2 - \|y'\|^2 - 2\|x'\|\eta u_2$

. . . . Then the numerator becomes.

$$-\frac{u_3}{\|z'\|^2}\left(\eta\|z'\|u_2 + \eta\|x'\|u_2 + \|x'\|^2 - \|z'\|^2 - 2\|x'\|\eta u_2\right).$$

$$= -\frac{u_3}{\|z'\|^2}\left(\|x'\|^2 - \|z'\|^2 - \eta u_2\left(\|x'\| - \|z'\|\right)\right)$$

$$\text{Num.} = -\frac{u_3}{\|z'\|^2}\left(\|x'\| - \|z'\|\right)\left(\|x'\| + \|z'\| - \eta u_2\right) \qquad (\dagger)$$

Next we turn to the denominator

$$1 + d_3 \cdot D_3 = 1 + \frac{x'}{\|x'\|}\cdot\frac{z'}{\|z'\|}$$

$$= \frac{1}{\|z'\|}\left(\|z'\| + d_3 \cdot \left(\|x'\| d_3 + \eta(u \times d_1)\right)\right)$$

$$= \frac{1}{\|z'\|}\left(\|z'\| + \|x'\| - \eta u_2\right) \qquad (\ddagger)$$

Substituting (†) and (‡) in (*) gives.

$$W_r(z) - W_r(x) = \frac{1}{2\pi}\int_a^b \frac{\frac{u_3}{\|z'\|^2}\left(\|z'\| - \|x'\|\right)\left(\cancel{\|x'\| + \|z'\| - \eta u_2}\right)}{\frac{1}{\|z'\|}\left(\cancel{\|z'\| + \|x'\| - \eta u_2}\right)}\, ds.$$

$$= \frac{1}{2\pi}\int_a^b u_3\left(1 - \frac{\|x'\|}{\|z'\|}\right) ds.$$