Differential Geometry of Framed Curves
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Session 3: Exercises T. Lessinnes

In this exercise session, s stands for a generic parameterisation of curves. In particular, it is not necessarily the arc-length.

1. Composition of Darboux vectors

Given a curve $\mathbf{r}(s)$ and two orthonormal framings $\{\mathbf{d}_i(s)\}$ and $\{\mathbf{D}_i(s)\}$ where $\mathbf{D}_i$ and $\mathbf{d}_i$ are column vectors, their two direction cosine matrices $\mathbf{d}$ and $\mathbf{D}$ are related by a simple rotation

$$\mathbf{D} = \mathbf{d}\,Q, \tag{1}$$

where each matrix can depend upon s. The relation (1) implies that $Q_{ij} = \mathbf{D}_i \cdot \mathbf{d}_j$.

Let $\mathbf{u}(s) = \mathsf{u}_i(s)\,\mathbf{d}_i(s)$ be the Darboux vector associated with the frame $\{\mathbf{d}_i\}$ with components u satisfying

$$\mathsf{u}^\times = \mathbf{d}^T\mathbf{d}',$$

and let $\mathbf{U}(s) = \mathsf{U}_i(s)\,\mathbf{D}_i(s)$ be the Darboux vector associated with the frame $\{\mathbf{D}_i\}$ with components U satisfying

$$\mathsf{U}^\times = \mathbf{D}^T\mathbf{D}'.$$

a) Show that

$$\mathsf{U}^\times = Q^T\,\mathsf{u}^\times\,Q + Q^T\,Q'.$$

b) Accordingly, show that

$$\mathbf{U} = \mathbf{u} + \mathbf{D}_i\,\mathsf{p}_i, \tag{2}$$

where the components p respect $\mathsf{p}^\times = Q^TQ'$.

c) Simplify (2) for the case when the two frames $\{\mathbf{D}_i\}$ and $\{\mathbf{d}_i\}$ share a common vector $\mathbf{D}_3(s) = \mathbf{d}_3(s)$, as would arise in the case of any two adapted framings.

d) In the further case when $\{\mathbf{D}_i\}$ is the Frenet frame, so that $\mathbf{D}_3 = \mathbf{t} = \mathbf{d}_3$, find the explicit expressions of the components of $\mathbf{u}$ in the Frenet frame in function of the curvature κ and torsion τ of the curve $\mathbf{r}$.

2. Factorisation of curves in $SE(3)$

Let $\mathcal{X}(s)$ and $\mathcal{Y}(s)$ be two curves in $SE(3)$ with homogeneous coordinates

$$\mathcal{X}(s) = \begin{pmatrix} X & \mathbf{x} \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad \mathcal{Y}(s) = \begin{pmatrix} Y & \mathbf{y} \\ 0 & 1 \end{pmatrix}.$$

Define a third curve in $SE(3)$ via

$$\mathcal{Z}(s) = \mathcal{X}(s)\mathcal{Y}(s). \tag{3}$$

There exist vectors u, v, w, a, b and c such that

$$\mathcal{X}' = \mathcal{X}\begin{pmatrix} \mathsf{u}^\times & \mathsf{a} \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{Y}' = \mathcal{Y}\begin{pmatrix} \mathsf{v}^\times & \mathsf{b} \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{Z}' = \mathcal{Z}\begin{pmatrix} \mathsf{w}^\times & \mathsf{c} \\ 0 & 0 \end{pmatrix}.$$

In parallel to what you have already seen for curves in $SO(3)$, find an expression for w and c as a function of $\mathcal{X}$, $\mathcal{Y}$, u, v, a and b.

3. Offset of a curve in $\mathbb{R}^3$

Assume that

$$\mathcal{X}(s) = \begin{pmatrix} X(s) & \mathbf{x}(s) \\ 0 & 1 \end{pmatrix},$$

is the $SE(3)$ curve corresponding to $\mathbf{x}(s)$, a prescribed curve in $\mathbb{R}^3$ equipped with the adapted frame $X(s) = [\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3]$ where s is not necessarily arc-length.

Given a real number $\epsilon > 0$, the curve

$$\mathbf{z}(s) = \mathbf{x}(s) + \epsilon\, \mathbf{d}_1(s),$$

in $\mathbb{R}^3$ is called an offset of $\mathbf{x}$.

- Under what condition can you guarantee that $\mathbf{z}'(s) \neq \mathbf{0}$ for all s?
- Show that if $\mathbf{z}'(s) \neq \mathbf{0}$ for all s, it is possible to equip the $\mathbb{R}^3$ curve $\mathbf{z}(s)$ with an adapted frame

$$\mathcal{Z}(s) = \begin{pmatrix} D(s) & \mathbf{z}(s) \\ 0 & 1 \end{pmatrix},$$

  where $D(s) = \begin{pmatrix} \mathbf{D}_1(s) & \mathbf{D}_2(s) & \mathbf{D}_3(s) \end{pmatrix}$ and such that $\mathbf{D}_1(s) = \mathbf{d}_1(s)$.
- Find the curve $\mathcal{Y}(s) \in SE(3)$ such that

$$\mathcal{Z}(s) = \mathcal{X}(s)\mathcal{Y}(s).$$

  Compute the explicit form of w and c from exercise 1.
- $\left[\text{hint: } Y \text{ is of the form } Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.\right]$