Differential Geometry of Framed Curves
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Session 4: Exercises T. Lessinnes

Given two smooth closed oriented curves C_1 (or $\mathbf{x}(s)$) and C_2 (or $\mathbf{y}(\sigma)$) in $\mathbb{R}^3$ such that $C_1 \cap C_2 = \{\varnothing\}$ (that is there no intersection between C_1 and C_2), we defined their Linking number Lk as

$$\mathrm{Lk}(C_1,C_2) = \frac{1}{4\pi}\int_{C_1}\int_{C_2} \frac{(\mathbf{y}(\sigma)-\mathbf{x}(s))\cdot(\mathbf{y}'(\sigma)\times\mathbf{x}'(s))}{||\mathbf{y}(\sigma)-\mathbf{x}(s)||^3}\,d\sigma\, ds.$$

Here σ and s are not necessarily arc-length parameterisations. As shown in class the value of Lk is unaffected by orientation-preserving reparametrisations of either curve; and the sign of Lk is switched if the orientation of one curve is switched.

Note that while the non-intersection of $\mathbf{y}(\sigma)$ and $\mathbf{x}(s)$ is an important hypothesis, the self-intersection of $\mathbf{y}(\sigma)$ with itself, or $\mathbf{x}(s)$ with itself, is not a significant difficulty.

1 Link as a signed area

Introduce the unit vector field $\mathbf{e}(\sigma, s) = \frac{\mathbf{y}(\sigma)-\mathbf{x}(s)}{||\mathbf{y}(\sigma)-\mathbf{x}(s)||}$ and notice that because the curves $\mathbf{y}$ and $\mathbf{x}$ do not intersect, $\mathbf{e}(\sigma, s)$ is well-defined and smooth for all σ and s. It is also periodic because $\mathbf{y}(\sigma)$ and $\mathbf{x}(s)$ are periodic.

If the triple bracket $[\mathbf{a},\mathbf{b},\mathbf{c}]$ denotes the scalar triple product $[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, show that

$$[\mathbf{e},\mathbf{e}_s,\mathbf{e}_\sigma] = \frac{(\mathbf{y}(\sigma)-\mathbf{x}(s))\cdot(\mathbf{y}'(\sigma)\times\mathbf{x}'(s))}{||\mathbf{y}(\sigma)-\mathbf{x}(s)||^3},$$

where $\mathbf{e}_\sigma$ and $\mathbf{e}_s$ are partial derivatives of $\mathbf{e}(\sigma, s)$ (at fixed $\mathbf{y}$ and $\mathbf{x}$).

Accordingly, the Link integral can be rewritten as:

$$\frac{1}{4\pi}\int_{C_1}\int_{C_2} [\mathbf{e},\mathbf{e}_s,\mathbf{e}_\sigma]\, d\sigma\, ds. \tag{1}$$

Show that therefore, Lk is the signed surface area of a (often multi-covered) portion of a sphere. We will discuss this further in next week's lecture.

2 Homotopy invariance

First order variations $\mathbf{x}(s;\epsilon) = \mathbf{x}(s) + \epsilon\,\delta\mathbf{x}(s)$ and $\mathbf{y}(\sigma;\epsilon) = \mathbf{y}(\sigma) + \epsilon\,\delta\mathbf{y}(\sigma)$ of the curves $\mathbf{x}$ and $\mathbf{y}$ generate a first order variation $\mathbf{e}(s,\sigma;\epsilon) = \mathbf{e}(s,\sigma) + \epsilon\,\delta\mathbf{e}(s,\sigma)$ of the unit vector field $\mathbf{e}(s,\sigma)$. Compute

$$\delta\mathbf{e} = \left.\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}.$$

Then show that the corresponding variation of the Link integral, i.e.

$$\delta\mathrm{Lk} = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathrm{Lk}(\mathbf{x}+\epsilon\,\delta\mathbf{x}, \mathbf{y}+\epsilon\,\delta\mathbf{y})\right|_{\epsilon=0},$$

is identically zero for all doubly-periodic unit vector fields $\mathbf{e}$.

[Hints: Notice that any derivative of $[\mathbf{a},\mathbf{b},\mathbf{c}]$ satisfies the 'product rule': $[\mathbf{a},\mathbf{b},\mathbf{c}]' = [\mathbf{a}',\mathbf{b},\mathbf{c}] + [\mathbf{a},\mathbf{b}',\mathbf{c}] + [\mathbf{a},\mathbf{b},\mathbf{c}']$. You must integrate by parts (in s on the $\delta\mathbf{e}_s$ and in σ on the $\delta\mathbf{e}_\sigma$ term) and use periodicity to conclude that no boundary term arises. You must also use skew-symmetry properties of the triple product, and the fact that because $\mathbf{e}(\sigma, s)$ is a unit vector field $\mathbf{e}\cdot\mathbf{e}_s = \mathbf{e}\cdot\mathbf{e}_\sigma = \mathbf{e}\cdot\delta\mathbf{e} = 0$, or in other words $\mathbf{e}_s$, $\mathbf{e}_\sigma$ and $\delta\mathbf{e}$ are co-planar so that $[\mathbf{e}_s,\mathbf{e}_\sigma,\delta\mathbf{e}] = 0$.]

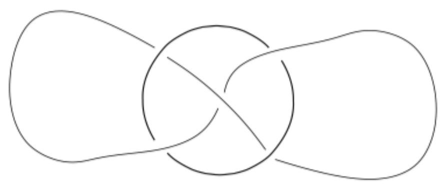
This computation is valid at all non-intersecting $\mathbf{y}$ and $\mathbf{x}$ (so that $\mathbf{e}$ is smooth), and suffices to show that Lk is a homotopy invariant.

3 Unknotted curves and the Whitehead link

If $\mathbf{y}(\sigma)$ and $\mathbf{x}(s)$ can be moved by homotopy, while keeping $C_1 \cap C_2 = \{\varnothing\}$, to opposite sides of a plane (in particular if $\mathbf{y}$ and $\mathbf{x}$ are not linked to each other in an intuitive sense), then $\mathrm{Lk}(\mathbf{y}, \mathbf{x}) = 0$. This is obviously true if we consider the definition of Lk based on signed crossings.

To prove this from the point of view of the integral definition, show that Lk can be bounded from above by an arbitrarily small number by the homotopy in which $\mathbf{x}(s)$ (say) is moved to infinity by translation along the normal to the separating plane.

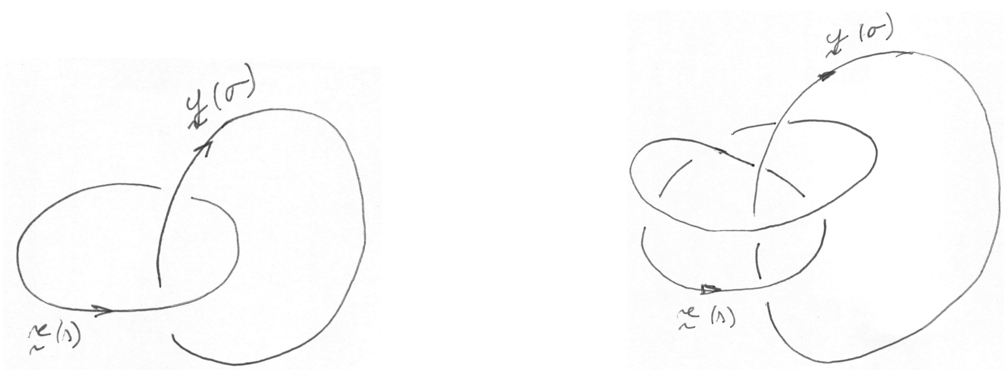
Unfortunately the converse is not true. Use the method of counting signed crossings to show that the Whitehead link shown in the figure below, and for any choices of orientations, has $\mathrm{Lk} = 0$ despite the fact that the two strands are physically linked: they cannot be moved arbitrarily far apart under a homotopy respecting $C_1 \cap C_1 = \{\varnothing\}$, and $C_2 \cap C_2 = \{\varnothing\}$. How could you use the result of Problem 2 to show



(without counting signed crossings) that the Whitehead link integral vanishes?

4 The Hopf link

The Hopf link is represented on the left of the following figure.



Evaluate its Link integral by explicit integration after choosing explicit parameterizations where the second loop $\mathbf{y}(\sigma)$ is formed by a straight segment $[-L, L]$ of the z-axis plus a smooth closure lying outside the ball of radius L and the unit circle $\mathbf{x}(s)$ is in the xy plane. You need to show

1. that the contribution to the Lk integral from the part of the curve $y(\sigma)$ outside the ball of radius L is arbitrarily small for $L \to \infty$ (which is a homotopy under which Lk is invariant).

2. that the remaining part of the integral concerning the Link between the straight part of $\mathbf{y}$ and the circle $\mathbf{x}$ is an integer. What is the interpretation of this integer?
(Hint: Use substitution by $\sinh z$ and the fact that $\tanh' z = \frac{1}{\cosh^2 z}$ and $\tanh z \to 1$ for $z \to \infty$.)

How would you use today's results to compute Lk for the link on the right of the figure?