Differential Geometry of Framed Curves
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Session 5: Exercises — T. Lessinnes

Given two smooth closed oriented curves $\mathbf{x}(s)$ and $\mathbf{y}(\sigma)$ parameterised by their respective arc-length and such that $\mathbf{x}''(s)$ and $\mathbf{y}''(\sigma)$ never vanish, consider the surface $\mathbf{r}(s,\sigma) = \mathbf{y}(\sigma) - \mathbf{x}(s)$. Whenever $\mathbf{r}_s \wedge \mathbf{r}_\sigma \neq 0$, the unit normal $\mathbf{n}$ to $\mathbf{r}$ is defined by

$$\mathbf{n}(s,\sigma) = \frac{\mathbf{r}_s \wedge \mathbf{r}_\sigma}{||\mathbf{r}_s \wedge \mathbf{r}_\sigma||}. \tag{1}$$

1. **Taylor expansion warm up.** Compute the Taylor expansion of $\mathbf{r}$ around a prescribed point (s_0, σ_0) up to second order. Then, express your result as a function of the Frenet frames and curvatures of $\mathbf{x}$ and $\mathbf{y}$. Also compute the Taylor expansion of $\mathbf{r}_s \wedge \mathbf{r}_\sigma$ to first order.

2. **In general, it is not possible to define a continuous field of unit normals on r.** Prove that if there exists a point (s_0, σ_0) such that the tangent to $\mathbf{x}$ at s_0 is parallel to the tangent to $\mathbf{y}$ at σ_0, then it is impossible to complete the definition (1) of $\mathbf{n}$ in a continuous way. What do you think $\mathbf{r}$ looks like close to such a point?

3. **Such pathological points are isolated provided that $\mathbf{x}_{ss}(s_0) \wedge \mathbf{y}_{\sigma\sigma}(\sigma_0) \neq 0$.** Given a point (s_0, σ_0) such that the tangent to $\mathbf{x}$ at s_0 is parallel to the tangent to $\mathbf{y}$ at σ_0 and $\mathbf{x}_{ss}(s_0) \wedge \mathbf{y}_{\sigma\sigma}(\sigma_0) \neq 0$, prove that it is not possible to choose a positive number $a > 0$ and a regular curve

$$\gamma : t \in (-a, a) \to \big(s(t), \sigma(t)\big),$$

such that both $\gamma(0) = (s_0, \sigma_0)$ and $\mathbf{r}_s\big(\gamma(t)\big) \wedge \mathbf{r}_\sigma\big(\gamma(t)\big) = 0$ for all $t \in (-a, a)$.

4. **Provided that the two curves x and y never share an osculating plane, points subtending tangent rays from the origin form smooth curves on r.** Define the function $f(s,\sigma) = \mathbf{r}(s,\sigma) \cdot \big(\mathbf{r}_s(s,\sigma) \wedge \mathbf{r}_\sigma(s,\sigma)\big)$. Show that if $f(s_0, \sigma_0) = 0$, then there exists an (unique) open curve γ such that (s_0, σ_0) is in the image of γ and $f(\gamma) = 0$.

5. **Provided that the two curves x and y do not share an osculating plane, the pathological points are contained in curves on r subtending tangent rays from the origin.** Given a point (s_0, σ_0) such that the tangent to $\mathbf{x}$ at s_0 is parallel to the tangent to $\mathbf{y}$ at σ_0 and $\mathbf{x}_{ss}(s_0) \wedge \mathbf{y}_{\sigma\sigma}(\sigma_0) \neq 0$, show that there exists a curve $\gamma(t) = \big(s(t), \sigma(t)\big)$ and a positive number a such that $f\big(\gamma(t)\big) = 0$ for all $t \in (-a, a)$.