Differential Geometry of Framed Curves
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Session 7: Exercises T. Lessinnes

The writhe integral of a single curve C_1 (or $\mathbf{x}(s)$) is defined as the double integral

$$Wr(C_1) = \frac{1}{4\pi} \int_{C_1} \int_{C_1} \frac{(\mathbf{x}(\sigma) - \mathbf{x}(s)) \cdot (\mathbf{x}'(\sigma) \times \mathbf{x}'(s))}{||\mathbf{x}(\sigma) - \mathbf{x}(s)||^3} \mathrm{d}\sigma \, \mathrm{d}s. \tag{1}$$

We denote the integrand of the writhe integral by

$$I_{Wr}(\sigma, s) = \frac{(\mathbf{x}(\sigma) - \mathbf{x}(s)) \cdot (\mathbf{x}'(\sigma) \times \mathbf{x}'(s))}{||\mathbf{x}(\sigma) - \mathbf{x}(s)||^3}.$$

We will discuss the importance and applications of this integral in the next classes. In this exercise set we consider some properties of the Writhe integral.

Problems

1. Show that the Writhe integral is even about $s = \sigma$, that is $I_{Wr}(\sigma, s) = I_{Wr}(s, \sigma)$ for all $\sigma, s \in C_1$. (Easy, but important.)

2. Show that the Writhe of a curve is invariant under

- translations $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$, $\mathbf{a} \in \mathbb{R}^3$,
- rotations $\mathbf{x} \mapsto R\mathbf{x}$, $R \in SO(3)$, and
- dilations $\mathbf{x} \mapsto \lambda\mathbf{x}$, $0 < \lambda \in \mathbb{R}$

of the curve. What happens to the Writhe of a curve if $\mathbf{x} \mapsto R\mathbf{x}$ with $R \in O(3)$ and $|R| = -1$?

3. Show that the integrand of the Writhe vanishes pointwise for a planar (non-self-intersecting) curve, so that the Writhe of a planar curve is zero. How could you use exercise 2 to prove this last claim without knowing that I_{Wr} vanishes identically in the plane?

4. Suppose that there exists a curve $\mathbf{x}(s)$ with $s \in [0, L]$ such that

- $\mathbf{x}'' \in C^1$ and $\mathbf{x}''(s) \neq 0$ for all s,
- $\mathbf{x}(s) = \mathbf{x}(\sigma) \Rightarrow s = \sigma$,
- The unit vector $\mathbf{e}(s, \sigma) = \frac{\mathbf{x}(s) - \mathbf{x}(\sigma)}{||\mathbf{x}(s) - \mathbf{x}(\sigma)||}$ satisfies $\mathbf{e} \cdot \big(\mathbf{e}_s \times \mathbf{e}_\sigma\big)(s, \sigma) = 0$ for all s and σ in $[0, L]$.

Prove that $\mathbf{x}$ must then be planar.

5. Define the non-symmetric doublet function $d(s, \sigma)$ as the diameter of the circle through $\mathbf{x}(s)$ and $\mathbf{x}(\sigma)$ and tangent to the curve at $\mathbf{x}(\sigma)$. Show that if $\mathbf{x}$ is parameterised by arclength, then

$$Wr = \frac{1}{4\pi} \int \int \mathrm{Sign}\Big[\mathbf{e} \cdot (\mathbf{e}_s \times \mathbf{e}_\sigma)\Big] \, \frac{\sin\psi}{d(s, \sigma)\, d(\sigma, s)} \, ds \, d\sigma, \tag{2}$$

where ψ is the angle between the planes subtended by $\mathbf{e}$ and $\mathbf{x}'(\sigma)$ and by $\mathbf{e}$ and $\mathbf{x}'(s)$.

6. Although the course focuses on closed curves, the double integral appearing in the definition (1) of the Writhe of a curve can be computed for open curves as well. Consider the circular helix of radius r and pitch $p\,\frac{r}{2\pi}$ parameterised by

$$\mathbf{x}(s) = \big(r\cos s,\, r\sin s,\, r\,p\,s\big),$$

where s spans $[0, L]$. Note that s is not the arc-length along the curve. By explicitly computing (1), show that the Writhe of $\mathbf{x}$ scales like

$$Wr(\mathbf{x}) \sim \frac{p}{2\pi}\left(\frac{1}{|p|} - \frac{1}{\sqrt{1+p^2}}\right)L \qquad \text{as } L \to +\infty.$$

[Hint: Computing the derivative of $\frac{w}{\sqrt{\sin^2 w + p^2 w^2}}$ w.r.t. w should prove useful.]