Differential Geometry of Framed Curves
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Session 10: Exercises — VERSION 2 — T. Lessinnes

1 Framings of closed curves (revisiting material from last lecture)

Let $\mathbf{x} : s \in [0, L] \mapsto \mathbf{x}(s) \in \mathbb{R}^3$ be a regular, closed, and C^3 curve.

1. Show that there exists a closed framing of $\mathbf{x}$.

a) Pick a framing, any framing. That is in fact, pick a function $\mathsf{u}_3(s)$ and an initial condition (see session 2).

b) Show how you can pick a different function U_3 so that the associated framing is closed.

2. We defined a Writhe framing as a framing the twist of which is $\mathsf{u}_3(s) = -\frac{1}{2} \int_0^L \mathrm{wr}(s, \sigma)\, d\sigma$. Show that a Writhe framing is always closed.

a) Define the register angle $\varphi(s)$ between a Writhe framing and the closed framing of question 1.

b) Use Lk = Tw + Wr to show that the difference of register between the beginning and end of $\mathbf{x}$ must be an integer multiple of $2\,\pi$.

3. Show that a Writhe framing always has zero Lk.

4. Show that the natural framings of a closed curves are closed if and only if the writhe is an integer.

5. More generally, show that if $D(s)$ is a natural framing of $\mathbf{x}$, then $D(L) = D(0)\, R_{3,\varphi}$ where

$$R_{3,\varphi} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad \varphi = 2\,\pi \left(Wr(\mathbf{x}) \bmod 1 \right).$$

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2 Writhe of a figure 8 curve (this is an example more than an exercise)

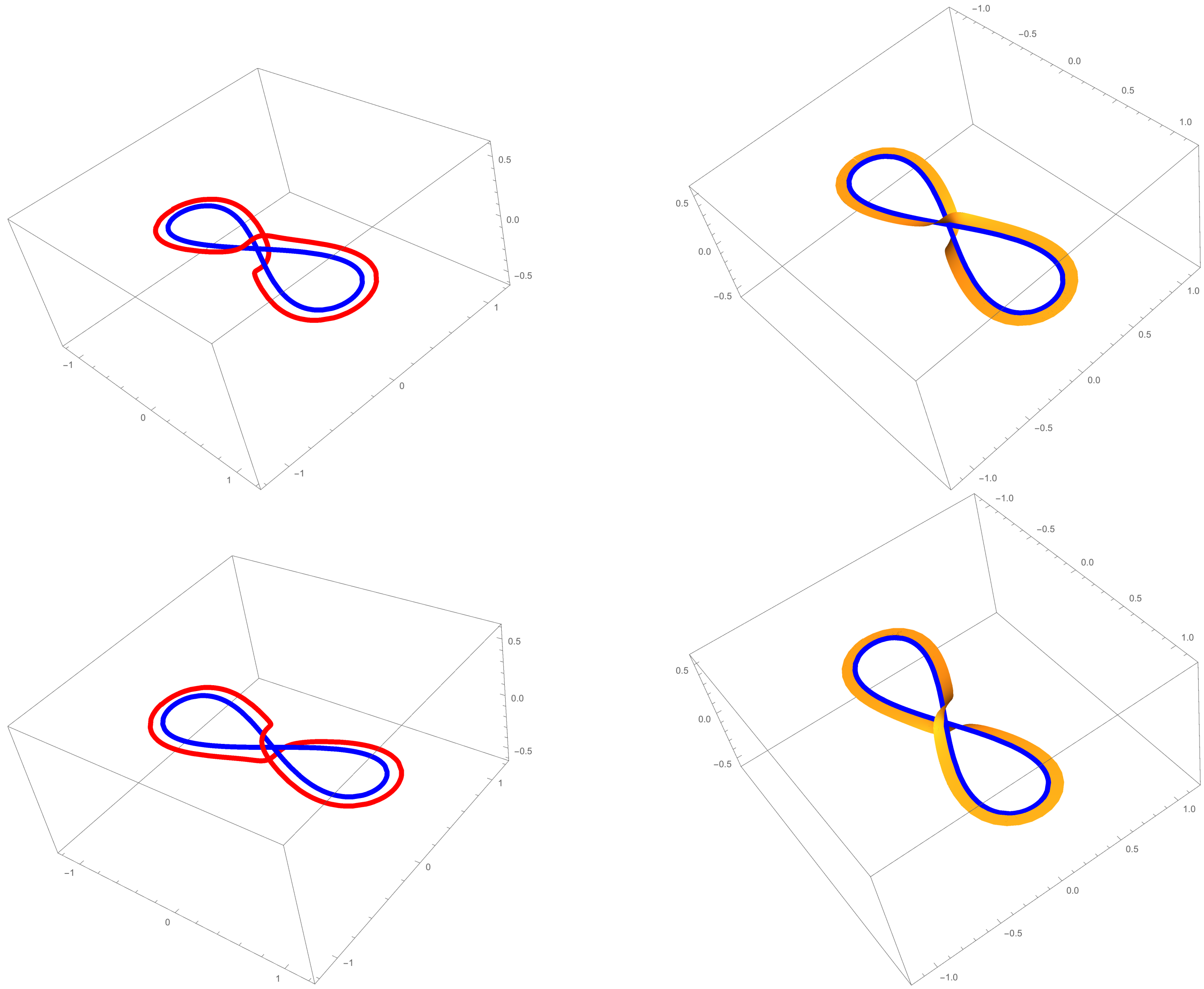

Figure 1: Figures of eight and their Writhe framing.

Consider the figure of eight curve

$$\mathbf{x}(t) = \begin{pmatrix} \cos t \\ \frac{\sin 2t}{2} \\ \nu \sin t \end{pmatrix}, \tag{1}$$

represented in Fig. 1. Note that when $\nu = 0$, $\mathbf{x}$ self-intersects so that Writhe is not defined. Here we study the limiting cases of $\nu \to 0$ from either side.

By direct application of the definition of Writhe, you could show (but beware: this is somewhat tedious) that

$$Wr(\mathbf{x};\, \nu) = \frac{\nu}{\pi} \int_0^{\pi} \frac{\cos\mu \quad d\mu}{\left(\nu^2 \,\cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2}\right) \sqrt{\nu^2 \,\cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2} + \cos^2 \mu}}. \tag{2}$$

1. Show that for all $\nu \in (0, 1)$ and for all $\mu \in (0, \pi/2)$, we have

$$\frac{\cos\mu}{\sqrt{\nu^2 \,\cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2} + \cos^2 \mu}} \leq 1, \tag{3}$$

and

$$1 - \left(\nu^2 \cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2}\right) \leq \frac{1}{\sqrt{\nu^2 \cos^2 \frac{\mu}{2} + \sin^2 \frac{\mu}{2} + \cos^2 \mu}}. \tag{4}$$

2. Accordingly, show that

$$\lim_{\nu \to 0^+} Wr(\mathbf{x};\, \nu) = 1. \tag{5}$$

3. What can you say about $\lim_{\nu \to 0^-} Wr(\mathbf{x};\, \nu)$?

The dependance of Wr on ν is shown in Fig. 2.

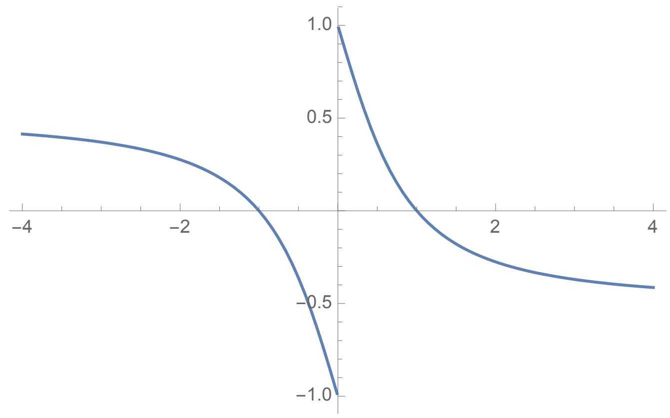

Figure 2: Writhe of the curve $\mathbf{x}$ of exercise 2 as a function of ν.

It is also possible to (numerically) compute the Writhe framing of each of these curves and those are displayed on Fig. 1.

3 Writhe Framing of the trefoil

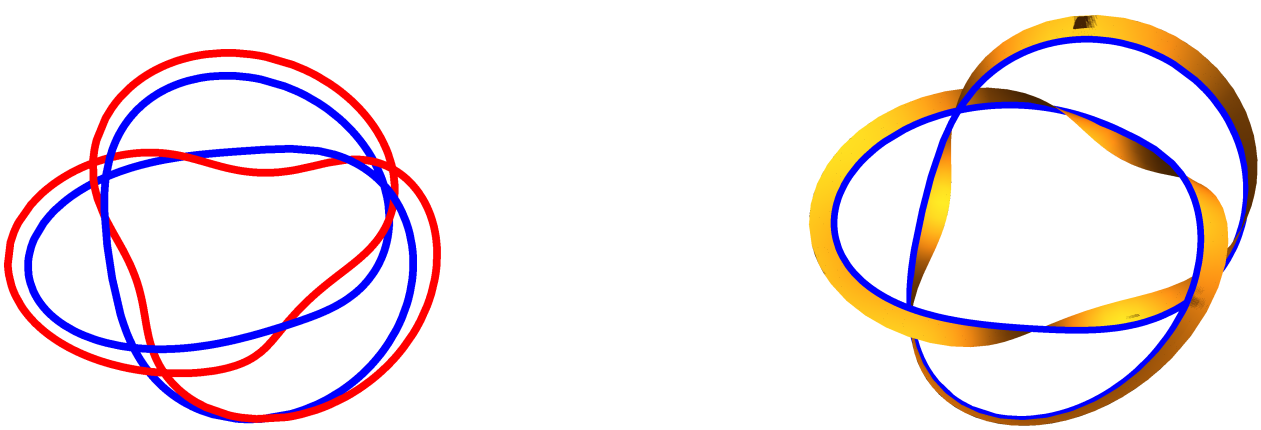

Figure 3: Writhe framing of the trefoil knot.

A Writhe framing of the trefoil is shown in Fig. 3. Show that $Lk = 0$ by counting crossings.

4 Writhe of an offset curve

Given a regular, closed curve $\mathbf{x} : s \in [a,b] \mapsto \mathbf{x}(s) \in \mathbb{R}^3$ and $X(t) = \big(\mathbf{d}_1(s) \;\; \mathbf{d}_2(s) \;\; \mathbf{d}_3(s)\big)$ an adapted and closed framing of $\mathbf{x}(s)$, let $\mathbf{u}(s) = \mathsf{u}_i(s)\,\mathbf{d}_i(s)$ be the Darboux vector of $X(s)$ and define the offset curve

$$\mathbf{z}(s) = \mathbf{x}(s) + \eta\,\mathbf{d}_1(s), \tag{6}$$

where η is sufficiently small such that for all $\epsilon \in (0,\eta]$ there is no intersection between the curves $\mathbf{x}(s)$ and the curve $\mathbf{x}(s) + \epsilon\,\mathbf{d}_1(s)$.

The aim of this exercise is to prove that

$$Wr(\mathbf{z}) = Wr(\mathbf{x}) + \frac{1}{2\pi}\int_a^b \mathsf{u}_3(s)\left(1 - \frac{||\mathbf{x}'(s)||}{||\mathbf{z}'(s)||}\right)\,ds. \tag{7}$$

This can be done in essentially two steps

1. Recall from session 3 that you know a particular adapted framing of $\mathbf{z}$ and compute the third component of the Darboux vector of that frame.

2. Apply $Lk = Tw + Wr$ together with the fact that $Lk(\mathbf{x},\mathbf{z}) = Lk(\mathbf{z},\mathbf{x})$.

To be continued... on week 14.