Differential Geometry of Framed Curves
Prof. John Maddocks
Session 11: Exercises T. Lessinnes

1 Yet another variant of the Euler-Rodrigues formula

Given a unit vector $\mathbf{n}$ along the axis of a right-handed rotation of angle ϕ, we have seen that the matrix $Q \in SO(3)$ associated with the rotation in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is given by

$$Q = \cos\phi \,\mathrm{Id} + (1-\cos\phi)\,\mathbf{n}\otimes\mathbf{n} + \sin\phi\,\mathbf{n}^\times. \tag{1}$$

Show that (1) can also be expressed as

$$Q = \mathrm{Id} + \sin\phi\,\mathbf{n}^\times + (1-\cos\phi)\,\mathbf{n}^\times\,\mathbf{n}^\times. \tag{2}$$

[Hint: First distribute the triple product $\mathbf{n}\times(\mathbf{n}\times\mathbf{v})$ for arbitrary vector $\mathbf{v}$.]

2 Euler-Rodrigues parameters

As discussed during last lecture, there are many ways to parameterize a general proper rotation matrix $Q \in SO(3)$ using only three or four parameters. Here we further study the four-dimensional parametrization in terms of Euler-Rodrigues parameters. Any element $q \in \mathbb{R}^4$ satisfying the condition

$$q\cdot q = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{3}$$

can be interpreted as a right-handed rotation of angle ϕ and the axis of which is along the unit vector $\mathbf{n}$, where ϕ and $\mathbf{n}$ solve

$$\cos\phi/2 = q_0, \qquad \text{and} \qquad \mathbf{n}\sin\phi/2 = \begin{pmatrix} q_1\\ q_2\\ q_3\end{pmatrix}.$$

Note that the condition (3) also means that $q \in S^3$, where S^3 is the unit sphere in $\mathbb{R}^4$.

(a) Show that the rotation matrix $Q \in SO(3)$ associated with q is

$$Q(q) = \mathrm{Id} + 2(q_0\,\mathbf{q}^\times + \mathbf{q}^\times\mathbf{q}^\times) = (q_0^2 - \mathbf{q}\cdot\mathbf{q})\,\mathrm{Id} + 2\,\mathbf{q}\otimes\mathbf{q} + 2\,q_0\,\mathbf{q}^\times, \tag{4}$$

where $\mathbf{q} = (q_1\, q_2\, q_3)^T$. Expand the very RHS of (4) to show that

$$Q(q) = \begin{pmatrix} q_1^2 - q_2^2 - q_3^2 + q_0^2 & 2(q_1q_2 - q_3q_0) & 2(q_1q_3 + q_2q_0)\\ 2(q_1q_2 + q_3q_0) & -q_1^2 + q_2^2 - q_3^2 + q_0^2 & 2(-q_1q_0 + q_2q_3)\\ 2(q_1q_3 - q_2q_0) & 2(q_1q_0 + q_2q_3) & -q_1^2 - q_2^2 + q_3^2 + q_0^2 \end{pmatrix}. \tag{5}$$

The formula (5) is a direct parameterisation of $SO(3)$ by elements of S^3. Next, we check that the matrix $Q(q)$ indeed has two expected properties.

(b) Given $q \in S^3$, $Q(q) \in SO(3)$ defined as in (a) and $\mathbf{q} = (q_1\, q_2\, q_3)^T$, assume that $\mathbf{q} \neq \mathbf{0}$ and show that $Q(q)\mathbf{q} = \mathbf{q}$; that is $\mathbf{q}$ defines the axis of rotation of Q. What can be said about Q if $\mathbf{q} = \mathbf{0}$?

(c) Let θ be the argument of one of the complex eigenvalue of $Q(q)$. Show that $\cos(\theta/2) = \pm q_0$.

Finally, here are a couple of results for later use

(d) Show that the application (5) is 2-to-1. That is any element $q \in S^3$ corresponds to a single element element of $SO(3)$ but for any element of $R \in SO(3)$ there are exactly two elements of S^3 sent on R by (5). What is the relation between those two elements of S^3.

(e) Given $q \in S^3$, show that $\{q, B_1\, q, B_2\, q, B_3\, q\}$ is an orthonormal basis for $\mathbb{R}^4$, where the matrices B_i $(i = 1, 2, 3)$ are defined by

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

$$B_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{7}$$

and

$$B_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{8}$$

3 Quaternion manipulation

We have seen that the set $\mathbb{H}$ of quaternions is obtained as a field extension of the real numbers to which the three elements i, j and k have been added. These three elements are such that they commute with real numbers and respect the following multiplication rules:

$$ijk = ii = jj = kk = -1. \tag{9}$$

1. Show that (9) together with the requirement that $\mathbb{H}$ forms a field imply $ij = k$ and $ji = -k$. What can you say about the other products of two elements in $\{i,\, j,\, k\}$?

2. Show that if you have two quaternions $q = (q_0,\, \mathbf{q})$ and $p = (p_0,\, \mathbf{p})$, then

$$q\, p = \Big(q_0\, p_0 - \mathbf{q} \cdot \mathbf{p},\; q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p}\Big). \tag{10}$$

3. Remember that a quaternion $q = (q_0,\, \mathbf{q})$ has conjugate $\overline{q} = (q_0, -\mathbf{q})$. Using formula (10), show that if r is a pure quaternion, then

$$q\, r\, \overline{q} = \Big(0,\; \big[(q\overline{q})(Id) + 2\, q_0\, \mathbf{q}^\times + 2\, \mathbf{q}^\times \mathbf{q}^\times\big]\; \mathbf{r}\Big). \tag{11}$$

Discuss similarities with (4).