1 Quaternions of particular curves in $SO(3)$

1.1 Quaternion associated with the Frenet frame of a helix

Come back to the helix defined during the first question of the exercise session 1:

$$\boldsymbol{\alpha}(s) = \left(a \cos\frac{s}{c},\, a\sin\frac{s}{c},\, \frac{b}{c}\, s\right), \tag{1}$$

where $a > 0$, $b > 0$ and $c = \sqrt{a^2 + b^2}$.

The Frenet frame $F(s)$ of $\boldsymbol{\alpha}(s)$ is a curve in $SO(3)$. Show that it can be described by the quaternion

$$q(s) = -\sin\frac{s}{2c}\sqrt{\frac{c+b}{2c}} - \sin\frac{s}{2c}\sqrt{\frac{c-b}{2c}}\, i + \cos\frac{s}{2c}\sqrt{\frac{c-b}{2c}}\, j + \cos\frac{s}{2c}\sqrt{\frac{c+b}{2c}}\, k. \tag{2}$$

1.2 Quaternion of the multiply covered circle

1. What is the quaternion describing a twist-less adapted frame to a n times covered circle (i.e the Frenet-Serret frame of a helix with 0 pitch)?

2. What is the quaternion describing an adapted frame to a n times covered circle if the local twist of the frame is prescribed by a function $\mathsf{u}_3(s)$ where s is the arc-length along the circle.

2 Cayley transforms

Let $N \in \mathbb{R}^{n\times n}$ such that $|I - N| \neq 0$. The Cayley transform of N is the matrix M defined by

$$M = (I + N)(I - N)^{-1}, \tag{3}$$

where I is the identity matrix in $\mathbb{R}^{n\times n}$.

2.1 A few general properties

1. Show that if M is the Cayley transform of some matrix N, then the matrix $I + M$ is invertible. [Hint: $I + M$ is not invertible iff $\exists \mathbf{v} \neq \mathbf{0} : (I + M)\mathbf{v} = \mathbf{0}$.]

2. Show that if M is the Cayley transform of some matrix N, then $N = (M + I)^{-1}(M - I)$.

3. Assume that Q is the Cayley transform of S. Show that $Q \in SO(n)$ if and only if S is skew. Is it true that for a general matrix $R \in SO(n)$ there exists a skew matrix U such that R is the Cayley transform of U?

2.2 The case $n = 3$

Assume that Q is the Cayley transform of a skew-symmetric matrix $S \in \mathbb{R}^{3\times 3}$.

1. Show that if $S = \mathbf{u}^\times$ for some vector $\mathbf{u}$, then $Q = \frac{1-\|\mathbf{u}\|^2}{1+\|\mathbf{u}\|^2} I + \frac{2}{1+\|\mathbf{u}\|^2}\mathbf{u}^\times + \frac{2}{1+\|\mathbf{u}\|^2}\mathbf{u}\otimes\mathbf{u}$. Interpreting Q as a rotation matrix, show that it corresponds to a right-handed rotation by an angle $2\,\mathrm{Arctan}(\|\mathbf{u}\|)$ about the axis along $\mathbf{u}/\|\mathbf{u}\|$.

2. Show that $S = (Q - Q^T)/(1 + \operatorname{tr} Q)$.

2.3 Composition of rotations

If $Q_1, Q_2 \in SO(3)$ with Cayley vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ – that is $Q_i = \mathrm{Cay}[\mathbf{u}_i^\times]$. Show that the Cayley vector of $Q_1 Q_2$ is

$$\frac{\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_1 \times \mathbf{u}_2}{1 - \mathbf{u}_1 \cdot \mathbf{u}_2}. \tag{4}$$

[Hint: There is an easy proof that uses the connection between Cayley vectors and quaternions and the composition formula for quaternions. There must also exist a linear algebra proof. Let us know if you find one...]

2.4 Cayley transforms in $SE(3)$

Show that if $\mathcal{Q}$ is the Cayley transform of $\mathcal{S} \in \mathbb{R}^{4\times 4}$, then $\mathcal{Q} \in SE(3)$ if and only if there exists two vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$\mathcal{S} = \begin{pmatrix} \mathbf{u}^\times & \mathbf{v} \\ 0 & 0 \end{pmatrix}.$$