Writhe of an offset curve

Given a regular, closed curve $\mathbf{x} : s \in [a,b] \mapsto \mathbf{x}(s) \in \mathbb{R}^3$ and $X(t) = \big(\mathbf{d}_1(s) \;\; \mathbf{d}_2(s) \;\; \mathbf{d}_3(s)\big)$ an adapted and closed framing of $\mathbf{x}(s)$, let $\mathbf{u}(s) = \mathsf{u}_i(s)\,\mathbf{d}_i(s)$ be the Darboux vector of $X(s)$ and define the offset curve

$$\mathbf{z}(s) = \mathbf{x}(s) + \eta\,\mathbf{d}_1(s), \tag{1}$$

where η is sufficiently small such that for all $\epsilon \in (0, \eta]$ there is no intersection between the curves $\mathbf{x}(s)$ and the curve $\mathbf{x}(s) + \epsilon\,\mathbf{d}_1(s)$.

Remember that we have seen in question 4 of session 10 that the Writhe of the offset curve $\mathbf{z}$ is

$$Wr(\mathbf{z}) = Wr(\mathbf{x}) + \frac{1}{2\pi}\int_a^b \mathsf{u}_3(s)\left(1 - \frac{||\mathbf{x}'(s)||}{||\mathbf{z}'(s)||}\right)\,ds. \tag{2}$$

Here we (2) that formula to find close form values of Wr for non-trivial curves.

A particular example

Consider the curve

$$\mathbf{z}(s) = \big(1 + \eta\,\cos(\omega(s))\big)\Big(\cos s\,\mathbf{e}_1 + \sin s\,\mathbf{e}_2\Big) - \eta\,\sin\omega(s)\,\mathbf{e}_3, \tag{3}$$

where the number $\eta \in (0,1)$ is a constant and the C^1 function ω is such that

$$\omega(2\,\pi) = k\,2\,\pi + \omega(0). \tag{4}$$

for some strictly positive integer k. This last condition ensures that $\mathbf{z}(s)$ is smoothly closed. Also it is easy to check that $\mathbf{z}(s)$ wraps k times around the unit circle in the $(\mathbf{e}_1, \mathbf{e}_2)$ plane. Examples of such curves $\mathbf{z}$ are given in the figure on next page.

- Find an explicit formula for the Writhe of $\mathbf{z}$ as a function of k and ω. You should be left with a single integral with ω and its first derivative appearing in the integrand.

Although your result can be integrated numerically, we next focus on a particular choice of ω and η such that the integration can be performed by hand. We pick (as was done to produce the figure)

$$\omega(s) = 2\pi\ \text{Ceil}\left(\frac{ks - \pi}{2\pi}\right) + 2\arctan\left[\sqrt{\frac{1+\eta}{1-\eta}}\,\tan\Big(\frac{2}{\eta}\sqrt{1-\eta^2}\,s\Big)\right], \tag{5}$$

where $\text{Ceil}(x) = \min\Big(\{z \in \mathbb{Z} : z \geq x\}\Big)$.

- Find η so that (5) respects the closure condition (4) for a given k (remember that $\omega(s)$ must be continuous).

- Show that for ω defined in (5) and with your choice of η,

$$Wr(\mathbf{z}) = k - \sqrt{\frac{16 + k^2}{17}}. \tag{6}$$

[Hint: First solve the differential equation $\omega'(s) = 4\,[1+\eta\,\cos\omega(s)]/\eta$ with initial condition $\omega(0) = 0$.]

Figure 1: Example of the curve $\mathbf{z}(s)$ defined in (3) with ω as defined in (5) for $k = 2$ (left), 10 (middle) and 18 (right).

Generalisation

Finally, show that Equation (2) is a special case of the following theorem by Fuller (fully proven by Aldinger *et al* in [1]):

Theorem 1. *Let $\mathbf{x}_0(t)$ and $\mathbf{x}_1(t)$ be two closed non self-intersecting space curves $\mathbf{x}_0$, $\mathbf{x}_1$: $[0,1] \mapsto \mathbb{R}^3$ of class C^3 with regular parameterization. Furthermore let $F : [0,1] \times [0,1] \mapsto \mathbb{R}^3$ be a C^0 deformation $(t, \lambda) \mapsto \mathbf{x}_\lambda(t)$ of $\mathbf{x}_0$ into $\mathbf{x}_1$ such that the $\mathbf{x}_\lambda$ are non self-intersecting curves of class C^1 and such that $\mathbf{t}_\lambda(t)$, the unit tangent vector to $\mathbf{x}_\lambda$ at t, changes continuously in λ.*

If $|\measuredangle\big(\mathbf{t}_1(t), \mathbf{t}_\lambda(t)\big)| < \pi$ for all $(t, \lambda) \in [0,1] \times [0,1]$ then:

$$Wr(\mathbf{x}_1) - Wr(\mathbf{x}_0) = \frac{1}{2\pi} \int_0^1 \frac{\mathbf{t}_0(t) \times \mathbf{t}_1(t)}{1 + \mathbf{t}_0(t) \cdot \mathbf{t}_1(t)} \cdot \frac{d}{dt}\Big(\mathbf{t}_0(t) + \mathbf{t}_1(t)\Big)\ dt. \tag{7}$$

Do you understand why (7) is much more general than (2)?

References

[1] J. Aldinger, I. Klapper, and M. Tabor. Formulae for the calculation and estimation of writhe. *Journal of Knot Theory and Its Ramifications*, 04(03):343–372, 1995.