# 1 Properties of the skew symmetric matrices

**1.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The vector product $\mathbf{u} \times \mathbf{v}$, in components, reads:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \tag{1}$$

From the equality above one can see that the following skew symmetric matrix

$$[\mathbf{u}\times] = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{2}$$

satisfies $\mathbf{u} \times \mathbf{v} = [\mathbf{u}\times]\mathbf{v}$.

**2.** Given any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, we can compute the following:

$$\begin{aligned} \boldsymbol{v}^T\left(|M|M^{-T}[\boldsymbol{u}\times]M^{-1}\right)\boldsymbol{w} &= |M|\boldsymbol{v}^T\left(M^{-T}[\boldsymbol{u}\times]M^{-1}\right)\boldsymbol{w} \\ &= |M|(M^{-1}\boldsymbol{v})^T[\boldsymbol{u}\times]M^{-1}\boldsymbol{w} \\ &= |M|(M^{-1}\boldsymbol{v})\cdot(\boldsymbol{u}\times M^{-1}\boldsymbol{w}) \\ &= |M|\left|\left[M^{-1}\boldsymbol{v} \vdots \boldsymbol{u} \vdots M^{-1}\boldsymbol{w}\right]\right| \\ &= |M||M^{-1}|\left|\left[\boldsymbol{v} \vdots M\boldsymbol{u} \vdots \boldsymbol{w}\right]\right| \\ &= \boldsymbol{v}\cdot(M\boldsymbol{u}\times\boldsymbol{w}) \\ &= \boldsymbol{v}^T[M\boldsymbol{u}\times]\boldsymbol{w}, \end{aligned}$$

As this is true for any $\boldsymbol{v}$ and $\boldsymbol{w}$, we can conclude that $[M\mathbf{u}\times] = |M|M^{-T}[\boldsymbol{u}\times]M^{-1}$. In the latter computations we denoted by $\left[\mathbf{a} \vdots \mathbf{b} \vdots \mathbf{c}\right]$ the three by three matrix which columns are the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

When $M \in SO(3)$, we have $[M\mathbf{u}\times] = M[\boldsymbol{u}\times]M^{-1}$.

**3.** For any $\mathbf{v} \in \mathbb{R}^3$, we have

$$[\mathbf{u}\times]^2\mathbf{v} = [\mathbf{u}\times]\left([\mathbf{u}\times]\,\mathbf{v}\right) = \mathbf{u}\times(\mathbf{u}\times\mathbf{v}) = (\mathbf{u}\cdot\mathbf{v})\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{v} = \left(\mathbf{u}\otimes\mathbf{u} - ||\mathbf{u}||^2\mathrm{I}\right)\mathbf{v}. \tag{3}$$

Since (3) holds for all $\mathbf{v}$ we have that $[\mathbf{u}\times]^2 = \mathbf{u}\otimes\mathbf{u} - ||\mathbf{u}||^2\mathrm{I}$.

# 2 Rotations in three dimensions

**1.** From the properties of the scalar product, $\forall \mathbf{w} \in \mathbb{R}^3$

$$\|Q\mathbf{w}\|^2 = Q\mathbf{w}\cdot Q\mathbf{w} = \mathbf{w}\cdot Q^TQ\mathbf{w} = \|\mathbf{w}\|^2 \tag{4}$$

where we have used the fact that $Q$ is a rotation matrix, i.e. $Q^TQ = \mathrm{I}$. If now $\lambda$ is an eigenvalue for $Q$, let $\mathbf{w}$ be the corresponding eigenvector

$$\|Q\mathbf{w}\| = \|\lambda\mathbf{w}\| = |\lambda|\|\mathbf{w}\| \tag{5}$$

but then from equation (4) we obtain

$$|\lambda| = 1 \tag{6}$$

and therefore $\lambda$ lies on the unit circle in the complex plane.

**2.** Note that the complex conjugate $\overline{\lambda}$ is an eigenvalue of $Q$ (with corresponding eigenvector $\overline{\mathbf{w}}$) whenever $\lambda$ is an eigenvalue of $Q$ (with corresponding eigenvector $\mathbf{w}$). This follows from $\overline{Q} = Q$. Explicitly,

$$Q\mathbf{w} = \lambda\mathbf{w} \Rightarrow Q\overline{\mathbf{w}} = \overline{Q\mathbf{w}} = \overline{\lambda\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}}.$$

As $Q$ has exactly 3 eigenvalues (counted according to multiplicity), we obtain from (6) that the set of eigenvalues of $Q$ is

$$\mathrm{eig}(Q) \; = \; \{e^{ix}, e^{-ix}, 1\} \quad \text{or} \quad \mathrm{eig}(Q) \; = \; \{e^{ix}, e^{-ix}, -1\}$$

for some $x \in [0, \pi]$. But for $\mathrm{eig}(Q) = \{e^{ix}, e^{-ix}, -1\}$ we get[1] $\det(Q) = -1$, so that we must have

$$\mathrm{eig}(Q) \; = \; \{e^{ix}, e^{-ix}, 1\} \quad \text{for } x \in [0, \pi]. \tag{7}$$

This also shows that $\lambda = 1$ cannot have multiplicity 2, but only 1 or 3. If $\lambda = 1$ has multiplicity 3, then $Q = \mathrm{Id}$, which therefore is the only case where there can be a non-unique axis of rotation.

For any eigenvalue $\lambda$ of $Q$, the inverse $\lambda^{-1}$ will be an eigenvalue of $Q^{-1} = Q^T$ corresponding to the same eigenvector. Therefore, $Q\mathbf{w} = \mathbf{w}$ immediately implies that $\mathbf{w}$ is an element of the nullspace of $S = (Q - Q^T)$, i.e. $S\mathbf{w} = 0$. On the other hand if $\mathbf{z}$ is the unique axial vector of the skew matrix $S$ (i.e. $S = [\mathbf{z}\times]$) , then

$$0 = S\mathbf{w} = \mathbf{z} \times \mathbf{w}, \tag{8}$$

so that $\mathbf{z}$ and $\mathbf{w}$ are parallel, i.e. the axial vector of the skew matrix is parallel to the axis of rotation of $Q$.

**3.** If $\mathbf{v}$ is any vector orthogonal to the axis of rotation $\mathbf{w}$, then

$$\begin{aligned} Q\mathbf{v} \cdot \mathbf{w} &= \mathbf{v} \cdot Q^T\mathbf{w} \\ &= \mathbf{v} \cdot \mathbf{w} \\ &= 0 \end{aligned} \tag{9}$$

Furthermore from (4) we can conclude that if $\mathbf{v}$ is a unit vector, then so is $Q\mathbf{v}$.

On the one hand, the angle $\theta$ between $Q\mathbf{v}$ and $\mathbf{v}$ obeys

$$Q\mathbf{v} \cdot \mathbf{v} = \|Q\mathbf{v}\| \, \|\mathbf{v}\| \cos\theta \overset{(4)}{=} \cos\theta. \tag{10}$$

[1] Here, we make use of

$$\det(A) = \prod_{i=1}^{n} \lambda_i,$$

for any matrix $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ repeated according to (algebraic) multiplicity.

On the other hand the trace of a matrix is the sum of its eigenvalues[2]. Then (7) implies

$$\operatorname{tr}(Q) = 1 + 2\cos x. \tag{11}$$

In the special cases $x = 0$ and $x = \pi$, the proof is immediate since $\cos x = \pm 1$ and $\mathbf{v}$ is itself an eigenvector and $Q\mathbf{v} = \pm\,\mathbf{v}$ so that $\cos\theta = \pm 1$.

Otherwise, $x \in (0, \pi)$ and we will show that $\cos x = \cos\theta$ holds in general. Along the way, we will need a number of properties regarding the complex eigenvectors of $Q$. We list them hereunder with their proof. Once and for all, $x \in (0, \pi)$ and $\mathbf{z} \in \mathbb{C}^3$ is a norm 1 eigenvector of $Q$ corresponding to the eigenvalue $e^{i\,x}$ from question 1.2. The hermitian product between complex vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{C}^3$ is noted $\langle \mathbf{a}, \mathbf{b}\rangle$. The scalar product between real vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$ is noted $\mathbf{x}\cdot\mathbf{y}$.

**Propostion 1.** *The conjugate $\overline{\mathbf{z}}$ of $\mathbf{z}$ is an eigenvector of $Q$ with eigenvalue $e^{-i\,x}$.*

*Proof.* See exercise 2.2. □

**Propostion 2.** *If $x \in (0, \pi)$, the eigenvector $\mathbf{z}$ is such that $\langle \mathbf{z}, \overline{\mathbf{z}}\rangle = 0$.*

*Proof.* Compute

$$e^{-i\,x}\,\langle \mathbf{z}, \overline{\mathbf{z}}\rangle = \langle e^{i\,x}\,\mathbf{z}, \overline{\mathbf{z}}\rangle = \langle Q\,\mathbf{z}, \overline{\mathbf{z}}\rangle = \langle \mathbf{z}, Q^T\,\overline{\mathbf{z}}\rangle = \langle \mathbf{z}, e^{i\,x}\,\overline{\mathbf{z}}\rangle = e^{i\,x}\,\langle \mathbf{z}, \overline{\mathbf{z}}\rangle. \tag{12}$$

And whenever $x \in (0, \pi)$, Eq. (12) implies $\langle \mathbf{z}, \overline{\mathbf{z}}\rangle = 0$. □

**Propostion 3.** *Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$ be respectively the real and imaginary part of the eigenvector $\mathbf{z} = \mathbf{x} + i\,\mathbf{y}$. If $x \in (0, \pi)$, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal: $\mathbf{x}\cdot\mathbf{y} = 0$. Furthermore,*

$$\mathbf{x}\cdot\mathbf{x} = \mathbf{y}\cdot\mathbf{y} = 1/2. \tag{13}$$

*Proof.* On the one hand, from Proposition 2 we have

$$\begin{aligned} 0 = \langle \mathbf{z}, \overline{\mathbf{z}}\rangle = \langle \mathbf{x} + i\,\mathbf{y}, \mathbf{x} - i\,\mathbf{y}\rangle &= \Big(\mathbf{x}\cdot\mathbf{x} - \mathbf{y}\cdot\mathbf{y}\Big) - 2\,i\,\Big(\mathbf{x}\cdot\mathbf{y}\Big), \\ \Rightarrow \mathbf{x}\cdot\mathbf{x} &= \mathbf{y}\cdot\mathbf{y}, \qquad \text{and} \qquad \mathbf{x}\cdot\mathbf{y} = 0. \end{aligned} \tag{14}$$

On the other hand, $\mathbf{z}$ is of norm 1 so that

$$\langle \mathbf{z}, \mathbf{z}\rangle = \mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} = 1 \overset{(14)}{\Rightarrow} \mathbf{x}\cdot\mathbf{x} = \mathbf{y}\cdot\mathbf{y} = \frac{1}{2}.$$

□

Now we are ready to proceed with the exercise. First we show that there exists a complex number $a \in \mathbb{C}$ such that $\mathbf{v} = a\,\mathbf{z} + \overline{a}\,\overline{\mathbf{z}}$. Indeed, since $\mathbf{z}$ and $\overline{\mathbf{z}}$ span the space orthogonal to $\mathbf{w}$ in $\mathbb{C}^3$, there exist complex numbers $a$, $b \in \mathbb{C}$ such that $\mathbf{v} = a\,\mathbf{z} + b\,\overline{\mathbf{z}}$. But then with $\mathbf{x}$ and $\mathbf{y}$ defined as in Proposition 3, the fact that $\mathbf{v}$ is a real vector implies

$$\big(\operatorname{Im}(a) + \operatorname{Im}(b)\big)\,\mathbf{x} + \big(\operatorname{Re}(a) - \operatorname{Re}(b)\big)\,\mathbf{y} = \mathbf{0} \Rightarrow b = \overline{a}, \tag{15}$$

[2] This comes from the fact that if $A \in \mathbb{R}^{n\times n}$ there exists $P \in SU(n)$ such that $P^{-1}AP$ is diagonal. Then $\operatorname{tr}(PAP^{-1})$ is the sum of the eigenvalues of $A$. But the trace is invariant under cyclic perturbations since $\operatorname{tr}(ABC) = A_{ij}B_{jk}C_{ki} = B_{jk}C_{ki}A_{ij} = \operatorname{tr}(BCA)$. So $\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(APP^{-1}) = \operatorname{tr} A$.

where the implication is due to Proposition 3: $\mathbf{x}$ and $\mathbf{y}$ are orthogonal and of non zero norm so that both brackets must vanish independently.

Then the fact that $\mathbf{v}$ is a unit vector implies that $|a|^2 = 1/2$. Indeed, $1 = \langle \mathbf{v}, \mathbf{v} \rangle = \langle a\mathbf{z} + \overline{a\mathbf{z}}, a\mathbf{z} + \overline{a\mathbf{z}} \rangle = |a|^2 + |\overline{a}|^2 = 2|a|^2$, where the third equality is due to Proposition 2.

Finally, simply compute

$$\begin{aligned} \cos\theta &\overset{(10)}{=} Q\,\mathbf{v}\cdot\mathbf{v} = \langle Q\,(a\,\mathbf{z} + \overline{a}\,\overline{\mathbf{z}}), a\,\mathbf{z} + \overline{a}\,\overline{\mathbf{z}} \rangle = \langle a\,e^{i\,x}\,\mathbf{z} + \overline{a}\,e^{-i\,x}\,\overline{\mathbf{z}}, a\,\mathbf{z} + \overline{a}\,\overline{\mathbf{z}} \rangle, \\ &\overset{Prop.(3)}{=} |a|^2(e^{-i\,x} + e^{i\,x}) = \frac{e^{-i\,x} + e^{i\,x}}{2} = \cos x. \end{aligned} \tag{16}$$

**4.** The idea of the proof is to decompose any vector $\mathbf{v} \in \mathbb{R}^3$, different from $\mathbf{u}$, in a component along $\mathbf{u}$ and an other component orthogonal to $\mathbf{u}$. We then apply the rotation $Q$ to the vector $\mathbf{v}$. We have that $\mathbf{v}$ can be written in the following manner:

$$\mathbf{v} = (\mathbf{u}\cdot\mathbf{v})\mathbf{u} + \mathbf{u}\times\mathbf{v}\times\mathbf{u} \tag{17}$$

We then apply to (17) the rotation $Q$:

$$Q\mathbf{v} = (\mathbf{u}\cdot\mathbf{v})\mathbf{u} + \cos(\phi)\mathbf{u}\times\mathbf{v}\times\mathbf{u} + \sin(\phi)\mathbf{u}\times\mathbf{v} \tag{18}$$

Next, we recall two properties of a double cross product

$$\mathbf{u}\times\mathbf{v}\times\mathbf{u} = -\mathbf{u}\times\mathbf{u}\times\mathbf{v} \tag{19}$$
$$\mathbf{u}\times\mathbf{u}\times\mathbf{v} = (\mathbf{u}\cdot\mathbf{v})\mathbf{u} - ||\mathbf{u}||^2\mathbf{v} \tag{20}$$

Using (19) and (20) in (18) we obtain

$$Q\mathbf{v} = ||\mathbf{u}||^2\mathbf{v} + \sin(\phi)\mathbf{u}\times\mathbf{v} + (1-\cos(\phi))\mathbf{u}\times\mathbf{u}\times\mathbf{v} \tag{21}$$
$$= \mathbf{v} + \sin(\phi)[\mathbf{u}\times]\mathbf{v} + (1-\cos(\phi))[\mathbf{u}\times]^2\mathbf{v} \tag{22}$$

As the latter expression is valid for any vector $\mathbf{v}$ we obtain that the matrix $Q$ can be expressed as

$$Q = \mathrm{I} + \sin(\phi)[\mathbf{u}\times] + (1-\cos(\phi))[\mathbf{u}\times]^2. \tag{23}$$

The second expression comes easily by using the result of the exercise 1.3 of this session in order to get

$$Q = \cos(\phi)\mathrm{I} + \sin(\phi)[\mathbf{u}\times] + (1-\cos(\phi))\mathbf{u}\otimes\mathbf{u} \tag{24}$$