

1 Properties of the skew symmetric matrices

1. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The vector product $\mathbf{u} \times \mathbf{v}$, in components, reads:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \quad (1)$$

From the equality above one can see that the following skew symmetric matrix

$$[\mathbf{u} \times] = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \quad (2)$$

satisfies $\mathbf{u} \times \mathbf{v} = [\mathbf{u} \times] \mathbf{v}$.

2. Given any two vectors \mathbf{v} and \mathbf{w} , we can compute the following:

$$\begin{aligned} \mathbf{v}^T (|M|M^{-T}[\mathbf{u} \times]M^{-1}) \mathbf{w} &= |M| \mathbf{v}^T (M^{-T}[\mathbf{u} \times]M^{-1}) \mathbf{w} \\ &= |M| (M^{-1} \mathbf{v})^T [\mathbf{u} \times] M^{-1} \mathbf{w} \\ &= |M| (M^{-1} \mathbf{v}) \cdot (\mathbf{u} \wedge M^{-1} \mathbf{w}) \\ &= |M| \left| \left[M^{-1} \mathbf{v} : \mathbf{u} : M^{-1} \mathbf{w} \right] \right| \\ &= |M| |M^{-1}| \left| \left[\mathbf{v} : M \mathbf{u} : \mathbf{w} \right] \right| \\ &= \mathbf{v} \cdot (M \mathbf{u} \times \mathbf{w}) \\ &= \mathbf{v}^T [M \mathbf{u} \times] \mathbf{w}, \end{aligned}$$

As this is true for any \mathbf{v} and \mathbf{w} , we can conclude that $[M \mathbf{u} \times] = |M| M^{-T} [\mathbf{u} \times] M^{-1}$. In the latter computations we denoted by $\left[\mathbf{a} : \mathbf{b} : \mathbf{c} \right]$ the three by three matrix which columns are the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} .

When $M \in SO(3)$, we have $[M \mathbf{u} \times] = M [\mathbf{u} \times] M^{-1}$.

3. For any $\mathbf{v} \in \mathbb{R}^3$, we have

$$[\mathbf{u} \times]^2 \mathbf{v} = [\mathbf{u} \times] ([\mathbf{u} \times] \mathbf{v}) = \mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} - (\mathbf{u} \cdot \mathbf{u}) \mathbf{v} = (\mathbf{u} \otimes \mathbf{u} - \|\mathbf{u}\|^2 \mathbf{I}) \mathbf{v}. \quad (3)$$

Since (3) holds for all \mathbf{v} we have that $[\mathbf{u} \times]^2 = \mathbf{u} \otimes \mathbf{u} - \|\mathbf{u}\|^2 \mathbf{I}$.

2 Rotations in three dimensions

1. From the properties of the scalar product, $\forall \mathbf{w} \in \mathbb{R}^3$

$$\|Q \mathbf{w}\|^2 = Q \mathbf{w} \cdot Q \mathbf{w} = \mathbf{w} \cdot Q^T Q \mathbf{w} = \|\mathbf{w}\|^2 \quad (4)$$

where we have used the fact that Q is a rotation matrix, i.e. $Q^T Q = I$. If now λ is an eigenvalue for Q , let \mathbf{w} be the corresponding eigenvector

$$\|Q\mathbf{w}\| = \|\lambda\mathbf{w}\| = |\lambda|\|\mathbf{w}\| \quad (5)$$

but then from equation (4) we obtain

$$|\lambda| = 1 \quad (6)$$

and therefore λ lies on the unit circle in the complex plane.

2. Note that the complex conjugate $\bar{\lambda}$ is an eigenvalue of Q (with corresponding eigenvector $\bar{\mathbf{w}}$) whenever λ is an eigenvalue of Q (with corresponding eigenvector \mathbf{w}). This follows from $\overline{Q\mathbf{w}} = Q\bar{\mathbf{w}}$. Explicitly,

$$Q\mathbf{w} = \lambda\mathbf{w} \Rightarrow Q\bar{\mathbf{w}} = \overline{Q\mathbf{w}} = \overline{\lambda\mathbf{w}} = \bar{\lambda}\bar{\mathbf{w}}.$$

As Q has exactly 3 eigenvalues (counted according to multiplicity), we obtain from (6) that the set of eigenvalues of Q is

$$\text{eig}(Q) = \{e^{ix}, e^{-ix}, 1\} \quad \text{or} \quad \text{eig}(Q) = \{e^{ix}, e^{-ix}, -1\}$$

for some $x \in [0, \pi]$. But for $\text{eig}(Q) = \{e^{ix}, e^{-ix}, -1\}$ we get¹ $\det(Q) = -1$, so that we must have

$$\text{eig}(Q) = \{e^{ix}, e^{-ix}, 1\} \quad \text{for } x \in [0, \pi]. \quad (7)$$

This also shows that $\lambda = 1$ cannot have multiplicity 2, but only 1 or 3. If $\lambda = 1$ has multiplicity 3, then $Q = \text{Id}$, which therefore is the only case where there can be a non-unique axis of rotation.

For any eigenvalue λ of Q , the inverse λ^{-1} will be an eigenvalue of $Q^{-1} = Q^T$ corresponding to the same eigenvector. Therefore, $Q\mathbf{w} = \mathbf{w}$ immediately implies that \mathbf{w} is an element of the nullspace of $S = (Q - Q^T)$, i.e. $S\mathbf{w} = 0$. On the other hand if \mathbf{z} is the unique axial vector of the skew matrix S (i.e. $S = [\mathbf{z} \times]$), then

$$0 = S\mathbf{w} = \mathbf{z} \times \mathbf{w}, \quad (8)$$

so that \mathbf{z} and \mathbf{w} are parallel, i.e. the axial vector of the skew matrix is parallel to the axis of rotation of Q .

3. If \mathbf{v} is any vector orthogonal to the axis of rotation \mathbf{w} , then

$$\begin{aligned} Q\mathbf{v} \cdot \mathbf{w} &= \mathbf{v} \cdot Q^T \mathbf{w} \\ &= \mathbf{v} \cdot \mathbf{w} \\ &= 0 \end{aligned} \quad (9)$$

Furthermore from (4) we can conclude that if \mathbf{v} is a unit vector, then so is $Q\mathbf{v}$.

On the one hand, the angle θ between $Q\mathbf{v}$ and \mathbf{v} obeys

$$Q\mathbf{v} \cdot \mathbf{v} = \|Q\mathbf{v}\| \|\mathbf{v}\| \cos \theta \stackrel{(4)}{=} \cos \theta. \quad (10)$$

¹Here, we make use of

$$\det(A) = \prod_{i=1}^n \lambda_i,$$

for any matrix $A \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ repeated according to (algebraic) multiplicity.

On the other hand the trace of a matrix is the sum of its eigenvalues². Then (7) implies

$$\operatorname{tr}(Q) = 1 + 2 \cos x. \quad (11)$$

In the special cases $x = 0$ and $x = \pi$, the proof is immediate since $\cos x = \pm 1$ and \mathbf{v} is itself an eigenvector and $Q\mathbf{v} = \pm \mathbf{v}$ so that $\cos \theta = \pm 1$.

Otherwise, $x \in (0, \pi)$ and we will show that $\cos x = \cos \theta$ holds in general. Along the way, we will need a number of properties regarding the complex eigenvectors of Q . We list them hereunder with their proof. Once and for all, $x \in (0, \pi)$ and $\mathbf{z} \in \mathbb{C}^3$ is a norm 1 eigenvector of Q corresponding to the eigenvalue e^{ix} from question 1.2. The hermitian product between complex vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$ is noted $\langle \mathbf{a}, \mathbf{b} \rangle$. The scalar product between real vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ is noted $\mathbf{x} \cdot \mathbf{y}$.

Proposition 1. *The conjugate $\bar{\mathbf{z}}$ of \mathbf{z} is an eigenvector of Q with eigenvalue e^{-ix} .*

Proof. See exercise 2.2. □

Proposition 2. *If $x \in (0, \pi)$, the eigenvector \mathbf{z} is such that $\langle \mathbf{z}, \bar{\mathbf{z}} \rangle = 0$.*

Proof. Compute

$$e^{-ix} \langle \mathbf{z}, \bar{\mathbf{z}} \rangle = \langle e^{ix} \mathbf{z}, \bar{\mathbf{z}} \rangle = \langle Q \mathbf{z}, \bar{\mathbf{z}} \rangle = \langle \mathbf{z}, Q^T \bar{\mathbf{z}} \rangle = \langle \mathbf{z}, e^{ix} \bar{\mathbf{z}} \rangle = e^{ix} \langle \mathbf{z}, \bar{\mathbf{z}} \rangle. \quad (12)$$

And whenever $x \in (0, \pi)$, Eq. (12) implies $\langle \mathbf{z}, \bar{\mathbf{z}} \rangle = 0$. □

Proposition 3. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ be respectively the real and imaginary part of the eigenvector $\mathbf{z} = \mathbf{x} + i \mathbf{y}$. If $x \in (0, \pi)$, then \mathbf{x} and \mathbf{y} are orthogonal: $\mathbf{x} \cdot \mathbf{y} = 0$. Furthermore,*

$$\mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y} = 1/2. \quad (13)$$

Proof. On the one hand, from Proposition 2 we have

$$\begin{aligned} 0 = \langle \mathbf{z}, \bar{\mathbf{z}} \rangle &= \langle \mathbf{x} + i \mathbf{y}, \mathbf{x} - i \mathbf{y} \rangle = (\mathbf{x} \cdot \mathbf{x} - \mathbf{y} \cdot \mathbf{y}) - 2i (\mathbf{x} \cdot \mathbf{y}), \\ &\Rightarrow \mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y}, \quad \text{and} \quad \mathbf{x} \cdot \mathbf{y} = 0. \end{aligned} \quad (14)$$

On the other hand, \mathbf{z} is of norm 1 so that

$$\langle \mathbf{z}, \mathbf{z} \rangle = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} = 1 \stackrel{(14)}{\Rightarrow} \mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y} = \frac{1}{2}. \quad \square$$

Now we are ready to proceed with the exercise. First we show that there exists a complex number $a \in \mathbb{C}$ such that $\mathbf{v} = a \mathbf{z} + \bar{a} \bar{\mathbf{z}}$. Indeed, since \mathbf{z} and $\bar{\mathbf{z}}$ span the space orthogonal to \mathbf{w} in \mathbb{C}^3 , there exist complex numbers $a, b \in \mathbb{C}$ such that $\mathbf{v} = a \mathbf{z} + b \bar{\mathbf{z}}$. But then with \mathbf{x} and \mathbf{y} defined as in Proposition 3, the fact that \mathbf{v} is a real vector implies

$$(\operatorname{Im}(a) + \operatorname{Im}(b)) \mathbf{x} + (\operatorname{Re}(a) - \operatorname{Re}(b)) \mathbf{y} = \mathbf{0} \Rightarrow b = \bar{a}, \quad (15)$$

²This comes from the fact that if $A \in \mathbb{R}^{n \times n}$ there exists $P \in SU(n)$ such that $P^{-1}AP$ is diagonal. Then $\operatorname{tr}(PAP^{-1})$ is the sum of the eigenvalues of A . But the trace is invariant under cyclic perturbations since $\operatorname{tr}(ABC) = \operatorname{tr}(A_{ij}B_{jk}C_{ki}) = \operatorname{tr}(B_{jk}C_{ki}A_{ij}) = \operatorname{tr}(BCA)$. So $\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(APP^{-1}) = \operatorname{tr} A$.

where the implication is due to Proposition 3: \mathbf{x} and \mathbf{y} are orthogonal and of non zero norm so that both brackets must vanish independently.

Then the fact that \mathbf{v} is a unit vector implies that $|a|^2 = 1/2$. Indeed, $1 = \langle \mathbf{v}, \mathbf{v} \rangle = \langle a\mathbf{z} + \bar{a}\bar{\mathbf{z}}, a\mathbf{z} + \bar{a}\bar{\mathbf{z}} \rangle = |a|^2 + |\bar{a}|^2 = 2|a|^2$, where the third equality is due to Proposition 2.

Finally, simply compute

$$\begin{aligned} \cos \theta &\stackrel{(10)}{=} Q \mathbf{v} \cdot \mathbf{v} = \langle Q(a\mathbf{z} + \bar{a}\bar{\mathbf{z}}), a\mathbf{z} + \bar{a}\bar{\mathbf{z}} \rangle = \langle a e^{ix} \mathbf{z} + \bar{a} e^{-ix} \bar{\mathbf{z}}, a\mathbf{z} + \bar{a}\bar{\mathbf{z}} \rangle, \\ &\stackrel{Prop.(3)}{=} |a|^2 (e^{-ix} + e^{ix}) = \frac{e^{-ix} + e^{ix}}{2} = \cos x. \end{aligned} \quad (16)$$

4. The idea of the proof is to decompose any vector $\mathbf{v} \in \mathbb{R}^3$, different from \mathbf{u} , in a component along \mathbf{u} and an other component orthogonal to \mathbf{u} . We then apply the rotation Q to the vector \mathbf{v} . We have that \mathbf{v} can be written in the following manner:

$$\mathbf{v} = (\mathbf{u} \cdot \mathbf{v})\mathbf{u} + \mathbf{u} \times \mathbf{v} \times \mathbf{u} \quad (17)$$

We then apply to (17) the rotation Q :

$$Q\mathbf{v} = (\mathbf{u} \cdot \mathbf{v})\mathbf{u} + \cos(\phi)\mathbf{u} \times \mathbf{v} \times \mathbf{u} + \sin(\phi)\mathbf{u} \times \mathbf{v} \quad (18)$$

Next, we recall two properties of a double cross product

$$\mathbf{u} \times \mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{u} \times \mathbf{v} \quad (19)$$

$$\mathbf{u} \times \mathbf{u} \times \mathbf{v} = (\mathbf{u} \cdot \mathbf{v})\mathbf{u} - \|\mathbf{u}\|^2 \mathbf{v} \quad (20)$$

Using (19) and (20) in (18) we obtain

$$Q\mathbf{v} = \|\mathbf{u}\|^2 \mathbf{v} + \sin(\phi)\mathbf{u} \times \mathbf{v} + (1 - \cos(\phi))\mathbf{u} \times \mathbf{u} \times \mathbf{v} \quad (21)$$

$$= \mathbf{v} + \sin(\phi)[\mathbf{u} \times] \mathbf{v} + (1 - \cos(\phi))[\mathbf{u} \times]^2 \mathbf{v} \quad (22)$$

As the latter expression is valid for any vector \mathbf{v} we obtain that the matrix Q can be expressed as

$$Q = I + \sin(\phi)[\mathbf{u} \times] + (1 - \cos(\phi))[\mathbf{u} \times]^2. \quad (23)$$

The second expression comes easily by using the result of the exercise 1.2 of this session in order to get

$$Q = \cos(\phi)I + \sin(\phi)[\mathbf{u} \times] + (1 - \cos(\phi))\mathbf{u} \otimes \mathbf{u} \quad (24)$$