1 Cayley transforms

1.1 A few general properties

1. We recall that $N \in \mathbb{R}^{n\times n}$ and that $|\mathrm{I} - N| \neq 0$.
By a direct computation we have:

$$\begin{aligned}(\mathrm{I}+N)(\mathrm{I}-N)^{-1} &= -(-2\mathrm{I}+(\mathrm{I}-N))(\mathrm{I}-N)^{-1} \quad (1)\\ &= -\mathrm{I}+2(\mathrm{I}-N)^{-1} \quad (2)\\ &= -(\mathrm{I}-N)^{-1}(-2\mathrm{I}+(\mathrm{I}-N)) \quad (3)\\ &= (\mathrm{I}-N)^{-1}(\mathrm{I}+N). \quad (4)\end{aligned}$$

2. *Ad absurdum*, if $(\mathrm{I}+M)$ is not invertible, then there exists a non-trivial (i.e. $\neq \mathbf{0}$) vector $\mathbf{v}$ such that $(\mathrm{I}+M)\,\mathbf{v} = \mathbf{0}$. This implies that

$$\begin{aligned}M\,\mathbf{v} = -\mathbf{v} \;&\Leftrightarrow\; (\mathrm{I}+N)(\mathrm{I}-N)^{-1}\,\mathbf{v} = -\mathbf{v}, \quad (5)\\ &\Leftrightarrow\; (\mathrm{I}-N)^{-1}(\mathrm{I}+N)\,\mathbf{v} = -\mathbf{v}, \quad (6)\\ &\Leftrightarrow\; (\mathrm{I}+N)\,\mathbf{v} = -(\mathrm{I}-N)\,\mathbf{v}, \quad (7)\\ &\Leftrightarrow\; \mathbf{v}+N\,\mathbf{v} = -\mathbf{v}+N\,\mathbf{v}, \quad \Leftrightarrow \quad \mathbf{v} = -\mathbf{v}, \quad (8)\end{aligned}$$

a contradiction.

3. We simply have

$$\begin{aligned}M = (\mathrm{I}+N)(\mathrm{I}-N)^{-1} \;&\Leftrightarrow\; M(\mathrm{I}-N) = (\mathrm{I}+N) \quad (9)\\ &\Leftrightarrow\; M-\mathrm{I} = (\mathrm{I}+M)N \overset{1.1.2}{\Leftrightarrow} N = (M+\mathrm{I})^{-1}(M-\mathrm{I}). \quad (10)\end{aligned}$$

4. By mimicking the above computation we have

$$\begin{aligned}M = (\mathrm{I}-P)(\mathrm{I}+P)^{-1} \;&\Leftrightarrow\; M(\mathrm{I}+P) = (\mathrm{I}-P) \quad (11)\\ &\Leftrightarrow\; \mathrm{I}-M = (M+\mathrm{I})P \overset{1.1.2}{\Leftrightarrow} P = (\mathrm{I}+M)^{-1}(\mathrm{I}-M). \quad (12)\end{aligned}$$

5. First note that if Q is a Cayley transform, then $(\mathrm{I}+Q)$ is invertible and so Q can not have -1 as an eigenvalue. This in turn implies $Q \notin O(n) \setminus SO(n)$ so that if Q is orthogonal, then it must be that $Q \in SO(n)$. Then we have

$$\begin{aligned}Q^TQ = \mathrm{I} \;&\Leftrightarrow\; (\mathrm{I}-S)^{-T}(\mathrm{I}+S)^T(\mathrm{I}+S)(\mathrm{I}-S)^{-1} = \mathrm{I}, \quad (13)\\ &\Leftrightarrow\; (\mathrm{I}+S)^T(\mathrm{I}+S) = (\mathrm{I}-S)^T(\mathrm{I}-S), \quad (14)\\ &\Leftrightarrow\; \mathrm{I}+S^T+S+S^TS = \mathrm{I}-S^T-S+S^TS, \quad (15)\\ &\Leftrightarrow\; S^T = -S. \quad (16)\end{aligned}$$

Finally, if $R \in SO(3)$ is a rotation matrix with rotation angle π there is not a skew symmetric matrix U such that R is the Cayley transform of U.

1.2 The case of $SO(3)$

1. We first need to compute $(\mathrm{I}-S)^{-1}$. For that we will first compute the characteristic polynomial of $S = [\mathbf{u}\times]$ by computing the following determinant

$$\begin{pmatrix} -\lambda & -u_3 & u_2 \\ u_3 & -\lambda & -u_1 \\ -u_2 & u_1 & -\lambda \end{pmatrix}. \tag{17}$$

We obtain that $-\lambda(\lambda^2 + ||\mathbf{u}||^2)$ is the characteristic polynomial of the skew symmetric matrix S. We first observe that the eigenvalues of a skew symmetric matrix $[\mathbf{u}\times]$ are always $0, \pm i||\mathbf{u}||$. For now on let define $u := ||\mathbf{u}||$.

Applying the Cayley-Hamilton theorem then, leads to

$$S^3 = -S\,u^2. \tag{18}$$

Next make the following *ansatz*: $(\mathrm{I} - S)^{-1} = \alpha \mathrm{I} + \beta S + \gamma S^2$.

Thus, we look for three numbers α, β and γ such that

$$(\mathrm{I} - S)\,(\alpha\,\mathrm{I} + \beta\,S + \gamma\,S^2) = \mathrm{I}, \tag{19}$$

$$\Leftrightarrow \quad (\alpha - 1)\,\mathrm{I} + (\beta - \alpha)\,S + (\gamma - \beta)\,S^2 - \gamma S^3 = 0, \tag{20}$$

$$\overset{(18)}{\Leftrightarrow} \quad (\alpha - 1)\,\mathrm{I} + (\beta - \alpha + \gamma\,u^2)\,S + (\gamma - \beta)\,S^2 = 0. \tag{21}$$

The condition (21) is satisfied by

$$\alpha = 1, \qquad \beta = \gamma = \frac{1}{1+u^2}. \tag{22}$$

Finally, compute

$$Q = (\mathrm{I} + S)\,(\mathrm{I} - S)^{-1}, \tag{23}$$

$$= (\mathrm{I} + S)\,\frac{1}{1+u^2}\Big((1+u^2)\,\mathrm{I} + S + S^2\Big), \tag{24}$$

$$= \frac{1}{1+u^2}\Big((1+u^2)\,\mathrm{I} + (2+u^2)\,S + 2\,S^2 + S^3\Big) \tag{25}$$

$$\overset{(18)}{=} \frac{1}{1+u^2}\Big((1+u^2)\,\mathrm{I} + 2\,S + 2\,S^2\Big), \tag{26}$$

$$\overset{(session\ 2, ex\ 1.3)}{=} \frac{1}{1+u^2}\Big((1-u^2)\,\mathrm{I} + 2\,[\mathbf{u}\times] + 2\,\mathbf{u}\otimes\mathbf{u}\Big). \tag{27}$$

As for the second part of the question, the Rodrigues formula directly gives the matrix of a right-handed rotation of angle ϕ and axis $\mathbf{n}$:

$$R(\phi, \mathbf{n}) = \cos\phi\,\mathrm{I} + (1 - \cos\phi)\,\mathbf{n}\otimes\mathbf{n} + \sin\phi\,[\mathbf{n}\times]. \tag{28}$$

We first substitute rewrite equation (27) as

$$Q = \frac{1-u^2}{1+u^2}\,\mathrm{I} + \frac{2u^2}{1+u^2}\,\frac{\mathbf{u}}{u}\otimes\frac{\mathbf{u}}{u} + \frac{2u}{1+u^2}[\frac{\mathbf{u}}{u}\times]. \tag{29}$$

In order to match the expression of R and Q we need now to express the angle ϕ in term of u. For that we recall the double angle formulas for cosinus and sinus:

$$\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}, \quad \text{and} \quad \sin\phi = \frac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}. \tag{30}$$

Next we make the following *ansatz*: $\phi = 2\,\mathrm{Arctan}(u)$.
To verify the latter *ansatz* we just need to observe that $u = \tan(\frac{\phi}{2})$ and that

$$\cos\phi = \frac{1-u^2}{1+u^2}, \quad (1-\cos\phi) = \frac{2u^2}{1+u^2}, \quad \text{and} \quad \sin\phi = \frac{2u}{1+u^2}. \tag{31}$$

Finally we obtain that

$$R(2\,\mathrm{Arctan}(u), \mathbf{u}/u) = \frac{1}{1+u^2}\Big[(1-u^2)\,\mathrm{I} + 2\,u^2\,\frac{\mathbf{u}}{u}\otimes\frac{\mathbf{u}}{u} + 2\,u\,[\frac{\mathbf{u}}{u}\times]\Big] = Q. \tag{32}$$

2. We will use the result of 1.1.3 to find S. To this intend, we first compute $(\mathrm{I}+Q)^{-1}$. Since $Q \in SO(3)$, the Cayley-Hamilton theorem gives

$$Q^3 - t\,Q^2 + t\,Q - \mathrm{I} = 0, \tag{33}$$

where $t := \mathrm{Trace}(Q)$. Multiplying (33) by Q^T yields

$$Q^2 = t\,Q - t\,\mathrm{I} + Q^T. \tag{34}$$

Next we try to find three numbers α, β and γ such that

$$(\mathrm{I}+Q)(\alpha\,Q^T + \beta\,\mathrm{I} + \gamma\,Q) = \mathrm{I}. \tag{35}$$

To this intend compute

$$\begin{aligned}
\text{Eq. (35)} \quad &\Leftrightarrow \quad \alpha\,Q^T + (\alpha+\beta-1)\,\mathrm{I} + (\gamma+\beta)\,Q + \gamma\,Q^2 = 0, \\
&\overset{(34)}{\Leftrightarrow} \quad (\alpha+\gamma)\,Q^T + (\alpha+\beta-1-t\,\gamma)\,\mathrm{I} + (\gamma\,(1+t)+\beta)\,Q = 0,
\end{aligned} \tag{36}$$

$$\Leftarrow \quad \begin{cases} \alpha+\gamma = 0, \\ \alpha+\beta-t\,\gamma = 1, \\ \beta+(1+t)\,\gamma = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \alpha = \dfrac{1}{2\,(1+t)}, \\ \beta = \dfrac{1+t}{2\,(1+t)}, \\ \gamma = \dfrac{-1}{2\,(1+t)}. \end{cases} \tag{37}$$

Accordingly, the final result in (37) contains the three numbers that we were looking for: we have shown that

$$(\mathrm{I}+Q)^{-1} = \frac{1}{2\,(1+t)}\Big(Q^T + (1+t)\,\mathrm{I} - Q\Big). \tag{38}$$

Now, making use of the result 1.1.3 of this sheet, we compute

$$\begin{aligned}
S \quad &= \quad (\mathrm{I}+Q)^{-1}\,(Q-\mathrm{I}) = \frac{1}{2\,(1+t)}\Big(Q^T + (1+t)\,\mathrm{I} - Q\Big)\,(Q-\mathrm{I}), && (39)\\
&= \quad \frac{1}{2\,(1+t)}\Big(-Q^T - t\,\mathrm{I} + (2+t)\,Q - Q^2\Big), && (40)\\
&\overset{(34)}{=} \quad \frac{Q-Q^T}{1+t} && (41)
\end{aligned}$$

1.3 The case of $SE(3)$

First, we prove that if $\mathcal{S}$ is of the form

$$\mathcal{S} = \begin{pmatrix} [\mathbf{u}\times] & \mathbf{v} \\ 0 & 0 \end{pmatrix}, \tag{42}$$

then its Cayley transform is in $SE(3)$. This is because

$$\begin{aligned} (\mathcal{I}+\mathcal{S})(\mathcal{I}-\mathcal{S})^{-1} &= \begin{pmatrix} \mathrm{I}+[\mathbf{u}\times] & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{I}-[\mathbf{u}\times] & -\mathbf{v} \\ 0 & 1 \end{pmatrix}^{-1}, & (43) \\ &= \begin{pmatrix} \mathrm{I}+[\mathbf{u}\times] & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\mathrm{I}-[\mathbf{u}\times])^{-1} & (\mathrm{I}-[\mathbf{u}\times])^{-1}\mathbf{v} \\ 0 & 1 \end{pmatrix} & (44) \\ &= \begin{pmatrix} Q & Q\mathbf{v}+\mathbf{v} \\ 0 & 1 \end{pmatrix}, & (45) \end{aligned}$$

where $Q := (\mathrm{I}+[\mathbf{u}\times])(\mathrm{I}-[\mathbf{u}\times])^{-1} \in SO(3)$ because it is the Cayley transform of a skew matrix. Accordingly, the RHS of (45) is in $SE(3)$.

Next, we show that if $\mathcal{Q}$ is the Cayley transform of some matrix $\mathcal{S} \in \mathbb{R}^{4\times 4}$ and $\mathcal{Q} \in SE(3)$, then $\mathcal{S}$ is of the form (42).

If $\mathcal{Q} \in SE(3)$ then there exist a matrix $Q \in SO(3)$ and a vector $\mathbf{q}$ such that

$$\mathcal{Q} = \begin{pmatrix} Q & \mathbf{q} \\ 0 & 1 \end{pmatrix}. \tag{46}$$

Furthermore, because $\mathcal{Q}$ is a Cayley transform, $\mathcal{Q}+\mathcal{I}$ must be invertible. This in turn implies that $Q+\mathrm{I}$ is invertible. Then after defining $S = (Q+\mathrm{I})^{-1}(Q-\mathrm{I})$ a similar argument to that developed in 1.1.2 (make sure you can do it) shows that $\mathrm{I}-S$ is invertible and a similar argument to 1.1.3 shows that Q is the Cayley transform of S. Finally, 1.1.5 ensures that S must be skew so there exists a vector $\mathbf{u}$ such that $S = [\mathbf{u}\times]$.

The result 1.1.3 implies

$$\begin{aligned} \mathcal{S} &= (\mathcal{Q}+\mathcal{I})^{-1}(\mathcal{Q}-\mathcal{I}) = \begin{pmatrix} Q+\mathrm{I} & \mathbf{q} \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} Q-\mathrm{I} & \mathbf{q} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (Q+\mathrm{I})^{-1} & -\frac{1}{2}(Q+\mathrm{I})^{-1}\mathbf{q} \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} Q-\mathrm{I} & \mathbf{q} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} [\mathbf{u}\times] & (Q+\mathrm{I})^{-1}\mathbf{q} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$