Mathematical Modelling of DNA
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Session 1: Exercises — A. Patelli

1 The length scales of DNA

Every diploid human cell[1], in the non-dividing state, contains twice the human genome ($\simeq 3 \cdot 10^9$ base-pairs of DNA) for a total of about $6 \cdot 10^9$ base-pairs of DNA. The diameter of a typical human nucleus cell is of the order of $10\mu\text{m} = 10^5\text{Å}$, where

$$\mu\text{m} = 10^{-6}\text{m}, \quad \text{nm} = 10^{-9}\text{m}, \quad \text{Å} = 10^{-10}\text{m}.$$

(a) By treating the DNA as a cylinder of length 3.4Å per base-pair, calculate the total length of DNA in any cell. Compare this to the diameter of the cell.

(b) By treating the DNA as a cylinder of diameter 20Å, calculate the total volume of DNA in any cell. Compare this to the total volume of the cell, if it is assumed to be spherical.

(This exercise was taken from the book 'Understanding DNA', C.R. Calladine, H.R. Drew)

2 Gaussian integrals

Let $\beta > 0$, $n \geq 0$ and a symmetric, positive - definite matrix $K \in I\!R^{n\times n}$ be given. Show that

(a) The one-dimensional Gaussian integral is

$$\int_{-\infty}^{\infty} e^{-\sigma^2} d\sigma = \sqrt{\pi}.$$

(b) The n-dimensional Gaussian integral is

$$Z^{(n)} := \int_{I\!R^n} e^{-\beta(x-\hat{x})\cdot K(x-\hat{x})} dx = \int_{I\!R^n} e^{-\beta x\cdot Kx} dx = \left(\frac{\pi}{\beta}\right)^{\frac{n}{2}} \sqrt{\det[K^{-1}]}.$$

(c)

$$\frac{1}{Z^{(n)}} \int_{I\!R^n} x_i e^{-\beta x\cdot Kx}\, dx = 0,$$

$$\frac{1}{Z^{(n)}} \int_{I\!R^n} x_i e^{-\beta(x-\hat{x})\cdot K(x-\hat{x})}\, dx = \hat{x}_i, \quad 1 \leq i \leq n.$$

(d)

$$\frac{1}{Z^{(n)}} \int_{I\!R^n} (x_i - \hat{x}_i)(x_j - \hat{x}_j)\, e^{-\beta(x-\hat{x})\cdot K(x-\hat{x})}\, dx$$

$$= \frac{1}{Z^{(n)}} \int_{I\!R^n} x_i x_j e^{-\beta x\cdot Kx}\, dx$$

$$= \frac{1}{2\beta} K^{-1}_{ij}, \quad (1 \leq i, j \leq n).$$

[Hint: For (a) calculate $\left(\int_{-\infty}^{\infty} e^{-\sigma^2} d\sigma\right)^2$. Then use (a) to solve (b)-(d).]

[1] A diploid cell contains two copies of each chromosome, plus a couple X X for women and a couple X Y for men.