Mathematical Modelling of DNA
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Session 2 A. Patelli

1 Properties of skew symmetric matrices

1. Given an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and a particular vector $\mathbf{u}$ in $\mathbb{R}^3$, write the skew symmetric matrix $[\mathbf{u}\times]$ that applied to any vector $\mathbf{v}$ gives $\mathbf{u} \times \mathbf{v}$.

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^{3\times 3}$$
$$\mathbf{u} \longmapsto [\mathbf{u}\times]$$

Is then a linear invertible mapping from $\mathbb{R}^3$ to 3×3 skew symmetric matrices such that $\mathbf{u} \times \mathbf{v} = [\mathbf{u}\times]\mathbf{v}$, $\forall \mathbf{v} \in \mathbb{R}^3$

2. Show that for any invertible matrix $M \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$[M\mathbf{u}\times] = |M|\, M^{-T}\, [\mathbf{u}\times]\, M^{-1}.$$

How does this formula simplify if $M \in \mathrm{SO}(3)$? The relation you will obtain is the change of basis formula.

3. Prove that, for any $\mathbf{u} \in \mathbb{R}^3$, we have

$$[\mathbf{u}\times]^2 = \mathbf{u} \otimes \mathbf{u} - |\mathbf{u}|^2 \mathrm{I}$$

[Hint: Distribute the triple vector product $\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$ for an arbitrary vector $\mathbf{v}$]

2 Rotations in three dimensions

Consider any matrix $Q \in SO(3)$.

1. Show that all the eigenvalues of Q are on the unit circle in the complex plane.

2. Show that Q always has an eigenvalue of unity, so that there is a unit vector $\mathbf{w}$ such that $Q\mathbf{w} = \mathbf{w}$. This vector defines the axis of rotation of Q and is parallel to the axial vector of the skew matrix $Q - Q^T$. Can a proper rotation have more than one axis?

3. Let $\mathbf{v}$ be any unit vector orthogonal to $\mathbf{w}$. Show that $Q\mathbf{v}$ is also a unit vector orthogonal to $\mathbf{w}$ and that the angle $0 \leq \theta \leq \pi$ between $\mathbf{v}$ and $Q\mathbf{v}$ satisfies the relation

$$1 + 2\cos\theta = \mathrm{tr}(Q). \tag{1}$$

[Hint: Express $\mathrm{tr}(Q)$ in terms of the eigenvalues.]

4. Let $Q \in SO(3)$ be a rotation matrix with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and denote by $\mathbf{u}$ its unit rotation axis and by ϕ its rotation angle.
Prove that Q can be written as

$$Q = \mathrm{I} + \sin\phi\, [\mathbf{u}\times] + (1 - \cos\phi)\, [\mathbf{u}\times]^2. \tag{2}$$

Moreover, show that (2) can also be expressed as

$$Q = \cos\phi\, \mathrm{I} + \sin\phi\, [\mathbf{u}\times] + (1 - \cos\phi)\, \mathbf{u} \otimes \mathbf{u}. \tag{3}$$

[Hint: For (2) $\Rightarrow$ (3) use exercise 1.3]

Relations (2) and (3) are matrix forms of the so-called Euler-Rodrigues formula, which give an expression for the rotation matrix Q about an axis $\mathbf{u}$ through an angle ϕ.