

1 Properties of skew symmetric matrices

1. Given an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and a particular vector \mathbf{u} in \mathbb{R}^3 , write the skew symmetric matrix $[\mathbf{u}\times]$ that applied to any vector \mathbf{v} gives $\mathbf{u} \wedge \mathbf{v}$.

$$\begin{aligned} \mathbb{R}^3 &\longrightarrow \mathbb{R}^{3\times 3} \\ \mathbf{u} &\longmapsto [\mathbf{u}\times] \end{aligned}$$

Is then a linear invertible mapping from \mathbb{R}^3 to 3×3 skew symmetric matrices such that $\mathbf{u} \wedge \mathbf{v} = [\mathbf{u}\times]\mathbf{v}$, $\forall \mathbf{v} \in \mathbb{R}^3$

2. Show that for any invertible matrix $M \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$[M\mathbf{u}\times] = |M| M^{-T} [\mathbf{u}\times] M^{-1}.$$

How does this formula simplify if $M \in SO(3)$? The relation you will obtain is the change of basis formula.

3. Prove that, for any $\mathbf{u} \in \mathbb{R}^3$, we have

$$[\mathbf{u}\times]^2 = \mathbf{u} \otimes \mathbf{u} - |\mathbf{u}|^2 \mathbf{I}$$

[Hint: Distribute the triple vector product $\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$ for an arbitrary vector \mathbf{v}]

2 Rotations in three dimensions

Consider any matrix $Q \in SO(3)$.

- Show that all the eigenvalues of Q are on the unit circle in the complex plane.
- Show that Q always has an eigenvalue of unity, so that there is a unit vector \mathbf{w} such that $Q\mathbf{w} = \mathbf{w}$. This vector defines the axis of rotation of Q and is parallel to the axial vector of the skew matrix $Q - Q^T$. Can a proper rotation have more than one axis?
- Let \mathbf{v} be any unit vector orthogonal to \mathbf{w} . Show that $Q\mathbf{v}$ is also a unit vector orthogonal to \mathbf{w} and that the angle $0 \leq \theta \leq \pi$ between \mathbf{v} and $Q\mathbf{v}$ satisfies the relation

$$1 + 2 \cos \theta = \text{tr}(Q). \tag{1}$$

[Hint: Express $\text{tr}(Q)$ in terms of the eigenvalues.]

- Let $Q \in SO(3)$ be a rotation matrix with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and denote by \mathbf{u} its unit rotation axis and by ϕ its rotation angle.

Prove that Q can be written as

$$Q = \mathbf{I} + \sin \phi [\mathbf{u}\times] + (1 - \cos \phi) [\mathbf{u}\times]^2. \tag{2}$$

Moreover, show that (2) can also be expressed as

$$Q = \cos \phi \mathbf{I} + (1 - \cos \phi) \mathbf{u} \otimes \mathbf{u} + \sin \phi [\mathbf{u}\times]. \tag{3}$$

[Hint: For (2) \Rightarrow (3) use exercise 1.2]

Relations (2) and (3) are matrix forms of the so-called Euler-Rodrigues formula, which give an expression for the rotation matrix Q about an axis \mathbf{u} through an angle ϕ .