Mathematical Modelling of DNA
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Session 3 A. Patelli

1 Cayley transforms

Let $N \in \mathbb{R}^{n\times n}$ such that $|I - N| \neq 0$. The Cayley transform of N is the matrix M defined by

$$M = (\mathrm{I} + N)(\mathrm{I} - N)^{-1}, \tag{1}$$

where I is the identity matrix in $\mathbb{R}^{n\times n}$.

1.1 A few general properties

1. Show that $(\mathrm{I} + N)(\mathrm{I} - N)^{-1} = (\mathrm{I} - N)^{-1}(\mathrm{I} + N)$.

2. Show that if M is the Cayley transform of some matrix N, then the matrix $\mathrm{I}+M$ is invertible. [Hint: Use 1.1.1 and use that $\mathrm{I} + M$ is not invertible iff $\exists \mathbf{v} \neq \mathbf{0} : (\mathrm{I} + M)\mathbf{v} = \mathbf{0}$.]

3. Show that if M is the Cayley transform of some matrix N, then the inverse Cayley transform of N is $N = (M + \mathrm{I})^{-1}(M - \mathrm{I})$.

4. Replace $N = -P$ in (1) and compute the inverse Cayley transform of P.

5. Assume that Q is the Cayley transform of S. Show that $Q \in SO(n)$ if and only if S is skew. Is it true that for a general matrix $R \in SO(n)$ there exists a skew matrix U such that R is the Cayley transform of U?

1.2 The case of $SO(3)$

Assume that Q is the Cayley transform of a skew-symmetric matrix $S \in \mathbb{R}^{3\times 3}$.

1. Show that if $S = [\mathbf{u}\times]$ for some vector $\mathbf{u}$, then

$$Q = \frac{1 - \|\mathbf{u}\|^2}{1 + \|\mathbf{u}\|^2}\,\mathrm{I} + \frac{2}{1 + \|\mathbf{u}\|^2}[\mathbf{u}\times] + \frac{2}{1 + \|\mathbf{u}\|^2}\mathbf{u} \otimes \mathbf{u}. \tag{2}$$

Equation (2) is another version of the Euler–Rodrigues formulas (2) and (3) seen in exercise 2, session 2.
Interpreting Q as a rotation matrix, show that it corresponds to a right-handed rotation by an angle $2\,\mathrm{Arctan}(\|\mathbf{u}\|)$ about the axis along $\mathbf{u}/\|u\|$.

2. Provided that $\mathrm{tr}\, Q \neq -1$ (i.e. Q is not a rotation through π). Show that

$$S = (Q - Q^T)/(1 + \mathrm{tr}\, Q).$$

1.3 The case of $SE(3)$

We recall that an element $\mathcal{R} \in SE(3)$ has the following matrix form

$$\mathcal{R} = \begin{bmatrix} R & r \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4},$$

where $R \in SO(3)$ and $r \in \mathbb{R}^3$.

Show that if $\mathcal{Q}$ is the Cayley transform of $\mathcal{S} \in \mathbb{R}^{4\times 4}$, then $\mathcal{Q} \in SE(3)$ if and only if there exists two vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$\mathcal{S} = \begin{pmatrix} [\mathbf{u}\times] & \mathbf{v} \\ 0 & 0 \end{pmatrix}.$$