Faculté des Sciences de Base
Master in mathematics
Master in computational science and engineering
Scientific computing applied to linear algebra
May 10, 2011

Solution of Serie 11

Exercise 1 *2. A solution is given in the following figure.*

B	R	B	R
R	B	R	B
B	R	B	R
R	B	R	B

3. *There is no dependency in each equation: red nodes are updated only with black nodes values and black nodes only with red nodes values.*

4. *The double loops for updating the node values and also the loops for computing the residual can be easily parallelized by the directive* `#pragma omp parallel for private (i,j)` .

5. *For small N (e.g. $N = 25$), the sequential version is always better; the speedup of the parallel version is quite good for a number of processors ≤ 4 . For large N, the parallel version is better for a number of processors ≥ 2; the speedup is quite good for a processor number $\leq 8 - 12$. This behaviour is due to the few number of operations in the* `j` *loops. The increase of the number of iterations with N comes from the increase of the condition number of the matrix.*

Exercise 2 *1. The error $E(\mathbf{x})$ is defined by*

$$E(\mathbf{x}) = \frac{1}{2}\left(A(\hat{\mathbf{x}} - \mathbf{x}), (\hat{\mathbf{x}} - \mathbf{x})\right) = \frac{1}{2}\sum_{l,m}(\hat{x}_l - x_l)a_{lm}(\hat{x}_m - x_m).$$

where $A\hat{\mathbf{x}} = \mathbf{b}$. The i-th component of the gradient g_i of $E(\mathbf{x})$ is given by

$$(\nabla E(\mathbf{x}))_i = \frac{1}{2}\frac{\partial}{\partial x_i}\left(\sum_{l,m}(\hat{x}_l - x_l)A_{lm}(\hat{x}_m - x_m)\right) = \sum_m A_{im}(x_m - \hat{x}_m) = (A\mathbf{x} - \mathbf{b})_i = -r_i.$$

One rewrites the first version of the algorithm using $\mathbf{r}^k = -\mathbf{g}^k$ and with changing the signs of α_k and β_{k+1}:

$\alpha_k = \left(\mathbf{g}^k, \mathbf{p}^k\right)/(A\mathbf{p}^k, \mathbf{p}^k)$
$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \mathbf{p}^k$
$\mathbf{g}^{k+1} = A\mathbf{x}^{k+1} - \mathbf{b}$
$\beta_k = -(A\mathbf{p}^k, \mathbf{g}^{k+1})/(A\mathbf{p}^k, \mathbf{p}^k)$
$\mathbf{p}^{k+1} = \mathbf{g}^{k+1} + \beta_k \mathbf{p}^k$

2. *From the second and third equations, one finds that:*

$$\mathbf{g}^{k+1} = A\mathbf{x}^{k+1} - \mathbf{b} = A(\mathbf{x}^k - \alpha_k \mathbf{p}^k) - \mathbf{b} = \mathbf{g}^k - \alpha_k A\mathbf{p}^k, \tag{1}$$

that is the relationship of point 3.

3. *From the first and last equations, one obtains:*

$$\alpha_k = \frac{(\mathbf{g}^k, \mathbf{p}^k)}{(A\mathbf{p}^k, \mathbf{p}^k)} = \frac{\left\|\mathbf{g}^k\right\|^2}{(A\mathbf{p}^k, \mathbf{p}^k)} + \beta_{k+1}\frac{(\mathbf{g}^k, \mathbf{p}^{k-1})}{(A\mathbf{p}^k, \mathbf{p}^k)}.$$

But

$$\left(\mathbf{g}^k, \mathbf{p}^{k-1}\right) \stackrel{(1)}{=} \left(\mathbf{g}^{k-1}, \mathbf{p}^{k-1}\right) - \alpha_{k-1}\left(A\mathbf{p}^{k-1}, \mathbf{p}^{k-1}\right) = 0, \tag{2}$$

and then

$$\alpha_k = \frac{\left\|\mathbf{g}^k\right\|^2}{(A\mathbf{p}^k, \mathbf{p}^k)}. \tag{3}$$

4. *The fourth equation*

$$\beta_k = -\frac{(A\mathbf{p}^k, \mathbf{g}^{k+1})}{(A\mathbf{p}^k, \mathbf{p}^k)} \overset{(1)}{=} \frac{1}{\alpha_k} \frac{(\mathbf{g}^{k+1} - \mathbf{g}^k, \mathbf{g}^{k+1})}{(A\mathbf{p}^k, \mathbf{p}^k)} \overset{(3)}{=} \frac{\left\|\mathbf{g}^{k+1}\right\|^2}{\left\|\mathbf{g}^k\right\|^2} - \frac{(\mathbf{g}^k, \mathbf{g}^{k+1})}{\left\|\mathbf{g}^k\right\|^2}, \tag{4}$$

implies, using the fifth one:

$$(\mathbf{g}^k, \mathbf{g}^{k+1}) \overset{(1)}{=} \left\|\mathbf{g}^k\right\|^2 - \alpha_k(\mathbf{g}^k, A\mathbf{p}^k) = \left\|\mathbf{g}^k\right\|^2 - \alpha_k(\mathbf{p}^k, A\mathbf{p}^k) + \alpha_k\beta_k(\mathbf{p}^{k-1}, A\mathbf{p}^k) \overset{(3)}{=} \alpha_k\beta_k(\mathbf{p}^{k-1}, A\mathbf{p}^k)$$

and, again with the help of the fith equation:

$$(\mathbf{p}^{k-1}, A\mathbf{p}^k) = (A\mathbf{p}^{k-1}, \mathbf{p}^k) = \beta_k(A\mathbf{p}^{k-1}, \mathbf{p}^{k-1}) + (A\mathbf{p}^{k-1}, \mathbf{g}^k) = 0.$$

Hence

$$\beta_k = \frac{\left\|\mathbf{g}^{k+1}\right\|^2}{\left\|\mathbf{g}^k\right\|^2}.$$

5. *The second version requires only the operation* $(A\mathbf{p}^k, \mathbf{p}^k)$ *while in the first version one needs to compute* $(A\mathbf{p}^k, \mathbf{p}^k)$ *and* $(A\mathbf{p}^k, \mathbf{r}^{k+1})$*, that is one more matrix-vector multiplication.*