Faculté des Sciences de Base
Master in mathematics
Master in computational science and engineering
Scientific computing applied to linear algebra
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Serie 2

Exercise 1 *One defines the matrix*

$$A = \begin{pmatrix} \epsilon & -1 \\ 1 & 1 \end{pmatrix}.$$

1. *Using Matlab and for $\epsilon = 10^{-16}$, compute the condition number of A with the maximum norm, its LU decomposition without pivoting and the residuum $LU - A$. Then, compute the LU decomposition with pivoting and the residuum. Finally, use the latter decomposition for solving the linear system $Ax = b = A\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by doing the different steps of the forward-backward elimination.*

2. *Explain the above behavior by computing by hand the condition number and the LU decomposition without pivoting.*

Exercise 2 *It is sometimes useful to equilibrate a matrix before solving the corresponding linear system, like in the following example:*

$$A = \begin{pmatrix} 2 \cdot 10^9 & 10^9 \\ 10^{-9} & 2 \cdot 10^{-9} \end{pmatrix}.$$

With Matlab, compute the condition number of A using the maximum norm and the one of

$$B = \begin{pmatrix} 10^{-9} & 0 \\ 0 & 10^9 \end{pmatrix} A;$$

then solve the linear system $Ax = y = A\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ using B, and compute the residuum and the error.

Exercise 3 *Given $T > 0$, $y_0 \in \mathbb{R}$ and a continuous function $(t, y) \in [0, T] \times \mathbb{R} \mapsto f(t, y) \in \mathbb{R}$, the Cauchy problem consists in finding a function with continuous derivative $y : [0, T] \to \mathbb{R}$ such that*

$$y'(t) = f(t, y(t)), t \in (0, T],\ y(0) = y_0.$$

Assume that the derivative f_y of f with respect to y is continuous; then the problem has a unique solution.
Given a regular partition $t_i = ih, i = 0, 1, 2, ..., N$, $h = T/N$, of $[0, T]$, the forward Euler method yields an approximation $\widehat{y}_i$ of $y(t_i)$ defined by:

$$\widehat{y}_0 = y_0,\ \widehat{y}_i = \widehat{y}_{i-1} + hf(t_{i-1}, \widehat{y}_{i-1}), i = 1, 2, ..., N.$$

1. *One computes an approximate solution $\widehat{y}$ of the Cauchy problem on the computer; in this computation, the evaluation of the function f is exact, but the initial condition is represented by a perturbation $y_0 + \delta y_0$; set up, to first order, an expression for the condition number $\kappa(t)$ for the absolute error:*
$$|\widehat{y}(t) - y(t)| \leq \varkappa(t)\, |\delta y_0|\,.$$

2. *Use this expression for evaluating the condition number of the two problems defined by $f(t, y) = \alpha y + q(t)$ and $f(t, y) = -\alpha y + q(t), \alpha > 0$, with q a continuous function.*

3. *Program (for instance in Matlab) the forward Euler method for solving the problem defined by $f(t, y) = -y + t$, $y_0 = 5, T = 20$ with $N = 10$ steps; plot the exact and approximate solutions.*