Faculté des Sciences de Base
Master in mathematics
Master in computational science and engineering
Scientific computing applied to linear algebra
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Solution of serie 2

The scripts are on the moodle page of the week

Exercise 1 1. *The matrix is well conditioned, but the algorithm of decomposition without pivoting is unstable, because ϵ becomes a pivot. By exchanging the two rows, the pivots are of order 1.*

2. *One computes:*

$$A = \begin{pmatrix} \epsilon & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow \|A\|_\infty = \max_{1\le i\le 2} \sum_{j=1}^{2} |a_{ij}| = 2$$

$$A^{-1} = \frac{1}{1+\epsilon} \begin{pmatrix} 1 & 1 \\ -1 & \epsilon \end{pmatrix} \Rightarrow \|A^{-1}\|_\infty = \frac{2}{1+\epsilon}.$$

Hence, the condition number is given by

$$\varkappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = \frac{4}{1+\epsilon}.$$

One has:

$$A = LU = \begin{pmatrix} 1 & 0 \\ 1/\epsilon & 1 \end{pmatrix} \begin{pmatrix} \epsilon & -1 \\ 0 & 1+1/\epsilon \end{pmatrix};$$

because of rounding errors u_{22} is represented by $1/\epsilon$ on the computer; hence

$$A - \widehat{L}\widehat{U} = \begin{pmatrix} \epsilon & -1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \epsilon & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise 2 *The equilibrated problem is well conditioned and it can be solved with precision.*

Exercise 3 1. *The computed solution satisfies:*

$$\widehat{y}' = f(t, \widehat{y}),\ \widehat{y}(0) = y_0 + \delta y_0,$$

and hence:

$$\left(\widehat{y} - y\right)'(t) = \left(\widehat{y} - y\right)(t) f_y(t, y(t)) + O((\widehat{y} - y)^2),\ \ \left(\widehat{y} - y\right)(0) = \delta y_0.$$

Consequently, to 1st order, settingt $p(t) = f_y(t, y(t))$, one finds:

$$\left(\widehat{y} - y\right)(t) = \delta y_0 e^{P(t)},\ P(t) = \int_0^t p(x)dx,$$

and the condition number we are looking for is:

$$\varkappa(t) = e^{P(t)}.$$

2. *For the first problem, one has $\varkappa(t) = e^{\alpha t}$ and then, for t not close to zero, the problem is ill conditioned. On the contrary, for the second problem, the condition number $\varkappa(t) = e^{-\alpha t}$ decreases when t increases and one has $\varkappa(t) < 1$ pour $t > 0$.*

3. *The forward Euler method is conditionally stable. In fact, the stability condition in this case is given by h <2 and we notice that for $h = 2$ it produces oscillations. Hence, we try to solve a well conditioned problem with an unstable algorithm!*