Faculté des Sciences de Base
Master in mathematics
Master in computational science and engineering
Scientific computing applied to linear algebra
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Serie 3

Exercise 1 *The forward and backward errors can be quite different. Consider the linear system $Ax = b$, with:*

$$A = \begin{pmatrix} 99 & 98 \\ 100 & 99 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

the exact solution of which is

$$x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Let $\widehat{x}_1$ be the solution computed with Matlab and assume that two different numerical methods yield the following results:

$$\widehat{x}_2 = \begin{pmatrix} 2.97 \\ -2.99 \end{pmatrix}, \ \widehat{x}_3 = \begin{pmatrix} 1.01 \\ -0.99 \end{pmatrix}.$$

1. *In Matlab, compute the matrix condition number and for each approximation the error and the residuum.*
2. *For a given approximation $\widehat{x}$, let be a matrix E such that*

$$E\widehat{x} = r = b - A\widehat{x}. \tag{1}$$

Then, the approximation is obtained as the exact solution of the problem

$$(A + E)\widehat{x} = b. \tag{2}$$

A solution of (1)is given by

$$\overline{E} = \frac{r\widehat{x}^t}{\widehat{x}^t\widehat{x}}. \tag{3}$$

Use (1) and (3) for proving that, in the euclidean norm, one has:

$$\left\|\overline{E}\right\|_2 = \frac{\|r\|_2}{\|\widehat{x}\|_2} = \min_{E\widehat{x}=r} \|E\|_2 \,;$$

hence the norm of $\overline{E}$ yield an a posteriori measure of the backward error. Compute these matrices and their norm in Matlab for the three approximations.

3. *With the help of (1) and (3), show that the relative forward error is bounded like:*

$$\frac{\|x - \widehat{x}\|_2}{\|\widehat{x}\|_2} \leq \kappa(A)\frac{\left\|\overline{E}\right\|_2}{\|A\|_2}, \ \kappa(A) = \|A\|_2 \left\|A^{-1}\right\|_2 ;$$

in Matlab, compute this bound for the three approximations and compare with the computed values of the error.

Exercise 2 *Let be a vector $x \in \mathbb{R}^n$ and a matrix $A \in \mathcal{L}(\mathbb{R}^n)$. The maximum (or l_∞) and l_1 norms in $\mathbb{R}^n$ are defined by*

$$\|x\|_\infty = \max_{1\leq i\leq n} |x_i| \,, \ \|x\|_1 = \sum_{i=1}^{n} |x_i| \,.$$

Using the definition of the norm of $\mathcal{L}(\mathbb{R}^n)$ induced by the norm of $\mathbb{R}^n$:

$$\|A\| = \max_{\|x\|=1} \|Ax\| \,,$$

prove that:

1.

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|\,,$$

2.

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|\,,$$

3.

$$\left\|A^t\right\|_\infty = \|A\|_1\,.$$

Exercise 3 *Estimate in Matlab the normwise backward and forward errors for both linear systems: 1) with a random matrix of order $n = 100$ and 2) with the ill-conditioned matrix of order $n = 20$ in the file `A.mat`. In both cases, the exact solution shall be given by $x(i) = \sqrt{i}$ and the function `rcond` shall be used for estimating the inverse of the condition number. Compare the estimates to the "exact" backward and forward errors.*

Exercise 4 *Given $f \in C[0,1]$, one consider the problem of finding a function $u \in C^2[0,1]$ such that:*

$$\begin{aligned} -u''(t) &= f(t),\ 0 < t < 1,\\ u(0) &= u(1) = 0. \end{aligned}$$

With the help of the regular partition

$$t_i = ih,\ i=0,1,2,\ldots,n+1,\ h = 1/(n+1),$$

of $[0,1]$, one looks for a finite differences approximation of u, $u_i \simeq u(x_i)$, defined by the scheme

$$u''(t_i) \simeq -\frac{u_{i+1} + u_{i-1} - 2u_i}{h^2} = f(t_i),\ i=1,2,\ldots,n. \tag{4}$$

1. *Write the linear system (4) in matrix form $Ax = b$.*
2. *With Matlab, for $f(x) = \sin(\pi x)$, $n = 24$, compute the componentwise backward error*

$$\omega_\infty = \max_i \frac{|r_i|}{(|A|\,|\widehat{x}| + |b|)_i},$$

and estimate the forward error with the bound

$$\frac{\|\widehat{x} - x\|_\infty}{\|x\|_\infty} \le 2\varkappa_\infty(A)\omega_\infty.$$

Compare to the "exact solution" and discuss the results.