Faculté des Sciences de Base
Master in mathematics
Master in computational science and engineering
Scientific computing applied to linear algebra
March 15, 2011

Serie 4

Exercise 1 *Modify the Matlab scripts of PRECISE contained in the packed file* `precise.tbz2`*, for doing the perturbation analysis of the following linear systems* $Ax = b$*. Consult the document "Precise User's Guide" on the moodle site for more informations.*

1. *Dindong matrix of order* $n = 10$*:*

$$a_{ij} = \frac{1}{2(n-i-j)+3}, \qquad \mathbf{x}_i = \sqrt{i}, \qquad 1 \le i, j \le n, \qquad \mathbf{b} = A\mathbf{x}.$$

2. *IPJ! matrix of order* $n = 10$*:*

$$a_{ij} = (i+j)!, \qquad \mathbf{x}_i = \sqrt{i}, \qquad 1 \le i, j \le n, \qquad \mathbf{b} = A\mathbf{x}.$$

In particular, compute the condition number. Notice the different behaviors of the indicators $\mathcal{I}$ *et* $\mathcal{K}$ *according to a perturbation of the right-hand-side, the matrix or both. Look also to the difference of the normwise and componentwise perturbations. Observe the interval* $t \in [s, r]$ *on which* $\mathcal{I}$ *is constant. Deduce from that an estimate of the backward error and compare to the exact value. Check the quality of the estimate of the forward error* (E_r) *compared to the exact value* (E_x)*. Remember that the forward error is given by* $\mathcal{K}*$*(backward error), the backward error being estimated by* η_∞ *or* ω_∞*.*

Exercise 2 *Let* A *be a symmetric positive definite matrix, tridiagonal by blocks; all the blocks have the size* $m \times m$ *and there are* n *blocks in the diagonal. This matrix has the form:*

$$A = \begin{pmatrix} S_1 & T_2 & 0 & \cdots & 0 \\ T_2^t & S_2 & T_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & S_{n-1} & T_n \\ 0 & \cdots & 0 & T_n^t & S_n \end{pmatrix}.$$

The Crout decomposition writes $A = U^t D U$*, with:*

$$D = \begin{pmatrix} D_1 & 0 & 0 & \cdots & 0 \\ 0 & D_2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & D_{n-1} & 0 \\ 0 & \cdots & 0 & 0 & D_n \end{pmatrix}, \quad U = \begin{pmatrix} I & U_2 & 0 & \cdots & 0 \\ 0 & I & U_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & I & U_n \\ 0 & \cdots & 0 & 0 & I \end{pmatrix}.$$

1. *Write the algorithm using this decomposition to solve a linear system* $A\mathbf{x} = \mathbf{b}$*,*

$$\mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{pmatrix},$$

that is:

Block Thomas method:

- *compute* U_k *from* D_{k-1} *and* T_k *by writing* U_k *in place of* T_k,
- *compute* D_k *from* U_k, D_{k-1} *and* S_k *by writing* D_k *in place of* S_k,
- *compute the vectors* $\mathbf{y}_k$, $\mathbf{z}_k$ *such that* $U_k^t\mathbf{z}_k = \mathbf{b}_k$, $D\mathbf{y}_k = \mathbf{z}_k$ *(forwad elimination)*,
- *compute the solution* $\mathbf{x}$ *of the system by solving (backward elimination)* $U\mathbf{x} = \mathbf{y}$.

2. *Given the unit square* $\Omega = (0,1) \times (0,1)$,*one discretizes the problem which, given* $f \in C^0(\overline{\Omega})$, *consists in finding* $u \in C^2(\overline{\Omega})$ *such that*

$$\begin{aligned} -\triangle u(\mathbf{x}) &= f(\mathbf{x}), \ \mathbf{x} \in \Omega, \\ u(\mathbf{x}) &= 0, \ \mathbf{x} \in \partial\Omega, \end{aligned}$$

by the following difference scheme. With the help of the partition

$$x_i = ih, \ y_i = ih, \ i = 0, 1, ..., n+1, \ h = \frac{1}{n+1},$$

the approximation $u_{ij} \simeq u(x_i, y_j)$ *is given by*

$$-\frac{u_{i-1,j} + u_{i+1,j} + u_{i.j-1} + u_{i,j+1} - 4u_{ij}}{h^2} = f(x_i, y_j), \ 1 \leq i, j \leq n.$$

Show that the matrix of this linear system is symmetric positive definite and block tridiagonal.