Faculté des Sciences de Base
Master in mathematics
Master in computational science and engineering
Scientific computing applied to linear algebra
March 15, 2011

Solution of serie 4

Exercise 1

1. *The dingdong matrix is well conditioned (cond=2.42). The behavior under perturbations of the matrix or of the matrix and the right-hand-side together, are identical, either normwise or componentwise. For a perturbation of the right-hand-side only, the behavior changes: the value of $\mathcal{K}$ (conditioning) is larger and closer to the true value. The reliability interval $[s, r]$ is equal to the whole investigated domain, except in the case of componentwise perturbation of the right-hand-side, where s is slightly larger. The value of s close to 10^{-16} corresponds to the exact backward error ($\sim 2\cdot 10^{-16}$). The estimated forward error is very close to the exact value, except again for the componentwise perturbation of the right-hand-side. In conclusion, this problem is well conditioned and the solution method is reliable.*

2. *The IPJ! matrix is very ill conditioned (cond=2.2$\cdot 10^{20}$). Normwise, for perturbations of A or of A and b, there is no reliability interval. For a perturbation of b only, the value of K is a good estimate of the condition number and the forward error estimate is much larger than its exact value. For componentwise perturbations of A or of A and b together, the reliability interval is between 10^{-12} and 10^{-8}, which shows that the alorithm is not reliable for this problem . In this interval, the estimate of the forward error is good and the estimate of the condition number given by K is too low. Furthermore, the value 0.01 for the backward error is not correctly estimated by s. In conclusion, the problem is ill conditioned and the algorithm is not reliable in this case. Notice that the componentwise perturbations are better suited to this matrix, which has entries of order 1 to 10^{18}.*

Exercise 2

1. *By identifying U^tDU with A, one obtains:*

$$\begin{array}{rcl} D_1 &=& S_1 \\ D_iU_{i+1} &=& T_{i+1} \\ D_{i+1} + U_{i+1}^t D_i U_{i+1} &=& S_{i+1} \end{array} \Rightarrow \begin{array}{rcl} D_1 &=& S_1 \\ U_{i+1} &=& D_i^{-1}T_{i+1} \\ D_{i+1} &=& S_{i+1} - U_{i+1}^t T_{i+1} \end{array}$$

Hence, we can write the algorithm, assuming that $\mathbf{x} = \mathbf{b}$ and that S_i are stored in D_i:

- *Initialization:*
$$\mathbf{x}_1 := D_1^{-1}\mathbf{x}_1;$$
- *Decomposition, forward elimination (values of U_i are stored in T_i and original T_i is in V):*
Loop i=2 until n
$$\begin{aligned} &V := T_i,\ T_i := D_{i-1}^{-1}T_i, \\ &D_i := D_i - T_i^tV, \\ &\mathbf{x}_i := \mathbf{x}_i - V^t\mathbf{x}_{i-1},\ \mathbf{x}_i := D_i^{-1}\mathbf{x}_i; \end{aligned}$$
End loop
- *Backward elimination (values of U_i computed in previous step are stored in T_i):*
Loop i=n-1 until 1
$$\mathbf{x}_i := \mathbf{x}_i - T_i\mathbf{x}_{i+1}.$$
End loop

2. *Let us define a vector of dimension n^2: $u_{i+(j-1)n} = u_{ij}$, and another one $f_{i+(j-1)n} = f(x_i, y_j)$. Then, we can write the problem as*

$$4u_{i+(j-1)n} - u_{i-1+(j-1)n} - u_{i+1+(j-1)n} - u_{i+(j-2)n} - u_{i+jn} = h^2 f_{i+(j-1)n},$$

that is a linear system $A\mathbf{u} = h^2\mathbf{f}$.

The matrix A has the structure of the preceeding point with:

$$S_i = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 4 & -1 \\ 0 & \cdots & 0 & -1 & 4 \end{pmatrix}, \ 1 \le i \le n$$

and

$$T_{i+1} = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}, \ 1 \le i \le n-1.$$

For the nonzero vector

$$\mathbf{w} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n^2} \end{pmatrix}$$

we compute $\mathbf{w} \cdot A\mathbf{w}$. One has:

$$\begin{aligned}
\mathbf{w} \cdot A\mathbf{w} &= 4\sum_{i=1}^{n^2} \alpha_i^2 - \sum_{i=2}^{n^2} \alpha_i\alpha_{i-1} - \sum_{i=1}^{n^2-1} \alpha_i\alpha_{i+1} - \sum_{i=n+1}^{n^2} \alpha_i\alpha_{i-n} - \sum_{i=1}^{n^2-n} \alpha_i\alpha_{i+n} \\
&= 2\alpha_1^2 + 2\alpha_{n^2}^2 + 2\sum_{i=1}^{n^2-1} \alpha_i^2 + 2\sum_{i=1}^{n^2-1} \alpha_{i+1}^2 - 2\sum_{i=1}^{n^2-1} \alpha_i\alpha_{i+1} - 2\sum_{i=1}^{n^2-n} \alpha_i\alpha_{i+n} \\
&= 2\alpha_1^2 + 2\alpha_{n^2}^2 + \sum_{i=1}^{n^2-1} (\alpha_i - \alpha_{i+1})^2 + \sum_{i=1}^{n^2-1} \alpha_i^2 + \sum_{i=1}^{n^2-1} \alpha_{i+1}^2 - 2\sum_{i=1}^{n^2-n} \alpha_i\alpha_{i+n}.
\end{aligned}$$

The three last terms of the above equation become:

$$\begin{aligned}
&\sum_{i=1}^{n^2-1} \alpha_i^2 + \sum_{i=1}^{n^2-1} \alpha_{i+1}^2 - 2\sum_{i=1}^{n^2-n} \alpha_i\alpha_{i+n} \\
&= \sum_{i=n^2-n+1}^{n^2-1} \alpha_i^2 + \sum_{i=1}^{n^2-n} \alpha_i^2 + \sum_{i=1}^{n-1} \alpha_{i+1}^2 + \sum_{i=1}^{n^2-n} \alpha_{i+n}^2 - 2\sum_{i=1}^{n^2-n} \alpha_i\alpha_{i+n} \\
&= \sum_{i=n^2-n+1}^{n^2-1} \alpha_i^2 + \sum_{i=1}^{n-1} \alpha_{i+1}^2 + \sum_{i=1}^{n^2-n} (\alpha_i - \alpha_{i+n})^2 > 0.
\end{aligned}$$

Then $\mathbf{w} \cdot A\mathbf{w} > 0$ and A is (symmetric) positive definite.