

Energy transfers in shell models for magnetohydrodynamics turbulence

Thomas Lessinnes,¹ Daniele Carati,¹ and Mahendra K. Verma²

¹*Physique Statistique et Plasmas, CP231, Faculté des Sciences, Université Libre de Bruxelles, B-1050 Bruxelles, Belgium*

²*Department of Physics, Indian Institute of Technology, Kanpur 208016, India*

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A systematic procedure to derive shell models for magnetohydrodynamic turbulence is proposed. It takes into account the conservation of ideal quadratic invariants such as the total energy, the cross helicity, and the magnetic helicity, as well as the conservation of the magnetic energy by the advection term in the induction equation. This approach also leads to simple expressions for the energy exchanges as well as to unambiguous definitions for the energy fluxes. When applied to the existing shell models with nonlinear interactions limited to the nearest-neighbor shells, this procedure reproduces well-known models but suggests a reinterpretation of the energy fluxes.

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I. INTRODUCTION

Understanding the existence and the dynamics of the magnetic field of the Earth, of the Sun, and, in general, of other celestial bodies remains one of the most challenging problems of classical physics. Astronomical and geophysical observations have provided many insights into these phenomena [1–3]. Laboratory experiments [4,5] have confirmed that generation of magnetic field (dynamo) can take place under various circumstances and lead to a variety of complex behaviors. However, analytical approaches of this problem are extremely complicated while numerical efforts are limited to a range of parameter space that is often quite distant from the realistic systems. For instance, in certain astrophysical bodies as well as in laboratory experiments, the kinematic viscosity ν of the fluid is 6 orders of magnitude smaller than its resistivity η . The two dissipation processes therefore take place at very different time scales. This property makes direct numerical simulation of dynamo intractable. Due to this reason we resort to simplified models.

Shell models specifically belong to this class of simplified approaches [6]. They have been constructed to describe interactions among various scales without any reference to the geometric structure of the problem. They were first introduced for fluid turbulence with the quite successful GOY (Gledzer, Ohkitani and Yamada) shell model [7,8] and have been extended to magnetohydrodynamic (MHD) turbulence [9–12]. In shell models, drastically reduced degrees of freedom (usually only one complex number) are used to describe the entire information provided by a shell of Fourier modes in wave-number space. This approach reduces the description of turbulence from a partial differential equation to a reduced set of ordinary differential equations and provides a simplified tool for studying the energy and helicity exchanges between different scales at a significantly reduced numerical cost.

The present work aims at deriving the expressions for the energy fluxes and the energy exchanges for MHD turbulence in a systematic and consistent manner. Then we apply this scheme to study energy transfers in a shell model of MHD. This approach follows quite closely the previous efforts in which fluxes and energy exchanges have been identified for

the complete MHD equation. However, in these works [13,14], energy exchanges between two degrees of freedom have been determined from the triadic interactions up to an indeterminate circulating energy transfer. The strategy adopted in the present paper is somewhat different. Here, we derive the energy-transfer formulas from the energy equations by identifying the terms that participate in these transfers. This process also involves various symmetries and conservation laws of the ideal (dissipationless) equations. In particular, the energy transfer from the magnetic field in a shell to the magnetic field in another shell is driven by the convective term of the induction equation that conserves the total magnetic energy. The identification of this convective term in the shell model is one of the main improvements of the present approach. It is actually required to define physically meaningful shell to shell energy transfers. One of the main advantages of the present formalism is that we need not worry about the indeterminate circulating transfer appearing in the related past work by Verma and co-workers [13,14].

The dynamo process involves growth of magnetic energy that is supplied from the kinetic energy by the nonlinear interactions. As we will show in the paper, a clear and unambiguous identification of the various energy fluxes and energy exchanges between the velocity and the magnetic fields is very important in the study of dynamo effects. This is one of the main motivations for the development of the present approach. The approach is also explicitly applied in Sec. IV to the derivation of the GOY shell model to MHD [12].

The outline of the paper is as follows. A general formalism for expressing the various constraints satisfied by the nonlinearities in the shell models is discussed in Sec. II. It is shown in Sec. III that this formalism can be adapted nicely to the derivation of explicit expressions for the energy fluxes as well as for the shell-to-shell energy exchanges in shell model. In Sec. IV, we apply the formalism to the GOY shell model for MHD turbulence [12] and study the energy fluxes for MHD turbulence. In Sec. V, we present our conclusions.

II. SHELL MODELS OF MHD TURBULENCE

Shell models were first introduced for fluid turbulence (see, for example, [7,8,15]). They can be seen as a drastic

simplification of the Navier-Stokes or the MHD equations which, assuming periodic boundary conditions, are expressed in Fourier space as follows:

$$\frac{d\mathbf{u}_{\mathbf{k}}}{dt} = \mathbf{n}_{\mathbf{k}}(\mathbf{u}, \mathbf{u}) - \mathbf{n}_{\mathbf{k}}(\mathbf{b}, \mathbf{b}) - \nu k^2 \mathbf{u}_{\mathbf{k}} + \mathbf{f}_{\mathbf{k}}, \quad (1)$$

$$\frac{d\mathbf{b}_{\mathbf{k}}}{dt} = \mathbf{n}_{\mathbf{k}}(\mathbf{u}, \mathbf{b}) - \mathbf{n}_{\mathbf{k}}(\mathbf{b}, \mathbf{u}) - \eta k^2 \mathbf{b}_{\mathbf{k}}, \quad (2)$$

where $\mathbf{u}_{\mathbf{k}}$ and $\mathbf{b}_{\mathbf{k}}$ are the velocity and magnetic field Fourier modes, respectively, with wave vector \mathbf{k} . The norm of this wave vector is $k=|\mathbf{k}|$. The viscosity ν and the magnetic diffusivity η are responsible for the dissipative effects in these equations while energy is injected through the forcing term $\mathbf{f}_{\mathbf{k}}$. The nonlinear term is defined by

$$\mathbf{n}_{\mathbf{k}}(\mathbf{x}, \mathbf{y}) = i\mathbf{P}(\mathbf{k}) \cdot \sum_{\mathbf{p}+\mathbf{q}=-\mathbf{k}} (\mathbf{k} \cdot \mathbf{x}_{\mathbf{q}}^*) \mathbf{y}_{\mathbf{p}}^*, \quad (3)$$

where \mathbf{x} and \mathbf{y} can be either the velocity or the magnetic field. The tensor \mathbf{P} is defined as

$$P_{ij}(\mathbf{k}) = \frac{k^2 \delta_{ij} - k_i k_j}{k^2}. \quad (4)$$

It projects any field to its divergence-free part and it is used since only incompressible flows are considered in this study ($\nabla \cdot \mathbf{u}=0$). In the velocity equation, the projection of the nonlinear terms using the tensor (4) replaces the introduction of the pressure term. In the magnetic field equation, the nonlinear terms are usually not projected to their divergence-free parts. Indeed, the nondivergence-free parts of the two nonlinear terms cancel each other and the constraint $\nabla \cdot \mathbf{b}=0$ is automatically satisfied. The writing of the nonlinear term in the magnetic field equation using the form (3) has been used to stress and explore the inner symmetries in the MHD equations.

The incompressible MHD equations are known to conserve the total energy, the cross helicity, and the magnetic helicity. The conservation of these quantities plays a central role for the derivation of shell models. Similarly, the conservations of both the kinetic helicity and the kinetic energy in absence of magnetic field are used to simplify further the shell model for MHD. There is however another property that has not been exploited so far: the conservation of magnetic energy by the first nonlinear term in the magnetic field equation. Indeed, assuming periodic boundary conditions, it is easy to prove that

$$\sum_{\mathbf{k}} \mathbf{n}_{\mathbf{k}}(\mathbf{u}, \mathbf{b}) \cdot \mathbf{b}_{\mathbf{k}}^* = 0. \quad (5)$$

The identification of a similar term in shell models for MHD will prove to be very useful in determining the energy exchanges and the energy fluxes in the shell model.

The equations of the evolution of the variables in a shell model are designed to mimic as much as possible the MHD equations (1) and (2). In order to build the shell model using a systematic procedure, we first introduce the partition of the Fourier space into shells s_i defined as the regions $|\mathbf{k}| \in [k_{i-1}, k_i]$, where $k_i = k_0 \lambda^i$. In this definition, k_0 corre-

sponds to the smallest wave vector. The number of shells is denoted by N , so that the wave vectors larger than $k_0 \lambda^{N-1}$ are not included in the model. Any observable that would be represented in the original MHD equation by its Fourier modes $\mathbf{x}_{\mathbf{k}}$ is described in the framework of the shell model by a vector of complex numbers noted \mathbf{X} . Each component x_i of this vector summarizes the information from all the modes $\mathbf{x}_{\mathbf{k}}$ corresponding to the shell s_i . It is also very useful to introduce the vector \mathbf{X}_i for which all components but the i th are zero

$$\mathbf{X} = (x_1, x_2, \dots, x_N) \in \mathbb{C}^N, \quad (6)$$

$$\mathbf{X}_i = (0, 0, \dots, 0, x_i, 0, \dots, 0) \in \mathbb{C}^N, \quad (7)$$

$$\mathbf{X} = \sum_{i=1}^N \mathbf{X}_i, \quad (8)$$

where the expansion (8) is a direct consequence of the definition of \mathbf{X}_i .

In the following, the scalar product of two real fields will be needed for defining various quantities such as kinetic and magnetic energies, cross helicity, and kinetic and magnetic helicities. Using the Parseval's identity, the shell model version of this physical space scalar product is expressed as follows:

$$\langle \mathbf{X} | \mathbf{Y} \rangle \equiv \sum_{i=1}^N \frac{1}{2} (x_i y_i^* + y_i x_i^*). \quad (9)$$

Due to the nonlinear evolution of the velocity and the magnetic field in the MHD equations, any attempt to design a mathematical procedure that would reduce the description of these fields to two vectors of complex numbers \mathbf{U} and \mathbf{B} must lead to closure issues. In the derivation of a shell model, the shell variables are usually not seen as projected versions of the original MHD variables and their evolution is not derived directly from the MHD equations (1) and (2). The evolution equations for \mathbf{U} and \mathbf{B} are rather postulated *a priori*, but a number of constraints are imposed on the shell model. In this section, the models are built by imposing on the evolution equations for these vectors as many constraints as possible derived from conservation properties of each of the terms appearing in the original MHD equations.

Property 1. The nonlinear term in the evolution equation for \mathbf{U} is a sum of two quadratic terms: The first one depends on \mathbf{U} only and conserves the kinetic energy \mathcal{E}^U and the kinetic helicity \mathcal{H}^k independently of the value of the field \mathbf{B} ; the second term depends on \mathbf{B} only.

Property 2. The nonlinear term in the evolution equation for \mathbf{B} is a sum of two bilinear terms. The first one must conserve the magnetic energy \mathcal{E}^B independently of the value of the field \mathbf{U} .

Property 3. The full nonlinear expression in both the equations for \mathbf{U} and \mathbf{B} changes sign under the exchange $\mathbf{U} \leftrightarrow \mathbf{B}$.

The dynamical system for the shell vectors can therefore be written as

$$d_i \mathbf{U} = \mathbf{Q}(\mathbf{U}, \mathbf{U}) - \mathbf{Q}(\mathbf{B}, \mathbf{B}) - \nu \mathbf{D}(\mathbf{U}) + \mathbf{F}, \quad (10)$$

$$d_t \mathbf{B} = \mathbf{W}(\mathbf{U}, \mathbf{B}) - \mathbf{W}(\mathbf{B}, \mathbf{U}) - \eta \mathbf{D}(\mathbf{B}), \quad (11)$$

where the term proportional to ν models the viscous effect, the term proportional to η models the Joule effect, and \mathbf{F} stands for the forcing. The linear operator \mathbf{D} is defined as follows:

$$\mathbf{D}(\mathbf{X}) = (k_1^2 x_1, k_2^2 x_2, \dots, k_N^2 x_N) \in \mathbb{C}^N. \quad (12)$$

Now, the conservation laws must be enforced. Assuming incompressibility, in the ideal limit and in absence of forcing ($\mathbf{F}, \nu, \eta \rightarrow 0$), the model is expected to conserve the total energy $\mathcal{E}^{tot} = \mathcal{E}^U + \mathcal{E}^B$, the cross helicity \mathcal{H}^c , and the magnetic helicity \mathcal{H}^m . In terms of shell variables of the model, the energies and the cross helicity are defined for the original MHD equation as

$$\mathcal{E}^U = \frac{1}{2} \langle \mathbf{U} | \mathbf{U} \rangle, \quad (13)$$

$$\mathcal{E}^B = \frac{1}{2} \langle \mathbf{B} | \mathbf{B} \rangle, \quad (14)$$

$$\mathcal{H}^c = \langle \mathbf{U} | \mathbf{B} \rangle. \quad (15)$$

The definitions of the kinetic helicity and the magnetic helicity require the expressions for the vorticity $\mathbf{O} = (o_1, \dots, o_N)$ and the magnetic potential vector $\mathbf{A} = (a_1, \dots, a_N)$. These quantities are not trivially defined in shell models since they require the use of the curl operator. Nevertheless, they should be linear function of the velocity and magnetic field, respectively. The kinetic helicity and the magnetic helicity are then defined as follows:

$$\mathcal{H}^k = \langle \mathbf{U} | \mathbf{O} \rangle, \quad (16)$$

$$\mathcal{H}^m = \langle \mathbf{A} | \mathbf{B} \rangle. \quad (17)$$

In terms of conservation laws, property 1 imposes the following constraints that correspond to the conservation of the kinetic energy and the kinetic helicity, respectively, by the first quadratic term in the U equation:

$$\langle \mathbf{Q}(\mathbf{U}, \mathbf{U}) | \mathbf{U} \rangle = 0 \quad \forall \mathbf{U}, \quad (18)$$

$$\langle \mathbf{Q}(\mathbf{U}, \mathbf{U}) | \mathbf{O} \rangle = 0 \quad \forall \mathbf{U}. \quad (19)$$

Here, the notation “ $\forall \mathbf{U}$ ” must be understood as “for all possible values of the shell variables \mathbf{U} as well as \mathbf{O} that is defined by \mathbf{U} .” The conservation of the magnetic energy by the first quadratic term in the \mathbf{B} equation (property 2) imposes

$$\langle \mathbf{W}(\mathbf{U}, \mathbf{B}) | \mathbf{B} \rangle = 0 \quad \forall \mathbf{U}, \mathbf{B}. \quad (20)$$

The conservations of the total energy and of the cross helicity, respectively, correspond to

$$\begin{aligned} & \langle \mathbf{Q}(\mathbf{U}, \mathbf{U}) - \mathbf{Q}(\mathbf{B}, \mathbf{B}) | \mathbf{U} \rangle + \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) - \mathbf{W}(\mathbf{B}, \mathbf{U}) | \mathbf{B} \rangle \\ & = 0 \quad \forall \mathbf{U}, \mathbf{B}, \end{aligned} \quad (21)$$

$$\begin{aligned} & \langle \mathbf{Q}(\mathbf{U}, \mathbf{U}) - \mathbf{Q}(\mathbf{B}, \mathbf{B}) | \mathbf{B} \rangle + \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) - \mathbf{W}(\mathbf{B}, \mathbf{U}) | \mathbf{U} \rangle \\ & = 0 \quad \forall \mathbf{U}, \mathbf{B}. \end{aligned} \quad (22)$$

These two constraints are equivalent since the second is obtained simply from the first under the exchange $(\mathbf{U}, \mathbf{B}) \rightarrow (\mathbf{B}, \mathbf{U})$. Hence, the general procedure adopted here shows that in the ideal limit, for a shell model with the structures (10) and (11), the conservation of the total energy \mathcal{E}^{tot} implies the conservation of the cross helicity \mathcal{H}^c and vice versa. Moreover, taking into account the constraints (18) and (20), the conservation of the total energy and cross helicity reduces to

$$\langle \mathbf{Q}(\mathbf{B}, \mathbf{B}) | \mathbf{U} \rangle + \langle \mathbf{W}(\mathbf{B}, \mathbf{U}) | \mathbf{B} \rangle = 0 \quad \forall \mathbf{U}, \mathbf{B}. \quad (23)$$

Finally, the conservation of the magnetic helicity imposes the condition

$$\langle \mathbf{W}(\mathbf{U}, \mathbf{B}) | \mathbf{A} \rangle + \langle \mathbf{W}(\mathbf{B}, \mathbf{U}) | \mathbf{A} \rangle = 0 \quad \forall \mathbf{U}, \mathbf{B}. \quad (24)$$

Again, the notation “ $\forall \mathbf{U}, \mathbf{B}$ ” must be understood as “for all possible values of the shell variables \mathbf{U} and \mathbf{B} as well as \mathbf{O} and \mathbf{A} that are defined by \mathbf{U} and \mathbf{B} , respectively.” The specific form of the nonlinear terms in the general shell models (10) and (11) cannot be defined further without giving explicit definitions for \mathbf{O} and \mathbf{A} . The choice of the interactions retained in the nonlinear terms (for example, first neighboring shell or distant shell interactions [16,17]) must also be made explicit in order to reach the final form of the shell model. An example will be treated in Sec. IV.

If the shell model has to reproduce all the symmetries of the original MHD equation, the following equality could also be imposed:

$$\mathbf{W}(\mathbf{X}, \mathbf{X}) = \mathbf{Q}(\mathbf{X}, \mathbf{X}). \quad (25)$$

It is a consequence of the particular way of writing the MHD equations in which all nonlinear terms, including those appearing in the magnetic field equation, are made explicitly divergence free through the application of the projection operation (4). In the example treated in Sec. IV, this equality appears as a direct consequence of the other constraints imposed on the structure of the shell model. Nevertheless, if the present approach is applied to more complex shell models for MHD, it might be interesting to keep the equality (25) in mind in order to simplify the nonlinearities as much as possible.

III. ENERGY FLUXES AND ENERGY EXCHANGES

A. Evolution equations for the shell energies

The kinetic and magnetic energies associated with the shell s_n are defined as $e_n^u = \langle \mathbf{U}_n | \mathbf{U}_n \rangle / 2$ and $e_n^b = \langle \mathbf{B}_n | \mathbf{B}_n \rangle / 2$. The evolution equations for these quantities are easily obtained in the inviscid and unforced limit

$$d_t e_n^u = T_n^u = \langle \mathbf{Q}(\mathbf{U}, \mathbf{U}) - \mathbf{Q}(\mathbf{B}, \mathbf{B}) | \mathbf{U}_n \rangle, \quad (26)$$

$$d_t e_n^b = T_n^b = \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) - \mathbf{W}(\mathbf{B}, \mathbf{U}) | \mathbf{B}_n \rangle. \quad (27)$$

The quantity T_n^u corresponds to the energy transferred into the velocity field in shell s_n and coming from either the ve-

locity or the magnetic fields. Since the first term of Eq. (26) conserves the total kinetic energy [cf. Eq. (18)], it is identified as the rate of energy T_n^{uu} flowing from the complete velocity field into the velocity field in the n th shell. The second term of Eq. (26) must then account for the energy coming from the magnetic field (T_n^{ub}), i.e.,

$$T_n^{uu} = \langle \mathbf{Q}(\mathbf{U}, \mathbf{U}) | \mathbf{U}_n \rangle, \quad (28)$$

$$T_n^{ub} = - \langle \mathbf{Q}(\mathbf{B}, \mathbf{B}) | \mathbf{U}_n \rangle. \quad (29)$$

Similarly, T_n^b corresponds to the energy transferred into the magnetic field in shell s_n and coming from either the velocity field or the magnetic field. The first term of Eq. (27) conserves the total magnetic energy [cf. Eq. (20)] and is identified with the rate of energy flowing from the complete magnetic field to the magnetic field of the n th shell. The second term of Eq. (27) corresponds to the energy flowing to the B_n shell from the complete velocity field, i.e.,

$$T_n^{bb} = \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) | \mathbf{B}_n \rangle, \quad (30)$$

$$T_n^{bu} = - \langle \mathbf{W}(\mathbf{B}, \mathbf{U}) | \mathbf{B}_n \rangle. \quad (31)$$

With this notation, the evolution equations for e_n^u and e_n^b become (with dissipative and forcing terms)

$$d_t e_n^u = T_n^{uu} + T_n^{ub} - 2\nu k_n^2 e_n^u + P_n^f, \quad (32)$$

$$d_t e_n^b = T_n^{bb} + T_n^{bu} - 2\eta k_n^2 e_n^b, \quad (33)$$

where $P_n^f = \langle \mathbf{F} | \mathbf{U}_n \rangle$ is the kinetic-energy injection rate into the shell s_n due to the external forcing.

It is also convenient to introduce the following decomposition of the vectors of shell variables:

$$\mathbf{X}_i^< = (x_1, x_2, \dots, x_{i-1}, x_i, 0, \dots, 0) \in \mathbb{C}^N, \quad (34)$$

$$\mathbf{X}_i^> = (0, 0, \dots, 0, x_{i+1}, x_{i+2}, \dots, x_N) \in \mathbb{C}^N, \quad (35)$$

$$\mathbf{X} = \mathbf{X}_i^< + \mathbf{X}_i^>. \quad (36)$$

where i can take any value between 1 and N . The kinetic energy contained in the vector $\mathbf{U}_n^<$ is simply given by $E_n^{U^<} = \langle \mathbf{U}_n^< | \mathbf{U}_n^< \rangle / 2 = \sum_{j=1}^n e_j^u$. The magnetic energy contained in the vector $\mathbf{B}_n^<$ is defined similarly. The evolutions of these quantities are easily derived from the relations (32) and (33),

$$d_t E_n^{U^<} = \sum_{j=1}^n T_j^{uu} + \sum_{j=1}^n T_j^{ub} - D_{vm}^< + P_n^{f^<}, \quad (37)$$

$$d_t E_n^{B^<} = \sum_{j=1}^n T_j^{bb} + \sum_{j=1}^n T_j^{bu} - D_{\eta m}^<, \quad (38)$$

where $P_n^{f^<} = \langle \mathbf{F} | \mathbf{U}_n^< \rangle$ is the injection rate of energy in $\mathbf{U}_n^<$ due to the forcing and $D_{vm}^< = \nu \langle \mathbf{D}(\mathbf{U}) | \mathbf{U}_n^< \rangle$ and $D_{\eta m}^< = \eta \langle \mathbf{D}(\mathbf{B}) | \mathbf{B}_n^< \rangle$ are the dissipative terms for $U_n^<$ and $B_n^<$, respectively.

B. Energy fluxes

The nonlinear terms in the Eqs. (37) and (38) correspond to the nonlinear energy fluxes that enter or leave the sphere of radius $k_0 \lambda^n$. These fluxes can be further specified. Indeed, the first sum in the right-hand side of the Eq. (37) comes from the quadratic $\mathbf{Q}(\mathbf{U}, \mathbf{U})$ term which conserves the total kinetic energy. Hence, this first sum must correspond to the kinetic-energy flux $\Pi_{U^<}^{U^>}(n)$ from $\mathbf{U}_n^>$ to $\mathbf{U}_n^<$,

$$\Pi_{U^<}^{U^>}(n) = \sum_{j=1}^n T_j^{uu} = \langle \mathbf{Q}(\mathbf{U}, \mathbf{U}) | \mathbf{U}_n^< \rangle. \quad (39)$$

The antisymmetry property for the fluxes can be used to define the opposite transfer: $\Pi_{U^>}^{U^<}(n) = -\Pi_{U^<}^{U^>}(n)$. It simply expresses that the energy gained by $\mathbf{U}_i^<$ due to the nonlinear interaction is equal and opposite to the energy lost by $\mathbf{U}_i^>$. The magnetic energy fluxes can be similarly defined as

$$\Pi_{B^<}^{B^>}(n) = \sum_{j=1}^n T_j^{bb} = \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) | \mathbf{B}_n^< \rangle. \quad (40)$$

It should be noted that the definition of this flux and the property 2 of Sec. II are intimately linked. Indeed, in order to define a flux between $\mathbf{B}^<$ and $\mathbf{B}^>$, a channel of interactions that conserve the magnetic energy must be identified. For instance, the flux (40) is defined as the sum of the increases of magnetic energy in shells corresponding to wave numbers smaller than k_n due to this specific channel, $\mathbf{W}(\mathbf{U}, \mathbf{B})$. Physically, it should be equivalently defined as the energy leaving the magnetic field with large wave numbers because of this same channel

$$\Pi_{B^<}^{B^>}(n) = - \sum_{j=n+1}^N T_j^{bb} = - \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) | \mathbf{B}_n^> \rangle. \quad (41)$$

Clearly,

$$\Pi_{B^>}^{B^<}(n) = \sum_{j=n+1}^N T_j^{bb} = \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) | \mathbf{B}_n^> \rangle. \quad (42)$$

These two definitions must be equivalent. Therefore, if this flux is physically well defined, the following property comes from Eqs. (9), (36), and (41):

$$\begin{aligned} \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) | \mathbf{B}_n^< \rangle &= - \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) | \mathbf{B}_n^> \rangle \Rightarrow \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) | \mathbf{B}_n^< + \mathbf{B}_n^> \rangle = 0 \\ &\Rightarrow \langle \mathbf{W}(\mathbf{U}, \mathbf{B}) | \mathbf{B} \rangle = 0. \end{aligned} \quad (43)$$

In other words, defining an energy flux between $\mathbf{B}^<$ and $\mathbf{B}^>$ due to a specific nonlinear term is only possible if this nonlinear term conserves the magnetic energy. Slightly anticipating on Sec. IV, it is because the fluxes defined in [12] do not meet this property that our approach leads to a different definition of the various fluxes.

The ‘‘cross’’ fluxes between the velocity and the magnetic field can also be defined systematically. The second sum in the right-hand side of Eq. (38) corresponds to the flux of energy from $\mathbf{B}_n^<$ to \mathbf{U} and readily leads to the following definitions:

$$\Pi_{B<}^U(n) = \sum_{j=1}^n T_j^{bu} = -\langle \mathbf{W}(\mathbf{B}, \mathbf{U}) | \mathbf{B}_n^{\leq} \rangle, \quad (44)$$

$$\Pi_{B>}^U(n) = \sum_{j=n+1}^N T_j^{bu} = \langle \mathbf{W}(\mathbf{B}, \mathbf{U}) | \mathbf{B}_n^{\geq} \rangle. \quad (45)$$

Since these terms are linear in \mathbf{U} , each of them can easily be split into two contributions related to \mathbf{U}_n^{\leq} and \mathbf{U}_n^{\geq} , respectively,

$$\Pi_{B<}^{U<}(n) = -\Pi_{U<}^{B<}(n) = -\langle \mathbf{W}(\mathbf{B}, \mathbf{U}_n^{\leq}) | \mathbf{B}_n^{\leq} \rangle, \quad (46)$$

$$\Pi_{B<}^{U>}(n) = -\Pi_{U>}^{B<}(n) = -\langle \mathbf{W}(\mathbf{B}, \mathbf{U}_n^{\geq}) | \mathbf{B}_n^{\leq} \rangle, \quad (47)$$

$$\Pi_{B>}^{U<}(n) = -\Pi_{U<}^{B>}(n) = -\langle \mathbf{W}(\mathbf{B}, \mathbf{U}_n^{\leq}) | \mathbf{B}_n^{\geq} \rangle, \quad (48)$$

$$\Pi_{B>}^{U>}(n) = -\Pi_{U>}^{B>}(n) = -\langle \mathbf{W}(\mathbf{B}, \mathbf{U}_n^{\geq}) | \mathbf{B}_n^{\geq} \rangle. \quad (49)$$

The formulas (39)–(49) show that the various fluxes can be defined univocally, almost independently of the structure of the shell model as long as the terms conserving kinetic and magnetic energies have been identified. It must be stressed that, at this stage, the exact expressions for the nonlinear terms \mathbf{Q} and \mathbf{W} are not needed.

C. Shell-to-shell energy exchanges

The expression for some of the energy exchanges between two shells may be derived from the above analysis. For instance, the quantity T_n^{bu} has been identified as the energy flux from the entire velocity field to the magnetic field associated to the shell s_n . The expansion (8) for \mathbf{U} can be inserted into the term T_n^{bu} and leads to

$$T_n^{bu} = \sum_{m=1}^N -\langle \mathbf{W}(\mathbf{B}, \mathbf{U}_m) | \mathbf{B}_n \rangle = \sum_{m=1}^N T_{nm}^{bu}, \quad (50)$$

where each term in this sum can now be identified as the shell-to-shell energy exchange rate from the velocity field in the shell s_m to the magnetic field in the shell s_n ,

$$T_{nm}^{bu} = -\langle \mathbf{W}(\mathbf{B}, \mathbf{U}_m) | \mathbf{B}_n \rangle. \quad (51)$$

Similarly, by inserting the expansion (8) for \mathbf{B} into the term T_n^{bb} , it is possible to identify the shell-to-shell energy exchange rate from the magnetic field in the shell s_m to the magnetic field in the shell s_n as follows:

$$T_{mn}^{bb} = \langle \mathbf{W}(\mathbf{U}, \mathbf{B}_m) | \mathbf{B}_n \rangle. \quad (52)$$

Since the quantities T_{nm}^{xy} are a shell-to-shell energy exchange rate (the notation xy is referred to as general exchange and it can take values uu , ub , bu , or bb), the following antisymmetry property is to be satisfied:

$$T_{mn}^{xy} = -T_{nm}^{yx}. \quad (53)$$

It is worth mentioning that the present analysis does not lead to a simple definition of the shell-to-shell kinetic-energy exchanges T_{nm}^{uu} . This is due to the presence of three velocity

variables in the expression for the \mathbf{U} -to- \mathbf{U} transfers that prevents a simple identification of the origin of the kinetic-energy flux. Nevertheless, considering the relation (25), the quantity T_n^{uu} (28) can be rewritten as follows:

$$T_n^{uu} = \langle \mathbf{W}(\mathbf{U}, \mathbf{U}) | \mathbf{U}_n \rangle, \quad (54)$$

and, by analogy with the expression (52), it is reasonable to adopt the following definition:

$$T_{nm}^{uu} = \langle \mathbf{W}(\mathbf{U}, \mathbf{U}_m) | \mathbf{U}_n \rangle. \quad (55)$$

The shell-to-shell energy exchanges give a more refined picture of the dynamics in the shell model than the fluxes. It is thus expected that these fluxes can be reconstructed from all the T_{nm}^{xy} . The general formulas are given by

$$\Pi_{X>}^{Y<}(n) = \sum_{i=n+1}^N \sum_{j=1}^n T_{ij}^{xy}, \quad (56)$$

$$\Pi_{X<}^{Y<}(n) = \sum_{i=1}^n \sum_{j=1}^n T_{ij}^{xy}, \quad (57)$$

$$\Pi_{X>}^{Y>}(n) = \sum_{i=n+1}^N \sum_{j=n+1}^N T_{ij}^{xy}. \quad (58)$$

As a direct consequence of the property (53), the same antisymmetry property holds for the energy fluxes. In the next section we will focus on a specific model adopted in [12]. We will derive the formulas for the energy fluxes and compute them numerically.

IV. GOY SHELL MODEL FOR MHD TURBULENCE

The results derived in Secs. II and III are valid for any shell model for MHD that use only one complex number per shell for each field (velocity and magnetic) and for which the properties (1–3) are satisfied. As long as the vorticity and the magnetic potential vector have not been defined explicitly, it is not possible to specify further the exact structure of the shell model, i.e., the structure of the nonlinear terms \mathbf{Q} and \mathbf{W} . In this section, we revisit the GOY-like shell model for MHD turbulence studied in [12] and apply the formalism discussed in Secs. II and III to this model. The choice to apply the formalism derived in the previous sections to the GOY model is motivated by the explicit computation of energy fluxes presented in [12] which allows a direct comparison. It does not mean that GOY models have to be considered as superior to other shell models in representing the phenomenology of MHD turbulence. The shell model is defined by the following expressions for the nonlinear \mathbf{Q} and \mathbf{W} terms:

$$q_n(X, X) = ik_n(\alpha_1 x_{n+1}^* x_{n+2}^* + \alpha_2 x_{n-1}^* x_{n+1}^* + \alpha_3 x_{n-2}^* x_{n-1}^*), \quad (59)$$

$$\begin{aligned}
 w_n(X, Y) = ik_n(\beta_1 x_{n+1}^* y_{n+2}^* + \beta_2 x_{n-1}^* y_{n+1}^* + \beta_3 x_{n-2}^* y_{n-1}^* \\
 + \beta_4 y_{n+1}^* x_{n+2}^* + \beta_5 y_{n-1}^* x_{n+1}^* + \beta_6 y_{n-2}^* x_{n-1}^*).
 \end{aligned}
 \quad (60)$$

This shell model is fully determined if the following definitions for the vorticity and the magnetic potential vector are also adopted:

$$o_i = (-1)^i u_i k_i, \quad (61)$$

$$a_i = (-1)^i b_i / k_i. \quad (62)$$

Imposing the conditions derived in the previous section from the various conservation laws [Eqs. (18)–(20), (23), and (24)] lead to the following values of the parameters α_i and β_i :

$$\alpha_2 = -\alpha_1 \frac{\lambda - 1}{\lambda^2} \quad \alpha_3 = -\alpha_1 \frac{1}{\lambda^3},$$

$$\beta_1 = \alpha_1 \frac{\lambda^2 + \lambda + 1}{2\lambda(\lambda + 1)} \quad \beta_2 = -\alpha_1 \frac{\lambda^2 - \lambda - 1}{2\lambda^2(\lambda + 1)},$$

$$\beta_3 = \alpha_1 \frac{\lambda^2 - \lambda - 1}{2\lambda^3(\lambda + 1)} \quad \beta_4 = \alpha_1 \frac{\lambda^2 + \lambda - 1}{2\lambda(\lambda + 1)},$$

$$\beta_5 = -\alpha_1 \frac{\lambda^2 + \lambda - 1}{2\lambda^2(\lambda + 1)} \quad \beta_6 = -\alpha_1 \frac{\lambda^2 + \lambda + 1}{2\lambda^3(\lambda + 1)}.$$

As discussed at the end of Sec. IV, this shell model also satisfies the constraint (25). It is indeed easy to verify that these parameters satisfy the following equalities: $\beta_1 + \beta_4 = \alpha_1$, $\beta_2 + \beta_5 = \alpha_2$, and $\beta_3 + \beta_6 = \alpha_3$.

In order to verify that the model derived here is exactly the same as the model discussed in [12], the dynamical system [Eqs. (10) and (11)] can then be rewritten after a few algebraic manipulations as

$$d_t \mu_n = ik_n [p_n(\mathbf{U}, \mathbf{U}) - p_n(\mathbf{B}, \mathbf{B})] - \nu k_n^2 \mu_n + f_n, \quad (63)$$

$$d_t b_n = ik_n [v_n(\mathbf{U}, \mathbf{B}) - v_n(\mathbf{B}, \mathbf{U})] - \eta k_n^2 b_n, \quad (64)$$

where

$$\begin{aligned}
 p_n(\mathbf{X}, \mathbf{X}) = \alpha_1 \left(x_{n+1}^* x_{n+2}^* - \frac{\lambda - 1}{\lambda^2} x_{n-1}^* x_{n+1}^* - \frac{1}{\lambda^3} x_{n-2}^* x_{n-1}^* \right),
 \end{aligned}
 \quad (65)$$

$$\begin{aligned}
 v_n(\mathbf{X}, \mathbf{Y}) = \frac{\alpha_1}{\lambda(\lambda + 1)} (x_{n+1}^* y_{n+2}^* + x_{n-1}^* y_{n+1}^* + x_{n-2}^* y_{n-1}^*).
 \end{aligned}
 \quad (66)$$

With these coefficients, the model [Eqs. (63) and (64)] is clearly the same as the one derived in [12]. Our interpretation of some of the shell-to-shell energy exchanges and the energy fluxes derived in the Sec. IV and computed in the next section differs from those of [12]. When we compare the two approaches carefully, we find that the velocity to velocity energy flux $\Pi_{U>}^{U<}$ and the total fluxes are the same

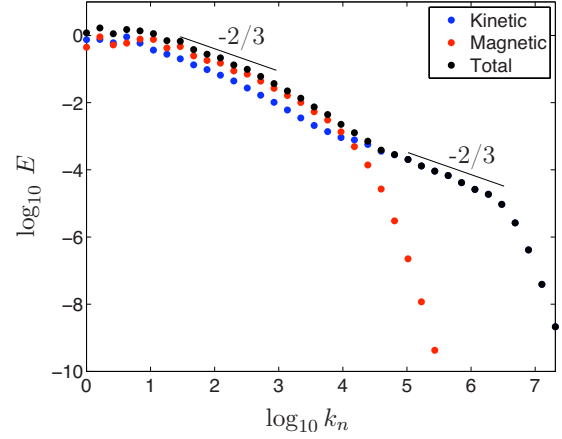


FIG. 1. (Color online) Kinetic and magnetic energy spectra $\nu=10^{-9}$ and $P_M=10^{-3}$.

for both the formalism, but other fluxes involving the magnetic field are different. This is due to the fact that the complete function \mathbf{W} is never computed in [12] because the property 2 was not used explicitly in the derivation of the shell model. In particular, the part of the bilinear term that conserves the magnetic energy in the magnetic field equation was not identified. It was not needed to derive completely the model coefficient. However, this identification is needed if the energy fluxes have to be defined unambiguously, which is the main objective of this work, but not of the approach developed in [12]. In the following section we compute the above fluxes using numerical simulation of shell model.

V. NUMERICAL RESULTS

We simulate the shell model [Eqs. (63) and (64)] with $\nu=10^{-9}$ and $\eta=10^{-6}$. The magnetic Prandtl number is then $P_M = \nu / \eta = 10^{-3}$. The shells ratio is taken to be the golden mean: $\lambda = (1 + \sqrt{5}) / 2$. This choice is the largest value of λ for which, when considering three consecutive shells (n , $n+1$, and $n+2$), the largest values of k in shell $n+2$ can be the longest side of a triangle while the two other sides can correspond to values of k from shells n and $n+1$. All k 's in shell $n+2$ then correspond to at least one triad. Moreover, this choice has also been adopted in order to have a direct comparison to the results presented by Stepanov and Plunian [12]. We take the number of shells as $N=36$ and apply non-helical forcing to s_4 , s_5 , and s_6 according to the scheme prescribed in [12] with an energy injection rate ϵ_i .

Although the shell variables experience a very complex evolution, the system reaches a statistically steady state fairly rapidly. We compute the energy spectra and energy fluxes under steady state by averaging over many time frames ($\sim 10^8$).

A. Energy spectra

We plot kinetic-energy and magnetic energy spectra in Fig. 1 for steady state. We observe that until $k \sim 10^4$ both the kinetic and magnetic energies show power-law behavior with $-2/3$ spectral exponent consistent with Kolmogorov's spec-

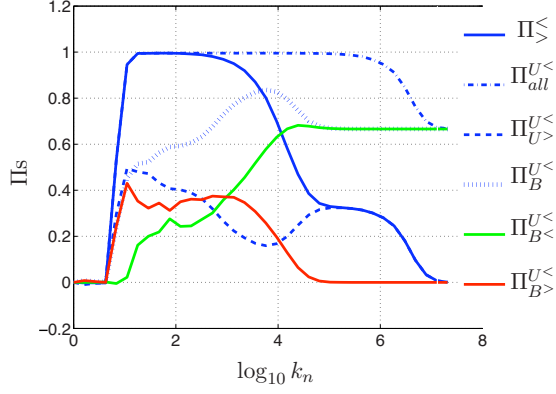


FIG. 2. (Color online) Energy fluxes in function of the logarithm of k_n , $\nu=10^{-9}$ and $P_M=10^{-3}$.

trum. After $k \sim 10^4$, the magnetic energy decays exponentially due to the Joule dissipation, while the kinetic energy continues to exhibit power-law behavior with the same spectral exponent of $-2/3$ till $k \sim k_\nu \sim 10^{6.5}$, where k_ν is the Kolmogorov's wave number. Note that the Kolmogorov's wavelengths for magnetic diffusion and thermal diffusion are $k_\eta = (\epsilon_i / \eta^3)^{1/4}$ and $k_\nu = (\epsilon_i / \nu^3)^{1/4}$, respectively, and they are quite close to our cutoffs of 10^4 and $10^{6.5}$. Up to slight differences due to the time steps or the initial conditions, steady-state spectra reported in the present simulations are in good agreement with the results presented by Stepanov and Plunian [12].

Although the spectra presented in Fig. 1 and in [12] are very similar, they are interpreted slightly differently for the low wave vector regime ($k \lesssim k_\eta$). In this regime, Plunian and Stepanov reported a “-1” spectral exponent for both velocity and magnetic fields. However, the results presented in Fig. 1 appear to be compatible with a “-2/3” exponent. This difference of interpretation is possibly due to the rather short range of wave vectors which makes the determination of the exponent quite difficult. The analysis of the energy fluxes however tends to support the $-2/3$ scaling in the range $k \lesssim k_\eta$.

B. Energy fluxes

According to the formulae (56)–(58), the energy fluxes are computed as functions of the shell index n . However, in this section, we present them as a function of wave number. The energy fluxes are proportional to the energy supply rate. Therefore, we report these fluxes in units of total-energy supply rate. First we focus on the various fluxes of kinetic energy leaving $U^<$. Three fluxes correspond to [Eqs. (56)–(58)], $\Pi_{U>}^{U<}(n)$, $\Pi_{B>}^{U<}(n)$, and $\Pi_{B<}^{U<}(n)$ and they are represented in Fig. 2. In addition, we also report the energy fluxes $\Pi_B^{U<}$ and $\Pi_{all}^{U<}$ that are defined as

$$\Pi_B^{U<} = \Pi_{B<}^{U<} + \Pi_{B>}^{U<}, \quad (67)$$

$$\Pi_{all}^{U<} = \Pi_{U>}^{U<} + \Pi_{B<}^{U<} + \Pi_{B>}^{U<} = \Pi_B^{U<} + \Pi_{U>}^{U<}. \quad (68)$$

Finally, the total-energy flux $\Pi_{>}^<$ leaving the sphere of wave vector smaller than k_n , from either the velocity or the magnetic field, is also presented in Fig. 2

$$\Pi_{>}^< = \Pi_{U>}^{U<} + \Pi_{B>}^{U<} + \Pi_{B<}^{U<} + \Pi_{B>}^{B<}. \quad (69)$$

We compute the energy fluxes for our shell model by averaging over many time frames once we reach steady state. The properties of these fluxes depend on the Prandtl number as described in earlier works [12]. In this paper we report these fluxes for $P_M=10^{-3}$ and they are illustrated in Fig. 2. Here we present all the energy fluxes and compare our results to those of Stepanov and Plunian [12]. We find that the energy fluxes $\Pi_{>}^<$ and $\Pi_{U>}^{U<}$ are in good agreement with the corresponding fluxes reported by Stepanov and Plunian [12]. However the energy fluxes from the velocity field to the magnetic field and vice versa do not match as our new definitions proposed in previous section differ from those of Stepanov and Plunian.

The energy fluxes can be interpreted in the following manner. The explanation of energy flux leaving $U^<$ ($\Pi_{all}^{U<}$) is relatively simple. Energy injected in the forcing range must flow out of the variable $U^<$ through the flux $\Pi_{all}^{U<}$ which redistributes energy from $U^<$ to the variables B and $U^>$ and through viscous dissipation. For $k_n \lesssim k_\nu \approx 10^{6.5}$, the viscous dissipation is negligible and the flux $\Pi_{all}^{U<}$ is independent of k_n and equal to ϵ_i . For $k_n \gtrsim k_\nu$, the dissipation of $U^<$ is significant and $\Pi_{all}^{U<}$ gradually decays. It is evident from Fig. 2 that the viscous dissipation rate $\epsilon_i u$ is 0.33 and the remaining energy supply ($1-0.33=0.67$) gets transferred to the magnetic energy that is finally dissipated through Joule heating.

Now we turn to the total-energy flux $\Pi_{>}^<$. We observe three plateaux that can be interpreted as follows. First, $\Pi_{>}^< \approx 1$ is flat in the range $k_n \lesssim k_\eta$ since the dissipation of both the magnetic and the kinetic energies is negligible in this range. The second range is $k_\eta \lesssim k_n \lesssim k_\nu$ where the magnetic energy dissipated through Joule heating. For our set of parameters, the Joule dissipation ϵ_η is approximately 0.67. In this range, kinetic energy is not dissipated strongly, as a result we observe Kolmogorov's spectrum for the kinetic energy. Hence our spectra and flux results are consistent. The third regime is $k > k_\nu$ where kinetic energy is dissipated. The amount of viscous dissipation is approximately 0.33.

The energy flux from $U^<$ to $B^>$ is denoted by $\Pi_{B>}^{U<}$. This flux is approximately 0.38 for lower wave numbers. The wave numbers in $B^>$ also receive significant energy from $U^>$ shells and it is approximately 0.46 for these wave numbers.

The energy flux $\Pi_{B<}^{U<}$, which is the energy-transfer rate from $U^<$ to $B^<$, increases in the range $k \lesssim k_\eta$. This is due to the fact that in this wave-number range, all $U^<$ shells transfer energy to $B^<$ for the parameters chosen by us. In the range $k > k_\eta$, the energy transfer from $U^<$ to $B^<$ is negligible because $B^<$ is very small; consequently $\Pi_{B<}^{U<}$ is constant in this range. The plateau value of 0.67 represents the total energy transferred from the velocity field to the magnetic field and is equal to ϵ_η .

The energy flux $\Pi_B^{U<}$ is the sum of $\Pi_{B<}^{U<}$ and $\Pi_{B>}^{U<}$. The flux $\Pi_{B>}^{U<}$ decreases gradually to zero near $k=k_\eta$ while $\Pi_{B<}^{U<}$ approaches the plateau near $k=k_\eta$. We observe that the quantity $\Pi_B^{U<}$ has a bump near k_η . Since $\Pi_{all}^{U<} = \Pi_{B<}^{U<} + \Pi_{B>}^{U<}$ and $\Pi_{all}^{U<}$ has been observed to be rather flat near k_η , the flux $\Pi_{B>}^{U<}$ has a downward bump near k_η .

In Fig. 3 we plot the energy fluxes leaving $U^<$, $U^>$, $B^<$, and $B^>$ spheres along with the energy dissipation. In each

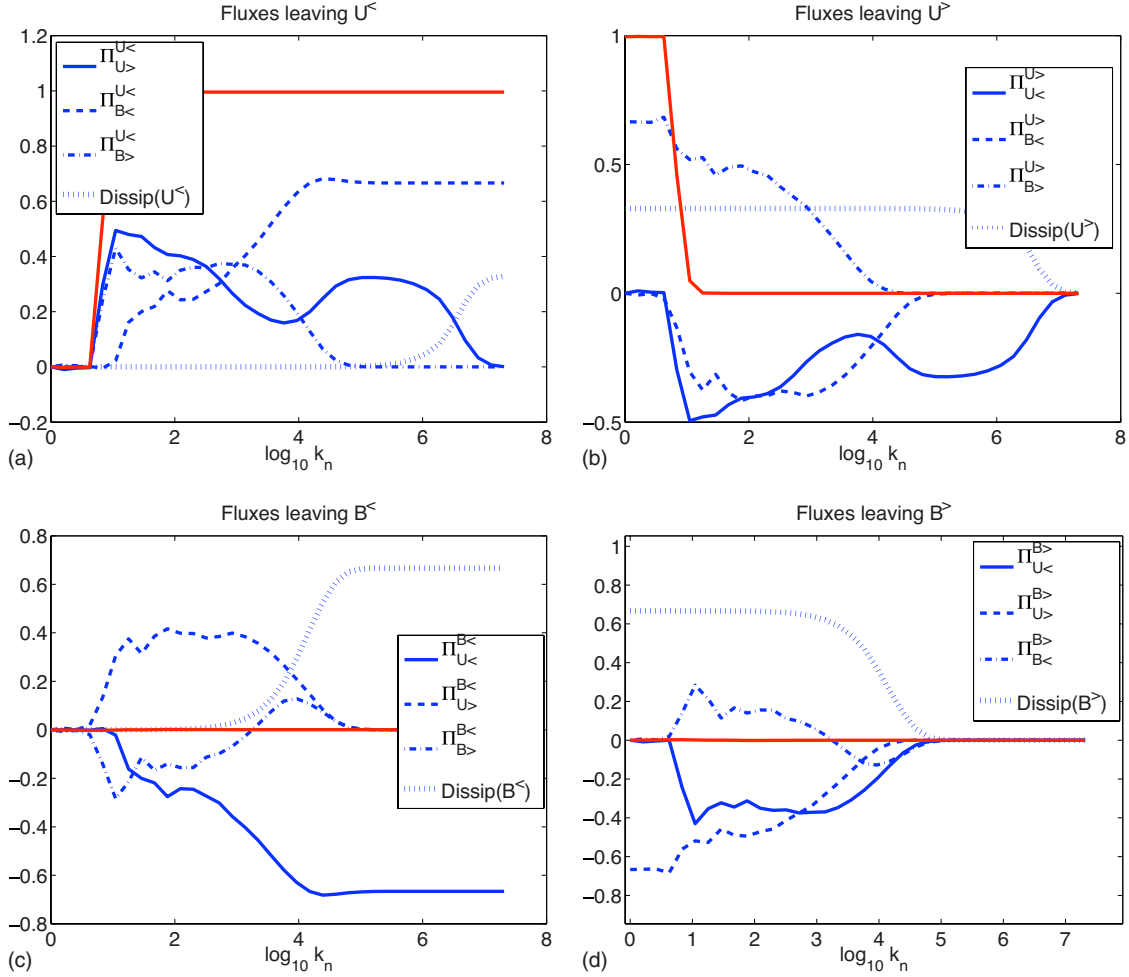


FIG. 3. (Color online) Fluxes of energy leaving each of the four regions of the variable’s space in function of the separating wave number k_n . Red (gray) line represents the rate of energy injection.

plot, the red (gray) line indicates the value of the overall energy flux leaving the corresponding region. It is 1.0 (energy injection) for $U^<$ and 0.0 for the other regions in steady state. In the next section we will study the energy flux from $B^<$ to $B^>$ ($\Pi_{B^>}^{B^<}$).

C. $\Pi_{B^>}^{B^<}$ and Reynolds number effects

We compute the magnetic energy flux $\Pi_{B^>}^{B^<}$ using the definition (42). Surprisingly we find that $\Pi_{B^>}^{B^<} < 0$ for the range $k \leq k_\eta$ indicating an inverse cascade of magnetic energy. These results are obviously different from the positive magnetic energy flux reported by Carati *et al.* [18] and Alexakis *et al.* [19] using direct numerical simulations (DNSs) of MHD equations. Note however that these DNS studies have been performed for $P_M=1$ and at a much lower Reynolds number ($\nu^{-1} = \eta^{-1} = 10^3$). In order to compare the shell-model results to DNS, we simulated the shell model [Eqs. (63) and (64)] for parameters close to those used in DNS [18] ($\lambda=2^{1/4}$, $\epsilon_i=1$, $\nu=10^{-3}$, $P_M=1$, and $N=38$). The forcing was kept similar to that of [12] in order to avoid dynamical alignment [20]. For these parameters, the magnetic energy flux of shell model and the DNS are in general agreement with each other.

In order to investigate whether the change of direction in the magnetic energy cascade is an effect of the magnetic Prandtl number or of the magnetic Reynolds number, we compute the flux $\Pi_{B^>}^{B^<}$ for various values of P_M and η while keeping the energy injection rate $\epsilon_i=1$ constant. In these simulations, the shell parameter λ has been set back to the value $(1 + \sqrt{5})/2$. The results have been displayed in Fig. 4. In shell models, the scale at which the energy is injected gives a typical length and the energy injection rate together with this typical length lead to a typical “velocity.” In our case, the corresponding estimation of the magnetic Reynolds number is $R_m \approx 10^{-1} / \eta$.

Our calculations show that the magnetic energy flux appears to depend mainly on the magnetic Reynolds number and not on the Prandtl number for a fixed η . The flux $\Pi_{B^>}^{B^<}$ seems to have two main features: a negative plateau for $k_n \leq k_\eta$ and a positive bump near magnetic dissipation wave number k_η . The negative-energy cascade of magnetic energy is absent in DNS. The appearance of negative-energy flux in the shell mode is rather surprising. These issues need further investigation.

In Fig. 5 we summarize all the flux results. Since $\Pi_{B^>}^{B^<}$ changes sign near k_η we compute the energy fluxes for two different wave numbers: $k_n=10^{2.27}$ and $k_n=10^{3.76}$. For

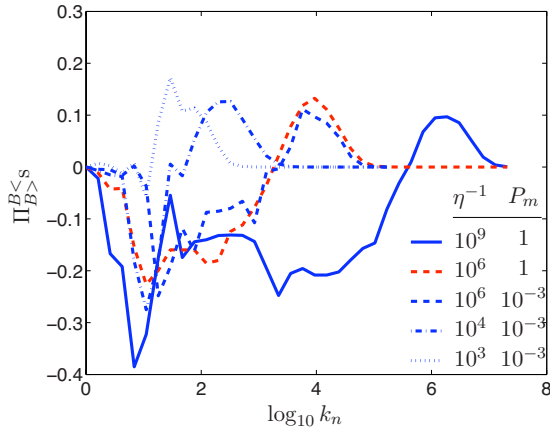


FIG. 4. (Color online) Magnetic-to-magnetic energy fluxes for various values of η and P_M in function of the logarithm of k_n . The forcing is applied in the region $10^{0.63} < k < 10^{1.25}$; $\lambda = (1 + \sqrt{5})/2$.

$k_n = 10^{2.27}$, $\Pi_{B>}^{B<}$ is negative. On the contrary, for $k_n = 10^{3.76}$, $\Pi_{B>}^{B<}$ is positive. The net dissipation of $B^<$ shells is 0.20 for the latter, contrary to zero dissipation for the former case. The difference in the two cases occurs because $\eta \gg \nu$.

The fluxes other than $\Pi_{B>}^{B<}$ are all positive in qualitative similarity to the DNS results. The values of these fluxes are different in the two wave-number regimes essentially due to the fact $\eta \gg \nu$.

VI. CONCLUSION

A general derivation of shell models for MHD has been proposed. The conservation of the traditional ideal invariants of three-dimensional MHD turbulence is expressed as general constraints that must be satisfied by the nonlinear terms in the shell model. The conservation of the kinetic helicity and kinetic energy by the hydrodynamic shell model in absence of magnetic field also leads to constraints on the nonlinearities. The similarity between the original MHD equations and the shell model is pushed one step further by identifying one term in the magnetic field equation in the shell model that conserves the magnetic energy. It corresponds to the advection of magnetic field by the velocity in the MHD equations. This identification is necessary to derive expressions for the transfers of magnetic energy between different shells. This procedure is presented using a very gen-

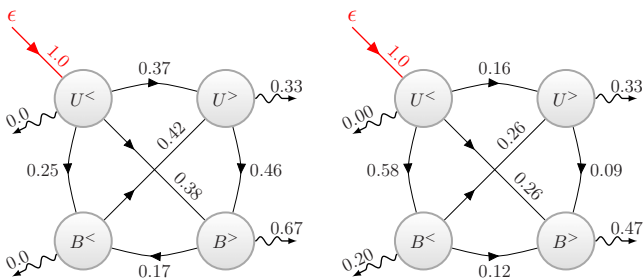


FIG. 5. (Color online) Energy fluxes in steady state for $\nu = 10^{-9}$ and $\eta = 10^{-6}$ and for $n = 12$ (left) and 19 (right) corresponding to $\log_{10} k_n = 2.30$ and 3.76 .

eral formalism which leads to a number of interesting results.

We show that the conservation of the cross helicity and the conservation of the total energy are equivalent in shell models. This equivalence is a direct consequence of the symmetries of the MHD equations expressed by the general properties 1–3 presented in Sec. II

The expressions for the energy fluxes that are valid independently of the specific structure of the nonlinear couplings between the shell variables have been derived. The knowledge of these fluxes is quite important when the shell models are used to explore dynamo regime. These expressions could even be used to derive shell models that would maximize or minimize certain energy transfers depending on the physics that has to be modeled.

Also, expressions for the shell-to-shell energy exchanges are derived. Like in the original MHD equations, the energy exchange mechanisms in shell models unavoidably involve three degrees of freedom (triadic interaction) [13,14,18,19,21–24]. It is thus not obvious to derive expression for shell-to-shell energy exchanges that are viewed as energy transfers between only two degrees of freedom. Nevertheless, the formalism presented in Sec. II yields a very natural identification of most of these energy exchanges. The only exception concerns the **U**-to-**U** energy exchanges. A simple expression is however also proposed for these quantities by analogy with the **B**-to-**B** energy exchanges.

Another property of the formalism presented here is the clear separation between the treatment of the conservation law and the assumptions that have to be made to define both the magnetic and the kinetic helicities. Because these helicities involve quantities that are defined using the curl operator, they are not very well adapted to shell models. It is thus quite appropriate to clearly present the expressions for the vorticity and the magnetic potential as additional assumptions required to fully specify the structure of the shell model.

The procedure has been applied to a specific class of shell models based on first neighbor couplings known as the GOY model. It has been shown that the general constraint naturally leads to the already derived GOY-MHD shell model [12]. However, the interpretation of the energy fluxes appears to be simpler in the present formalism. This model together with our flux definitions point out at a magnetic-to-magnetic inverse cascade of energy at high magnetic Reynolds number. It also reproduces the direct cascade observed in [18,19] for lower values of R_m . This intriguing aspect needs to be further studied with nonlocal shell models.

Several extensions to this work could be considered. Shell models using distant interactions between the shell variables [17] could be analyzed using the same formalism. Also, despite the fact that the presentation has been made for shell models with one complex number per shell and per field (velocity and magnetic), extending the present formalism to shell models with more degrees of freedom should be quite obvious. Finally, it would be interesting to explore other shell models based on alternative definitions for both the vorticity and the magnetic potential.

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- [1] H. K. Moffat, *Magnetic Fields Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, England, 1978).
 - [2] F. Krause and K. H. Radler, *Mean-Field Magnetohydrodynamics and Dynamo Theory*, Cambridge (Pergamon, New York, 1980).
 - [3] A. Brandenburg and K. Subramanian, *Phys. Rep.* **417**, 1 (2005).
 - [4] R. Monchaux *et al.*, *Phys. Rev. Lett.* **98**, 044502 (2007).
 - [5] M. Berhanu *et al.*, *Europhys. Lett.* **77**, 59001 (2007).
 - [6] L. Biferale, *Annu. Rev. Fluid Mech.* **35**, 441 (2003).
 - [7] E. B. Gledzer, *Sov. Phys. Dokl.* **18**, 216 (1973).
 - [8] M. Yamada and K. Ohkitani, *J. Phys. Soc. Jpn.* **56**, 4210 (1987).
 - [9] C. Gloaguen, J. Léorat, A. Pouquet, and R. Grappin, *Physica D* **17**, 154 (1985).
 - [10] P. Frick and S. Sokoloff, *Phys. Rev. E* **57**, 4155 (1998).
 - [11] T. Gilbert and D. Mitra, *Phys. Rev. E* **69**, 057301 (2004).
 - [12] R. Stepanov and F. Plunian, *J. Turbul.* **7**, 38 (2006).
 - [13] G. Dar, M. K. Verma, and V. Eswaran, *Physica D* **157**, 207 (2001).
 - [14] M. K. Verma, *Phys. Rep.* **401**, 229 (2004).
 - [15] V. S. L'vov, E. Podivilov, A. Pomyalov, I. Procaccia, and D. Vandembroucq, *Phys. Rev. E* **58**, 1811 (1998).
 - [16] F. Plunian and R. Stepanov, *New J. Phys.* **9**, 294 (2007).
 - [17] R. Stepanov and F. Plunian, *Astrophys. J.* **680**, 809 (2008).
 - [18] D. Carati, O. Debliquy, B. Knaepen, B. Teaca, and M. Verma, *J. Turbul.* **51**, 51 (2006).
 - [19] A. Alexakis, P. D. Mininni, and A. Pouquet, *Phys. Rev. E* **72**, 046301 (2005).
 - [20] P. Giuliani and V. Carbone, *Europhys. Lett.* **43**, 527 (1998).
 - [21] J. A. Domaradzki and R. S. Rogallo, *Phys. Fluids A* **2**, 413 (1990).
 - [22] F. Waleffe, *Phys. Fluids A* **4**, 350 (1992).
 - [23] O. Debliquy, M. K. Verma, and D. Carati, *Phys. Plasmas* **12**, 042309 (2005).
 - [24] P. D. Mininni, A. Alexakis, and A. Pouquet, *Phys. Rev. E* **72**, 046302 (2005).