The shape of a Möbius strip

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The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping[3] and paper crumpling[4,5]. This could give new insight into energy localization phenomena in unstretchable sheets[6], which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures[7–9].

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher[10]. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped $NbSe_3$ crystals under certain growth conditions involving a large temperature gradient[7,8]. The mechanism proposed by Tanda *et al.* to explain this behaviour is a combination of Se surface tension, which makes the crystal bend, and twisting as a result of bend–twist coupling due to the crystal nature of the ribbon. Recently, quantum eigenstates of a particle confined to the surface of a developable Möbius strip were computed[9] and the results compared with earlier calculations[11]. Curvature effects were found in the form of a splitting of the otherwise doubly degenerate ground-state wavefunction. Thus qualitative changes in the physical properties of Möbius strip structures (for instance nanostrips) may be anticipated and it is of physical interest to know the exact shape of a free-standing strip. It has also been theoretically predicted that a novel state appears in a superconducting Möbius strip placed in a magnetic field[12]. Möbius strip geometries have furthermore been proposed to create optical fibres with tuneable polarization[13].

The simplest geometrical model for a Möbius strip is the ruled surface swept out by a normal vector that makes half a turn as it traverses a closed path. A common paper Möbius strip (Fig. 1) is not well described by this model because the surface generated in the model need not be developable, meaning that it cannot be mapped isometrically (that is, with preservation of all intrinsic distances) to a plane strip. A paper strip is to a good approximation developable because bending a piece of paper is energetically

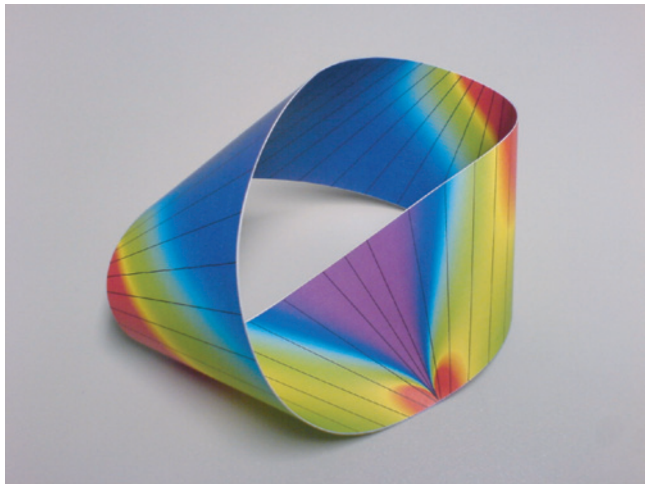

Figure 1 Photo of a paper Möbius strip of aspect ratio 2π. The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.

much cheaper than stretching it. The strip therefore deforms in such a way that its metrical properties are barely changed. It is reasonable to suggest that some nanostructures have the same elastic properties. A necessary and sufficient condition for a surface to be developable is that its gaussian curvature should everywhere vanish. Given a curve with non-vanishing curvature there exists a unique flat ruled surface (the so-called rectifying developable) on which this curve is a geodesic curve[14]. This property has been used to construct examples of analytic (and even algebraic) developable Möbius strips[15–18].

If $\mathbf{r}(s)$ is a parametrization of a curve then

$$\mathbf{x}(s,t) = \mathbf{r}(s) + t\,[\mathbf{b}(s) + \eta(s)\mathbf{t}(s)],$$
$$\tau(s) = \eta(s)\kappa(s), \quad s = [0, L], t = [-w, w]$$

is a parametrization of a strip with $\mathbf{r}$ as centreline and of length L and width $2w$, where $\mathbf{t}$ is the unit tangent vector, $\mathbf{b}$ the unit binormal, κ the curvature and τ the torsion of the centreline (see, for example, ref. 18). The parametrized lines $s =$ const. are the generators, which make an angle $\beta = \arctan(1/\eta)$ with the positive tangent direction. Thus the shape of a developable Möbius strip is completely determined by its centreline. We also recall that a

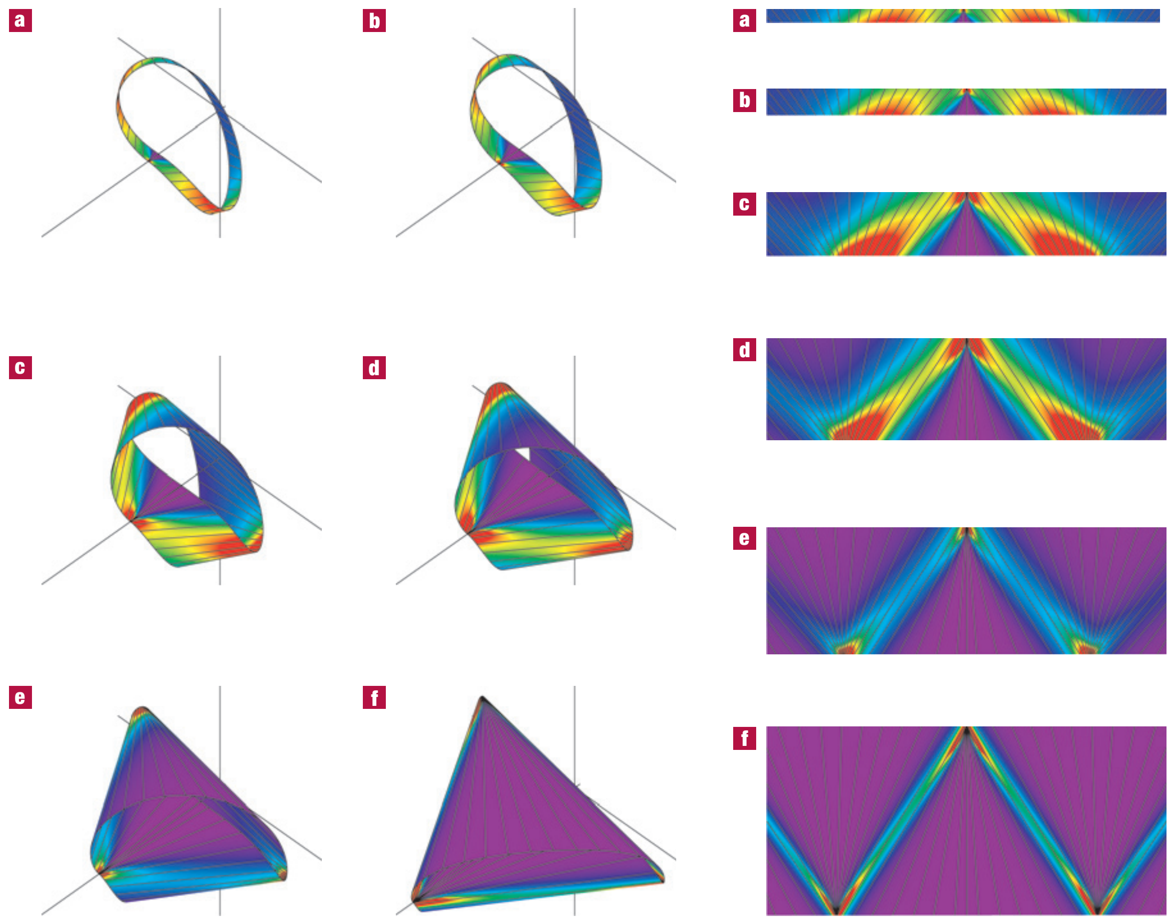

Figure 2 Computed Möbius strips. The left panel shows their three-dimensional shapes for $w = 0.1$ (**a**), 0.2 (**b**), 0.5 (**c**), 0.8 (**d**), 1.0 (**e**) and 1.5 (**f**), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from violet for regions of low bending to red for regions of high bending (scales are individually adjusted). Solution **c** may be compared with the paper model in Fig. 1 on which the generator field and density colouring have been printed.

regular curve in three dimensions is completely determined (up to euclidean motions) by its curvature and torsion as functions of arc length.

As simple experimentation shows, an actual material Möbius strip, made of inextensible material, when left to itself, adopts a characteristic shape independent of the type of material (sufficiently stiff for gravity to be ignorable). This shape minimizes the deformation energy, which is entirely due to bending. We shall assume the material to obey Hooke's linear law for bending. Because for a developable surface one of the principal curvatures is zero, the elastic energy is then proportional to the integral of the other principal curvature squared over the surface of the strip:

$$V = \frac{1}{2} D \int_0^L \int_{-w}^{w} \kappa_1^2(s,t)\,\mathrm{d}t\mathrm{d}s, \tag{1}$$

where $D = 2h^3E/[3(1-\nu^2)]$, with $2h$ the thickness of the strip and E and ν Young's modulus and Poisson's ratio of the material[19].

Sadowsky[1,2], as long ago as 1930, seems to have been the first to formulate the problem (open to this date) of finding the developable Möbius strip of minimal energy, albeit in the limit of an infinitely narrow strip ($w = 0$). He derived the equations for this special case (to our knowledge the only equilibrium equations for a developable elastic strip anywhere in the literature) but did not solve them. For the general case Wunderlich[15] reduced the two-dimensional integral to a one-dimensional integral over the centreline of the strip by integrating over the straight generator (that is, carrying out the t integration in (1)). The resulting functional is expressed in terms of the curvature and torsion of the centreline and their derivatives:

$$V = \frac{1}{2} Dw \int_0^L g(\kappa, \eta, \eta')\,\mathrm{d}s, \tag{2}$$

$$g(\kappa,\eta,\eta') = \kappa^2\left(1+\eta^2\right)^2 \frac{1}{w\eta'} \log\left(\frac{1+w\eta'}{1-w\eta'}\right),$$

where the prime denotes differentiation with respect to arc length s. In the limit of zero width this reduces to Sadowsky's functional

$$V_S = Dw \int_0^L \frac{\left(\kappa^2+\tau^2\right)^2}{\kappa^2}\,\mathrm{d}s.$$

Because D appears as an overall factor, equilibrium shapes will not depend on the material properties.

Energy minimization is thus turned into a one-dimensional variational problem represented in a form that is invariant under euclidean motions. The standard way of solving it, by

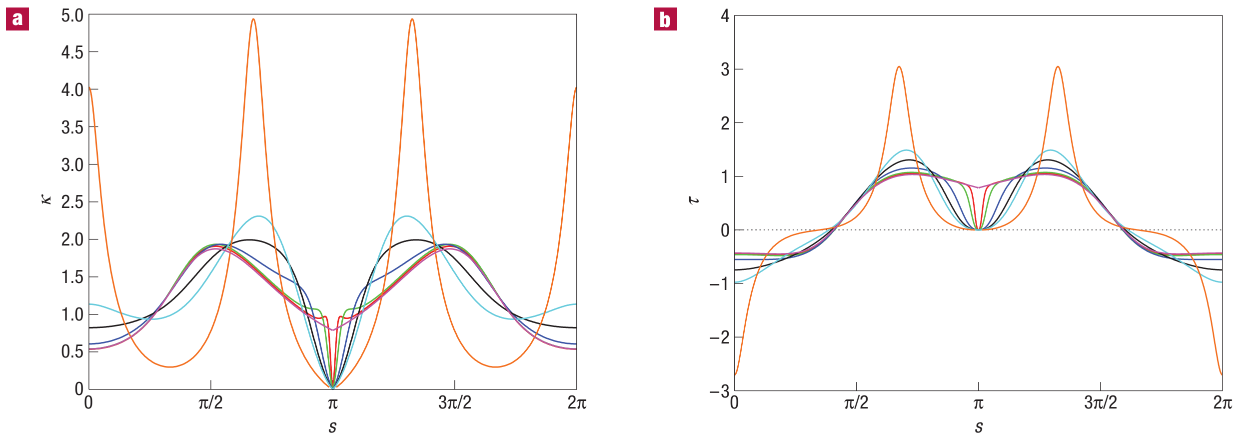

Figure 3 Curvature and torsion of a Möbius strip. Curvature κ (**a**) and torsion τ (**b**) for $w = 0$ (magenta), 0.1 (red), 0.2 (green), 0.5 (blue), 0.8 (black), 1.0 (cyan) and 1.5 (orange). At $s = \pi$ the principal normal changes to the opposite direction.

expressing the Lagrangian g in terms of $\mathbf{r}$ and its derivatives (or possibly introducing coordinates) and deriving the Euler–Lagrange equations, is a formidable task even with the use of modern symbolic computer software, and no equations for the finite-width case seem to exist in the literature. Here we use a powerful geometric approach on the basis of the variational bicomplex formalism[20,21], which enables us to obtain a manageable set of equations in invariant form almost immediately. This theory (apparently little known outside the mathematicians' community), when applied to variational problems for space curves, yields equilibrium equations for functionals of general type

$$\int_0^L f(\kappa, \tau, \kappa', \tau', \kappa'', \tau'', \ldots, \kappa^{(n)}, \tau^{(n)})\, \mathrm{d}s, \qquad (3)$$

involving derivatives up to any order (ref. 21, Ch. 2, Sec. C). A similar technique was applied in refs 22,23 to derive Euler–Lagrange equations for some simple Lagrangians f, but our current problem seems to be the first for which an invariant approach is essential to obtain a solution.

The equilibrium equations can be cast into the form of six balance equations for the components of the internal force $\mathbf{F}$ and moment $\mathbf{M}$ in the directions of the Frenet frame of tangent, principal normal and binormal, $\mathbf{F} = (F_t, F_n, F_b)^{\mathrm{T}}$, $\mathbf{M} = (M_t, M_n, M_b)^{\mathrm{T}}$, and two scalar equations that relate M_t and M_b to the Lagrangian g:

$$\mathbf{F}' + \boldsymbol{\omega} \times \mathbf{F} = \mathbf{0}, \quad \mathbf{M}' + \boldsymbol{\omega} \times \mathbf{M} + \mathbf{t} \times \mathbf{F} = \mathbf{0}, \qquad (4)$$

$$\partial_\kappa g + \eta M_t + M_b = 0, \quad \left(\partial_{\eta'} g\right)' - \partial_\eta g - \kappa M_t = 0, \qquad (5)$$

where $\boldsymbol{\omega} = \kappa(\eta, 0, 1)^{\mathrm{T}}$ is the Darboux (or curvature) vector. These equations follow from Proposition 2.16 in ref. 21. They can also be obtained by extending the theory of Sadowsky[1], on the basis of mechanical considerations, to functional (2). We note that in the variables (κ, η, η') the first equation in (5) is an algebraic equation. (In the limit $w = 0$ both equations in (5) become algebraic.) The equations have $|\mathbf{F}|^2$ and $\mathbf{F} \cdot \mathbf{M}$ as first integrals and are invariant under the involution $(\kappa, \eta, \eta') \to (\kappa, \eta, -\eta'), s \to -s$.

It has been shown[18] that along the centreline of a rectifying developable Möbius strip an odd number of switching points must occur where $\kappa = \eta = 0$ and the principal normal to the centreline flips (that is, makes a 180° turn). It follows that the strip must contain an umbilic line, that is, a line on which both principal curvatures vanish[24]. (Incidentally, if the initial strip is not a rectangle then a Möbius strip may be constructed that has no switching points[25].) To make the twisted nature of the Möbius strip precise we note that a closed centreline with a periodic twist rate (here $\tau(s)$) defines a closed cord[26], for which we can define a linking number Lk (ref. 26). Any ribbon of a cord of half-integer Lk is one sided. Any such ribbon with $Lk = \pm(1/2)$ gives a Möbius strip.

The centreline in three dimensions may be reconstructed from $(\kappa(s), \tau(s))$ by integrating the usual Frenet–Serret equations and the equation $\mathbf{r}' = \mathbf{t}$. Coupling these to (4), (5) we thus have a differential-algebraic system of equations for which we formulate a boundary-value problem for the Möbius strip by imposing boundary conditions at $s = 0$ and $s = L/2$ and selecting the solution with $Lk = 1/2$. The involution property is then used to obtain the solution on the full $[0, L]$ interval by suitable reflection. This yields a symmetric solution; it seems unlikely that non-symmetric solutions exist. A Möbius strip has chirality, meaning that it is not equivalent to its mirror image. This mirror image, having a link $Lk = -1/2$, is obtained by reflecting $\eta \to -\eta$, $\eta' \to -\eta'$.

Figure 2 shows numerically obtained solutions. There is only one physical parameter in the problem, namely the aspect ratio $L/2w$ of the strip. In the computations we have fixed $L = 2\pi$ and varied w. Also shown in the figures is the evolution along the strip of the straight generator. We note the points where the generators start to accumulate. At these points $|w\eta'| \to 1$ and the integrand in (2) (the energy density) diverges. Where this happens the generator rapidly sweeps through a nearly flat (violet) triangular region, a phenomenon readily observed in a paper Möbius strip (Fig. 1). We also observe two additional (milder) accumulations where no inflection occurs and the energy density remains finite. It can be shown that the energy density is monotonic along a generator. This implies that the (red) regions of high curvature cannot be connected by a generator, as a careful inspection confirms. Bounding the (violet) triangular (more precisely, trapezoidal) regions are two generators of constant curvature. These generators realize local minima for the angle β.

As w is increased the accumulations and associated triangular regions become more pronounced. At the critical value given by $w/L = \sqrt{3}/6$ the strip collapses into a triple-covered equilateral triangle[17,27]. The folding process as w is increased towards this flat triangular limit resembles the tightening of tubular knots as they approach the ideal shape of minimum length to diameter

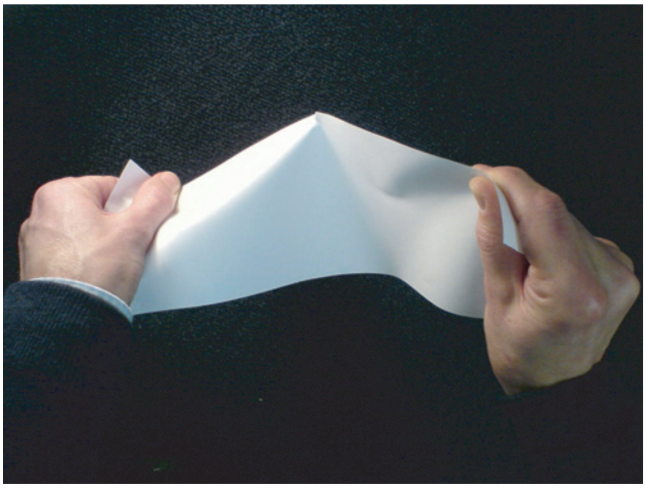

Figure 4 Tearing a piece of paper. In trying to tear a sheet of paper we create a deformation similar to what we see in the Möbius strip. A crack will start at the vertex, where the energy density diverges.

ratio[28]. In the flat limit the generators are divided into three groups, intersecting one another at three vertices. The bounding generators of constant curvature become the creases. It has been conjectured that a smooth developable Möbius strip can be isometrically embedded in $\mathbb{R}^3$ only if $w/L < \sqrt{3}/6$ (ref. 29), whereas it has been proven that a smooth developable Möbius strip can be immersed in $\mathbb{R}^3$ only if $w/L < 1/\pi$ (ref. 29). Interestingly, larger (in fact, arbitrarily large) values for the ratio w/L can be obtained if we allows for additional folding[27].

Figure 3 gives plots of curvature and torsion. The Randrup and Røgen property that $\kappa = \eta = 0$ at an odd number of points is confirmed in Fig. 3 and can also be seen in Fig. 2 at the centre of the images, where the generator makes an angle of 90° with the centreline. As the maximum value $w_c = \pi/\sqrt{3} = 1.8138\ldots$ is approached, both curvature and torsion become increasingly peaked about $s = 0$, $2\pi/3$ and $4\pi/3$. In the limit all bending and torsion is concentrated at the creases of the flat triangular shape. Going to the other extreme, we find that the solution in the limit of zero width has non-vanishing curvature, so that the Randrup and Røgen conditions are not satisfied. Given that the Frenet frame flips at $s = \pi$ (as enforced by the boundary conditions), this means that the curvature is discontinuous. In addition, η tends to unity, giving a limiting generator angle $\beta = 45°$. Both these properties were anticipated in ref. 1. This shows that the zero-width limit is singular and suggests that the Sadowsky problem has only a solution with discontinuous curvature.

In ref. 30 the shape of a Möbius strip was computed by using a thin anisotropic elastic rod model. Asymptotic equations were obtained for large values of the aspect ratio of the rod's cross-section. This limit corresponds to perfect alignment of the rod material frame and the Frenet frame, and the equilibrium equations are therefore the Euler–Lagrange equations for Lagrangian $f = (1/2)B\kappa^2 + (1/2)C\tau^2$ in (3), where B and C are the bending and torsional stiffnesses, respectively. The solution to these equations, however, even after the modifications made in ref. 30, does not satisfy the Randrup and Røgen conditions mentioned above, and therefore cannot serve as the centreline of a developable Möbius strip, even a narrow one (see the Supplementary Information).

The Möbius strip defines only one example of a boundary-value problem for twisted sheets. A natural generalization is to strips with linking numbers other than $\pm 1/2$. Our techniques can readily be applied to such problems and an example of a strip with $Lk = 3/2$ (also known from Escher's work[10]) is shown in the Supplementary Information. Clearly, the same equations (4) and (5) apply to non-closed strips. A further generalization would be to non-rectangular sheets, although it is not guaranteed that the t integration in (1) can be carried out, meaning that we might end up with a system of integro-differential equations instead of (4) and (5).

The geometrical features of Möbius strips observed here are seen more widely in problems of elastic sheets such as paper folding or crumpling and fabric draping. Crumpling of paper is dominated by bending along ridges bounding almost flat regions or facets[4,5], behaviour that we see back in the nearly flat triangular regions in Fig. 2. In fabric draping, triangular regions are seen to form that radiate out from (approximate) vertices. The formation of these flat triangular regions seems to be a generic feature of nature's response to twisting inextensible sheets. Analytical work on such sheets often assumes regions of localization of bending energy in the form of vertices of conical surfaces[3,6]. It is known that conical surfaces have infinite elastic energy within the linear elastic theory. The difficulties associated with this necessitate the introduction of a cut-off[3]. As the example of the Möbius strip shows, the consideration of non-conical developable elastic surfaces enables us to describe bending localization phenomena without the need for a cut-off. Importantly, our approach predicts the emergence of regions of high bending. Points of divergence of the bending energy may serve as indicators of positions where out-of-plane tearing (fracture failure mode III) is likely to be initiated. In this respect it is interesting to observe that when we try to tear a piece of paper (see Fig. 4) we intuitively apply a torsion, thereby creating intersecting creases as in the vertices of the central triangular domains in Fig. 2.

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