Incompletely parabolic systems from Friedrichs theory point of view*

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Abstract

An incompletely parabolic system of partial differential equations consist of one parabolic subsystem coupled to a hyperbolic subsystem. For the initial-boundary value problem, it has been shown, by requiring that a solution remains at every time bounded by the data, that boundary conditions which make both subsystems well-posed render the global system well-posed too. In this paper, we establish the same type of result with the help of the notion of semi-admissible boundary conditions in the theory of Friedrichs positive systems of differential equations. Our aim is not to obtain the same results of existence and uniqueness as those for the Cauchy problem, but rather to find a way to establish boundary conditions on the subsystems which can be used for the global system too. The theory is illustrated by the two examples of the compressible Navier-Stokes equations and of the hydrodynamic model for semiconductor devices, both in two space dimensions.

1 Introduction

Incompletely parabolic systems of partial differential equations consist of two subsystems, the first one being parabolic and the second one hyperbolic. Existence and uniqueness results for the Cauchy problem were obtained by Novik [8]. Later on, Strikwerda, in a paper quite difficult to read [10], studied boundary conditions for the global system being well-posed; the meaning of his results is that boundary conditions making the parabolic and the hyperbolic subsystems well-posed render the global system well-posed. These results have been often used but it seems that no other proof can be found in the literature. For instance, Strikwerda's theory was applied to get boundary condition for the viscous shallow-water equation and the compressible Navier-Stokes equations [5]; it was also used to obtain non-absorbing boundary conditions for the global system [6]. Beside pure fluid dynamics problems, an important application of the theory of incompletely parabolic systems consists in the so-called hydrodynamic model for semiconductor devices (for a clear introduction see [11]); in this context, the problem of setting up correct boundary conditions was partially studied in Ref. [12]; this can be done with the help of Friedrichs theory of symmetric positive systems, as suggested in Ref. [4]. The results we obtain from this theory are valid for any incompletely parabolic system and we illustrate them on the compressible Navier-Stokes equations and on the hydrodynamic model only to give examples. This work could be generalized to the study of more general boundary conditions.

In **Section 2**, we recall some definitions and characteristic features from the theory of incompletely parabolic systems; the theory of positive systems is briefly summarized in **Section 3**. This theory is used to establish boundary conditions for incompletely parabolic systems in one space dimension in **Section 4** and in two space dimensions in **Section 5**. Finally, **Section 6** is devoted to the specific examples of the compressible Navier-Stokes equations and of the hydrodynamic model for semiconductor devices; both cases are now the subject of numerous numerical simulations.

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2 Incompletely parabolic systems

We consider systems of linear partial differential equations in the variables $(t,\mathbf{x}) \in [0,\infty) \times \mathbf{R}^N$; we suppose that the matrices which define these systems are sufficiently regular functions of $(\mathbf{x},t)$, for instance C^1. Given N^2 matrices $P^{ij} \in \mathcal{L}(\mathbf{R}^n)$, the system

$$\partial_t \mathbf{u} - \sum_{i,j=1}^{N} P^{ij} \partial_{x_i} \partial_{x_j} \mathbf{u} = \mathbf{f} \tag{1}$$

is said to be **parabolic** iff there exists $\alpha > 0$ such that for every vector $\boldsymbol{\omega} = (\omega_1, \omega_2, .., \omega_N)^t \in \mathbf{R}^N$ and for every eigenvalue λ of the matrix $\sum_{i,j} P^{ij}\omega_i\omega_j$ one has $\operatorname{Re}\lambda > \alpha \parallel \omega \parallel^2$. A system of type (1) will be called **strongly parabolic** iff for all $\boldsymbol{\xi_1}, \boldsymbol{\xi_2}, ..., \boldsymbol{\xi_N} \in \mathbf{R}^n$ one has $\sum_{i,j} \boldsymbol{\xi}_i^t P^{ij} \boldsymbol{\xi}_j \geq \alpha \sum_i \boldsymbol{\xi}_i^t \; \boldsymbol{\xi}_i$. Given p matrices $Q^i \in \mathcal{L}(\mathbf{R}^n)$, the system

$$\partial_t \mathbf{u} + \sum_{i=1}^{N} Q^i \partial_{x_i} \mathbf{u} = \mathbf{f} \tag{2}$$

is **hyperbolic** iff for all $\boldsymbol{\omega} \in \mathbf{R}^N$ the matrix $\sum_i Q^i \omega_i$ is diagonalizable with real eigenvalues. It is a **strictly hyperbolic** system iff the eigenvalues are distinct.

We consider now matrices $C \in \mathcal{L}(\mathbf{R}^n)$, $B^{ij} \in \mathcal{L}(\mathbf{R}^n)$ of rank $m < n$, which we assume (after a possible change of basis) to have the form

$$B^{ij} = \begin{pmatrix} B_{11}^{ij} & 0 \\ 0 & 0 \end{pmatrix}, \; 1 \leq i,j \leq N, \tag{3}$$

and we suppose that the matrices $A^i \in \mathcal{L}(\mathbf{R}^n)$ are partitioned in the same way:

$$A^i = \begin{pmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{pmatrix}, \; 1 \leq i \leq N,$$

and the vectors $\mathbf{w} = (\mathbf{u}, \mathbf{v})$, $\mathbf{F} = (\mathbf{f}, \mathbf{g}) \in \mathbf{R}^n$ are written accordingly with $\mathbf{u}, \mathbf{f} \in \mathbf{R}^m$. The system

$$\partial_t \mathbf{w} + \sum_{i=1}^{N} A^i \partial_{x_i} \mathbf{w} - \sum_{i,j=1}^{N} B^{ij} \partial_{x_i} \partial_{x_j} \mathbf{w} + C\mathbf{w} = \mathbf{F} \tag{4}$$

is called **incompletely parabolic** (or parabolic-hyperbolic) iff

1) The system $\partial_t \mathbf{w} + \sum_i A^i \partial_{x_i} \mathbf{w} = \mathbf{F}$ is hyperbolic,

2) the system $\partial_t \mathbf{u} - \sum_{i,j} B_{11}^{ij} \partial_{x_i} \partial_{x_j} \mathbf{u} = \mathbf{f}$ is parabolic,

3) the system $\partial_t \mathbf{v} + \sum_i A_{22}^i \partial_{x_i} \mathbf{v} = \mathbf{g}$ is strictly hyperbolic.

In this paper, we suppose that the parabolic subsystem is strongly parabolic and we will furthermore assume that the system (4) is symmetrizable, i.e. that there exists a symmetric positive definite matrix $S \in \mathcal{L}(\mathbf{R}^n)$ such that the matrices SB^{ij} and $SA^i + \sum_j \partial_{x_j}(SB^{ji})$ are symmetric. In this case and after the change of variables $w \mapsto e^{\alpha t} w, \alpha > 0$, the system can be rewritten with symmetric matrices:

$$S\partial_t \mathbf{w} + \sum_{i=1}^{N} A^i \partial_{x_i} \mathbf{w} - \sum_{i,j=1}^{N} \partial_{x_i}(B^{ij} \partial_{x_j} \mathbf{w}) + (C + \alpha S)\mathbf{w} = \mathbf{F}. \tag{5}$$

It is precisely on this kind of system (5) that Strikwerda has worked [10], by studying essentially boundary conditions making the half-space problem well-posed.

Some results can be easily obtained and we think they give an interesting insight into the theory. Let us first consider the Cauchy problem with the norm

$$\parallel \mathbf{w}(t) \parallel = \int_{\mathbf{R}^N} \mathbf{w}(\mathbf{x},t)^t S(\mathbf{x},t) \mathbf{w}(\mathbf{x},t) d\mathbf{x};$$

if we say that the homogeneous ($\mathbf{F} = \mathbf{0}$) problem is well-posed if there exists a constant $C(T)$ such that $\parallel \mathbf{w}(T) \parallel \leq C(T) \parallel \mathbf{w}(0) \parallel$ for $T > 0$, we compute very easily from (5) if $\mathbf{w}$ vanishes at infinity:

$$\frac{d}{dt} \parallel \mathbf{w}(t) \parallel^2 + 2\int_{\mathbf{R}^N} \sum_{i,j=1}^{N} (\partial_{x_i} \mathbf{w}^t B^{ij} \partial_{x_j} \mathbf{w}) d\mathbf{x} + \int_{\mathbf{R}^N} \mathbf{w}^t K \mathbf{w} d\mathbf{x} = 0,$$

with $K = C + C^t + 2\alpha S - \partial_{x_i} A^i - \partial_t S$. The constant α can be chosen such that the last term is lower bounded by $\gamma \parallel \mathbf{w}(t) \parallel^2, \gamma > 0$, and under the only condition that the subsystem defined by the matrices B^{ij}_{11} is strongly parabolic, we get from $\frac{d}{dt} \parallel \mathbf{w}(t) \parallel^2 + \gamma \parallel \mathbf{w}(t) \parallel^2 \geq 0$ that the global system is well posed. Hence, in this particular case, global well-posedness is obtained from a condition on one subsystem only.

Let us consider now the same problem on the half-space $\Omega = \{\mathbf{x} = (x_1, \mathbf{y}) \in \mathbf{R}^N;\ x_1 > 0\}$; the energy equality writes in this case:

$$\frac{d}{dt} \parallel \mathbf{w}(t) \parallel^2 + 2\int_{\mathbf{R}^N} \sum_{i,j=1}^{N} (\partial_{x_i} \mathbf{w}^t B^{ij} \partial_{x_j} \mathbf{w}) d\mathbf{x} + \int_{\mathbf{R}^N} \mathbf{w}^t K \mathbf{w} d\mathbf{x}$$

$$+ \int_{\mathbf{R}^{N-1}} [2\mathbf{w}^t B^{11} \partial_{x_1} \mathbf{w} - \mathbf{w}^t A^1 \mathbf{w}](0, \mathbf{y}, t)\ d\mathbf{y} = 0.$$

If the subsystem defined by the B^{ij}_{11} is strongly parabolic, the global system will be well-posed if the boundary integral is nonnegative, i.e. with $\mathbf{w} = (\mathbf{u}, \mathbf{v})$, if

$$(\mathbf{w}^t, \partial_{x_1} \mathbf{u}^t) T \begin{pmatrix} \mathbf{w} \\ \partial_{x_1} \mathbf{u} \end{pmatrix} \leq 0,\ T = \begin{pmatrix} \multicolumn{2}{c}{A^1} & \begin{matrix} (-B^{11}_{11})^t \\ 0 \end{matrix} \\ -B^{11}_{11} & 0 & 0 \end{pmatrix}.$$

The following lemma, which is proved in Appendix 1, will give the number of sufficient conditions for the matrix T to be negative semi-definite.

Lemma 2.1 *Let $A \in \mathcal{L}(\mathbf{R}^n)$, $B_{11} \in \mathcal{L}(\mathbf{R}^m)$ be symmetric matrices with B_{11} positive definite. Set*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},\ A_{11} \in \mathcal{L}(\mathbf{R}^m),\ B = \begin{pmatrix} B_{11} & 0 \end{pmatrix} \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m),\ T = \begin{pmatrix} A & (-B)^t \\ -B & 0 \end{pmatrix}.$$

Then, the number of positive (resp. negative) eigenvalues of T is equal to m + the number of positive (resp. negative) eigenvalues of A_{22}.

Now, in order that $\boldsymbol{\xi}^t T \boldsymbol{\xi} \leq 0$, it is sufficient to make the orthogonal projection of $\boldsymbol{\xi}$ on the subspace spanned by the eigenvectors corresponding to the positive eigenvalues of T vanish. We clearly see on this example that the number of boundary conditions to impose on the global system are determined by the number of conditions for both subsystems separately to be well-posed.

3 Friedrichs systems

The original theory of Friedrichs on symmetric positive systems can be found in [3] and is briefly exposed in [2]; a good introduction can be found in [7] and generalizations a given in [9]. We recall here only what we need from this theory. We consider a system of n linear partial differential equations of order 1 in the variables $(x_0 = t, \mathbf{x}) \in \mathbf{R}^{N+1}$ defined with the help of the C^1 matrices $\mathcal{A}^i$, $i = 0, 1, ..., N$, and of the continuous matrix $\mathcal{K}$:

$$\sum_{i=0}^{N} \mathcal{A}^i \partial_{x_i} \mathbf{q} + \mathcal{K} \mathbf{q} = \mathbf{F}, \tag{6}$$

with a given function $\mathbf{F}$. A system of type (6) is called **symmetric positive** (or simply a Friedrichs system) iff

1) $\mathcal{A}^i = (\mathcal{A}^i)^t$, $i = 0, 1, ..., N$;

2) the matrix $\mathcal{C} = \mathcal{K} + \mathcal{K}^t - \sum_{i=0}^{N} \partial_{x_i} \mathcal{A}^i$ is uniformly positive definite.

Homogeneous conditions on the boundary $\partial\Omega$ of a (partially) bounded domain $\Omega \subset \mathbf{R}^{N+1}$ are prescribed in the following way. Let $\boldsymbol{\nu}$ be the unit outward normal to $\partial\Omega$; one defines the "boundary" matrix

$$\mathcal{B}(\mathbf{x}) = \sum_{i=0}^{N} (\nu_i \mathcal{A}^i)(\mathbf{x}),\ \ \mathbf{x} \in \partial\Omega, \tag{7}$$

and, with the help of a matrix $\mathcal{M} \in \mathcal{L}(\mathbf{R}^n)$, defined on $\partial\Omega$, the boundary conditions are written:

$$(\mathcal{B}(\mathbf{x}) - \mathcal{M}(\mathbf{x}))\mathbf{q}(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\Omega, \tag{8}$$

i.e. $\mathbf{q}$ belongs to the kernel of $\mathcal{B} - \mathcal{M}$ on the boundary. The condition (8) is **semi-admissible** iff the matrix $\mathcal{M}$ is **positive semi-definite**; this property ensures the existence of a solution of the positive system (6) as quoted in the following theorem.

Theorem 3.1 *[3] Assume that the system (6) is positive with $\mathbf{F} \in L^2(\Omega)^n$ and the boundary condition (8) is semi-admissible. Then (6) has at least one weak solution $\mathbf{q} \in L^2(\Omega)^n$ with $\| \mathbf{q} \|_{L^2(\Omega)^n} \leq C \| \mathbf{F} \|_{L^2(\Omega)^n}$.*

Remark 1 *The matrix $\mathcal{M} = \sqrt{\mathcal{B}^2}$ yields semi-admissible conditions and hence their exists at least one such matrix for any system. For instance, assume that (6) is a symmetric hyperbolic system (2) with $\mathcal{A}^0 = I$, $\mathcal{A}^i = Q^i, i = 1, 2, ...$; then, the boundary matrix for the true boundary condition (and not the initial one) is given by $\mathcal{B} = \sum_{i=1}^{N} \nu_i \mathcal{A}^i$. The boundary conditions specified by $\mathcal{M} = \sqrt{\mathcal{B}^2}$ in this case consist in annihilating the orthogonal projection of the unknown vector $\mathbf{q}$ onto the space spanned by the eigenvectors of $\mathcal{B}$ corresponding to negative eigenvalues.*

4 The one-dimensional case

For an incompletely parabolic system in one space dimension, we show that semi-admissible boundary conditions for both, the hyperbolic and the parabolic subsystems, are also semi-admissible for the global system. We assume that this latter is strongly parabolic and symmetrizable; in its symmetrized form and after the change of variables $\mathbf{w} \to e^{\alpha t}\mathbf{w}$, it reads:

$$S\partial_t \mathbf{w} + A\partial_x \mathbf{w} - \partial_x(B\partial_x \mathbf{w}) + (C + \alpha S)\mathbf{w} = F,\ \ (t, x) \in [0, T) \times (0, 1). \tag{9}$$

Like before, the partitioning of the matrix B induces a partitioning of the matrices and vectors. We set:

$$\begin{aligned} B &= \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix},\ S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^t & S_{22} \end{pmatrix},\ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^t & A_{22} \end{pmatrix}, \\ \mathbf{w} &= \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix},\ \mathbf{q} = \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \partial_x \mathbf{w}_1 \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{pmatrix}. \end{aligned}$$

The following relationships

$$-B_{11}\partial_x \mathbf{w}_1 + B_{11}\mathbf{q}_3 = \mathbf{0},\ \partial_x(B\partial_x \mathbf{w}) = \partial_x B\partial_x \mathbf{w} + B\partial_x^2 \mathbf{w},$$

allow us to obtain the first order system equivalent to (9):

$$\mathcal{S}\partial_t \mathbf{q} + \mathcal{A}\partial_x \mathbf{q} + \mathcal{K}\mathbf{q} = \mathbf{F}, \tag{10}$$

with

$$\mathcal{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},\ \mathcal{A} = \begin{pmatrix} 0 & A_{12} & -B_{11} \\ A_{12}^t & A_{22} & 0 \\ -B_{11} & 0 & 0 \end{pmatrix},\ \mathcal{K} = \begin{pmatrix} C + \alpha S & A_{11} - \partial_x B_{11} \\ 0 & 0 \\ 0 & B_{11} \end{pmatrix}. \tag{11}$$

In order to prove that (10) is a Friedrichs system, it is sufficient to show that the matrix

$$\mathcal{C} = \mathcal{K} + \mathcal{K}^t - \partial_t \mathcal{S} - \partial_x \mathcal{A} \stackrel{def}{=} \begin{pmatrix} \mathcal{C}_1 + \alpha S & \mathcal{C}_2^t \\ \mathcal{C}_2 & B_{11} \end{pmatrix} \tag{12}$$

is positive definite for a sufficiently large α. Making use of the Crout block factorization of $\mathcal{C}$ we get:

$$\mathcal{C} = \mathcal{L} \begin{pmatrix} \mathcal{C}_1 + \alpha S & 0 \\ 0 & B_{11} - \mathcal{C}_2(\mathcal{C}_1 + \alpha S)^{-1}\mathcal{C}_2^t \end{pmatrix} \mathcal{L}^t, \tag{13}$$

which proves, since $(\mathcal{C}_1 + \alpha S)^{-1} = (I + O(\alpha^{-1}))\alpha^{-1}S^{-1}$, that $\mathcal{C}$ is positive definite for α large. The same argument shows that the subsystem

$$\begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix} \partial_t \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_3 \end{pmatrix} + \begin{pmatrix} 0 & -B_{11} \\ -B_{11} & 0 \end{pmatrix} \partial_x \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_3 \end{pmatrix}$$

$$+ \begin{pmatrix} C_{11} + \alpha S_{11} & 0 \\ 0 & B_{11} \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}, \tag{14}$$

which is the first order system equivalent to the parabolic subsystem, is symmetric positive; it is very easy to see that the hyperbolic subsystem

$$S_{22}\partial_t \mathbf{q}_2 + A_{22}\partial_x \mathbf{q}_2 + (C_{22} + \alpha S_{22})\mathbf{q}_2 = \mathbf{g} \tag{15}$$

is also a Friedrichs system.

With regard to the initial condition at $t = 0$, the boundary matrix $\mathcal{B}$ (see (7)) is negative semi-definite for each system (10), (14) and (15). In this case [3], the only matrix giving semi-admissible boundary conditions is $\mathcal{M} = \sqrt{\mathcal{B}^2}$, yielding $\mathbf{q}_1(t = 0, x) = \mathbf{0}$ for (14), $\mathbf{q}_2(t = 0, x) = \mathbf{0}$ for (15) and finally $\mathbf{w}(t = 0, x) = \mathbf{0}$ for (10). Hence, the semi-admissible initial conditions for the subsystems produce a semi-admissible initial condition for the global system.

For the boundary conditions, we consider the point $x = 1$, the argument for $x = 0$ being analoguous. We choose Dirichlet boundary conditions for the subsystem (14) given by the positive semi-definite matrix

$$M_1 = \begin{pmatrix} 0 & -B_{11} \\ B_{11} & 0 \end{pmatrix} \tag{16}$$

and for (15) any semi-admissible boundary condition given by a positive semi-definite matrix M_2. For the global system (14) we impose the boundary conditions with the help of the matrix

$$\mathcal{M} = \begin{pmatrix} 0 & A_{12} & -B_{11} \\ -A_{12}^t & M_2 & 0 \\ B_{11} & 0 & 0 \end{pmatrix} \tag{17}$$

which is positive semi-definite. The condition (8) means that $\mathbf{q} \in Ker(\mathcal{A} - \mathcal{M})$ at $x = 1$; an easy computation shows that this last property is equivalent to $\mathbf{q}_1 = \mathbf{0}$, $\mathbf{q}_2 \in Ker(A_{22} - M_2)$ at the points $x = 1$. Thus, we have proved the following result.

Lemma 4.1 *Dirichlet boundary conditions for Subsystem (14) and any semi-admissible boundary conditions for Subsystem (15) are semi-admissible boundary conditions for the Friedrichs system (10).*

It seems difficult to obtain the same kind of result for boundary conditions different from the Dirichlet ones for (14).

Remark 2 *The Dirichlet boundary conditions for (14) are* ***admissible****, i.e. they are semi-admissible and furthermore one has $Ker(\mathcal{B} - \mathcal{M}) + Ker(\mathcal{B} + \mathcal{M}) = \mathbf{R}^{2m}$ (the kernels span the whole space). If the solution of a Friedrichs system is sufficiently regular, admissible boundary conditions are satisfied in a classical way and uniqueness is obtained [3]. It would be nice to get admissible conditions for (10) in the case where the matrix M_2 produces admissible conditions for (15). Unfortunately, this is difficult with the matrix given by (17) because of*

$$
\begin{aligned}
Ker(\mathcal{A}-\mathcal{M}) &= \{\mathbf{q} \in \mathbf{R}^{n+m} \mid \mathbf{q}_1 = \mathbf{0},\ \mathbf{q}_2 \in Ker(A_{22}-M_2)\},\\
Ker(\mathcal{A}+\mathcal{M}) &= \{\mathbf{q} \in \mathbf{R}^{n+m} \mid \mathbf{q}_2 \in Ker(A_{22}+M_2) \cap Ker(A_{12})\}.
\end{aligned}
$$

Sufficient conditions to get admissiblility are $A_{12} = 0$ or $Ker(A_{22}+M_2) \subset Ker(A_{12})$. The first case corresponds to the case where the global system is made of two decoupled subsystems and is not interesting; the second condition is not satisfied in general. It would be worthwile to look for another matrix $\mathcal{M}$ to prove this result.

Remark 3 *The number of admissible scalar boundary conditions of type (8) is equal to the number of negative eigenvalues of the matrix $\pm\mathcal{A}$ [1] and we get by Lemma 2.1 that this number is equal to m + the number of negative eigenvalues of $\pm A_{22}$.*

5 The 2D case

We follow the same arguments as in the preceeding section. We have essentially to define matrices and vectors related to the first order system of type (6) equivalent to (5) with $\mathbf{x} = (x_1, x_2) \in \Omega \subset \mathbf{R}^2$. We set

$$
\mathbf{q} = \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \partial_{x_1}\mathbf{w}_1 \\ \partial_{x_2}\mathbf{w}_1 \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \\ \mathbf{q}_4 \end{pmatrix},
$$

we write the mixed derivative as derivatives of $\mathbf{q}_3$, i.e. $\partial_{x_1}\partial_{x_2}\mathbf{w}_1 = \partial_{x_2}\partial_{x_1}\mathbf{w}_1 = \partial_{x_2}\mathbf{q}_3$ and we use

$$
\begin{aligned}
-B_{11}^{11}\partial_{x_1}\mathbf{w}_1 + B_{11}^{11}\mathbf{q}_3 - (B_{11}^{12}+B_{11}^{21})^t\partial_{x_2}\mathbf{w}_1 + (B_{11}^{12}+B_{11}^{21})^t\mathbf{q}_4 &= \mathbf{0},\\
-B_{11}^{22}\partial_{x_1}\mathbf{w}_1 + B_{11}^{22}\mathbf{q}_4 &= \mathbf{0},
\end{aligned}
$$

to get (6) with the matrices in $\mathcal{L}(\mathbf{R}^{n+2m})$:

$$
\begin{aligned}
\mathcal{A}^0 &= \mathcal{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},\ \mathcal{A}^1 = \begin{pmatrix} 0 & A_{12}^1 & -B_{11}^{11} & 0 \\ (A_{12}^1)^t & A_{22}^1 & 0 & 0 \\ -B_{11}^{11} & 0 & & \\ 0 & 0 & & 0 \end{pmatrix},\\
\mathcal{A}^2 &= \begin{pmatrix} 0 & A_{12}^2 & -(B_{11}^{12}+B_{11}^{21}) & -B_{11}^{22} \\ (A_{12}^2)^t & A_{22}^2 & 0 & 0 \\ -(B_{11}^{12}+B_{11}^{21})^t & 0 & & \\ -B_{11}^{22} & 0 & & 0 \end{pmatrix},\\
\mathcal{K} &= \begin{pmatrix} C+\alpha S & & A_{11}^1 - \sum_i \partial_{x_i}B_{11}^{i1} & A_{11}^2 - \sum_i \partial_{x_i}B_{11}^{i2} \\ & & 0 & 0 \\ & & B_{11}^{11} & (B_{11}^{12}+B_{11}^{21})^t \\ 0 & & 0 & B_{11}^{22} \end{pmatrix}.
\end{aligned}
$$

If the subsystem is strongly parabolic, the lower diagonal $2m \times 2m$ block of the matrix $\mathcal{K}+\mathcal{K}^t - \partial_t\mathcal{S} - \sum_i \partial_{x_i}\mathcal{A}^i$ is positive definite and this is also true for the global matrix. Hence, we have a Friedrichs system and both subsystems are positive too. At the boundary $\partial\Omega$, the matrix for boundary conditions is given by $\mathcal{B} = \nu_1 A^1 + \nu_2 A^2$ and the Dirichlet conditions for the "parabolic" subsystem are specified with the help of $M_1 \in \mathcal{L}(\mathbf{R}^{3m})$:

$$
M_1 = \nu_1 \begin{pmatrix} 0 & -B_{11}^{11} & 0 \\ B_{11}^{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \nu_2 \begin{pmatrix} 0 & -(B_{11}^{12}+B_{11}^{21}) & -B_{11}^{22} \\ (B_{11}^{12}+B_{11}^{21})^t & 0 & 0 \\ B_{11}^{22} & 0 & 0 \end{pmatrix}.
$$

Let the matrix M_2 generate semi-admissible conditions for the hyperbolic subsystem with boundary matrix $B_2 = \nu_1 A_{22}^1 + \nu_2 A_{22}^2$; then the matrix

$$
\mathcal{M} = \begin{pmatrix} 0 & \sum_i \nu_i A^i_{12} & -\nu_1 B^{11}_{11} - \nu_2(B^{12}_{11} + B^{21}_{11}) & -\nu_2 B^{22}_{11} \\ (\sum_i \nu_i A^i_{12})^t & M_2 & 0 & 0 \\ \nu_1 B^{11}_{11} + \nu_2(B^{12}_{11} + B^{21}_{11})^t & 0 & & \\ \nu_2 B^{22}_{11} & 0 & & 0 \end{pmatrix},
$$

yields semi-admissible boundary conditions for the global system which are the same as those for both subsystems.

We can again raise the questions of finding admissible boundary conditions for the hyperbolic subsystem which yields admissible conditions for the global system or to impose boundary conditions for the parabolic subsystem which are not of Dirichlet type.

6 Examples

Our first example consists of the compressible Navier-Stokes equations system in two space dimensions for the unknown velocity $\mathbf{u} = (u_1, u_{2)}$, temperature T and density ρ for a perfect gas, which can be written as [6]:

$$
\partial_t \mathbf{w} + \sum_{i=1}^{2} A^i(\mathbf{w}) \partial_{x_i} \mathbf{w} - \sum_{i,j=1}^{2} B^{ij} \partial_{x_i} \partial_{x_j} \mathbf{w} + C\mathbf{w} = \mathbf{0}, \tag{18}
$$

where

$$
\mathbf{w} = \begin{pmatrix} \mathbf{u} \\ T \\ \rho \end{pmatrix}, \; A^1 = \begin{pmatrix} u_1 & 0 & R & RT \\ 0 & u_1 & 0 & 0 \\ (\gamma-1)T & 0 & u_1 & 0 \\ 1 & 0 & 0 & u_1 \end{pmatrix}, \; A^2 = \begin{pmatrix} u_2 & 0 & 0 & 0 \\ 0 & u_2 & R & RT \\ 0 & (\gamma-1)T & u_2 & 0 \\ 0 & 1 & 0 & u_2 \end{pmatrix},
$$

$$
B^{11} = diag[\frac{4}{3}, 1, \frac{\gamma}{\mathrm{Pr}}, 0], \; B^{22} = diag[1, \frac{4}{3}, \frac{\gamma}{\mathrm{Pr}}, 0], \; \mathrm{B}^{12} = B^{21} = \begin{pmatrix} 0 & \frac{1}{6} & & 0 \\ \frac{1}{6} & 0 & & \\ & 0 & & 0 \end{pmatrix};
$$

the physical constants are the gas constant R, the Prandtl number Pr and the ratio of the specific heats γ. We linearize (18) by replacing the unknown $\mathbf{w}$ in the matrices by a given field $\overline{\mathbf{w}}$. With the help of the symmetrizer $S = diag[1, 1, R/(\gamma-1)\overline{T}, R\overline{T}]$, the system (18) can be brought into the form (6), with two positive subsystems. The three Dirichlet conditions for the parabolic one are conditions for the velocity and temperature. For the hyperbolic subsystem

$$
R\overline{T}(\partial_t \rho + \overline{u}_1 \partial_{x_1} \rho + \overline{u}_2 \partial_{x_2} \rho) = 0,
$$

one needs to specify the boundary condition $\rho = 0$ only on the part of $\partial\Omega$ where $B_2 = \mathbf{u} \cdot \boldsymbol{\nu} < 0$ (the inflow part) since the only matrix yielding a semi-admissble condition is $\sqrt{B_2^2}$.

The second example is the hydrodynamical model for semiconductor devices [11]; we consider the case where the electrons are the only charge carriers and assume the device is made of one semiconductor material. In this case, the conservation laws for the electron gas unknowns T (temperature), n (concentration), $\mathbf{u}$ (velocity) can be written in two space dimensions as follows:

$$
\begin{aligned}
\partial_t T + \mathbf{u} \cdot \boldsymbol{\nabla} T + \frac{2}{3} T \; div(\mathbf{u}) - \frac{2q}{3k} \mathbf{u} \cdot \mathbf{E} - \frac{2}{3kn} div(\kappa \boldsymbol{\nabla} T) &= \frac{2}{3k\mu} \mid \mathbf{u} \mid^2 - \frac{1}{\tau_w} \left(\frac{m}{3k} \mid \mathbf{u} \mid^2 + T - T_0 \right), \\
\partial_t n + div(n\mathbf{u}) &= 0, \\
\partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + \frac{k}{mn} \boldsymbol{\nabla}(nT) + \frac{q}{m} \mathbf{E} &= -\frac{q}{m\mu} \mathbf{u};
\end{aligned} \tag{19}
$$

the physical quantities are the electric charge q, the Boltzmann constant k, the thermal conductivity κ, the mobility μ, the energy relaxation time τ_w and the effective mass m. For the sake of simplicity, we assume all this quantities to be constants and write (19) under the matrix form

$$\partial_t \mathbf{w} + A^1(\mathbf{w})\partial_{x1}\mathbf{w} + A^2(\mathbf{w})\partial_{x2}\mathbf{w} - B(\mathbf{w})\partial^2_{x_1}\mathbf{w} - B\partial^2_{x2}\mathbf{w} + C(\mathbf{w})\mathbf{w} = \mathbf{F},$$

with

$$\begin{aligned}
\mathbf{w} &= \begin{pmatrix} T \\ n \\ \mathbf{u} \end{pmatrix}, \ \mathbf{F} = \begin{pmatrix} \frac{T_0}{\tau_w} \\ 0 \\ -\frac{q}{m}\mathbf{E} \end{pmatrix}, \ B = diag[\frac{2\kappa}{3kn}, 0, 0, 0], \\
A^1 &= \begin{pmatrix} u_1 & 0 & \frac{2}{3}T & 0 \\ 0 & u_1 & n & 0 \\ \frac{k}{m} & \frac{k}{mn}T & u_1 & 0 \\ 0 & 0 & 0 & u_1 \end{pmatrix}, A^2 = \begin{pmatrix} u_2 & 0 & 0 & \frac{2}{3}T \\ 0 & u_2 & 0 & n \\ 0 & 0 & u_2 & 0 \\ \frac{k}{m} & \frac{k}{mn}T & 0 & u_2 \end{pmatrix}, \\
C &= \begin{pmatrix} \frac{1}{\tau_w} & 0 & \frac{1}{3k}(-2qE_1 + (\frac{m}{\tau_w} - \frac{2}{\mu})u_1) & \frac{1}{3k}(-2qE_2 + (\frac{m}{\tau_w} - \frac{2}{\mu})u_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{q}{m\mu} & 0 \\ 0 & 0 & 0 & \frac{q}{m\mu} \end{pmatrix}.
\end{aligned}$$

Linearizing again by replacing the argument of the matrices by a given field $\overline{\mathbf{w}}$ yields an incompletely parabolic system of type (4), which can be put in symmetric form by multiplying by the matrix $S = diag[\frac{3k\overline{n}}{2m\overline{c}\overline{T}}, \frac{\overline{c}}{\overline{n}}, \frac{\overline{n}}{\overline{c}}, \frac{\overline{n}}{\overline{c}}]$, with the local sound velocity defined by $c = kT/m$. The matrix $\overline{B}_2 = \nu_1 A^1_{22}(\overline{\mathbf{w}}) + \nu_2 A^2_{22}(\overline{\mathbf{w}})$ has eigenvalues equal to $\overline{\mathbf{u}}\cdot\boldsymbol{\nu}$, $\overline{\mathbf{u}}\cdot\boldsymbol{\nu}\pm\overline{c}$. In the case of a supersonic outflow ($\overline{\mathbf{u}}\cdot\boldsymbol{\nu} > c$), each eigenvalue is positive, while for a subsonic inflow ($-c < \overline{\mathbf{u}}\cdot\boldsymbol{\nu} < 0$) there are two negative eigenvalues and three for a supersonic inflow ($\overline{\mathbf{u}}\cdot\boldsymbol{\nu} < -c$). The same properties are true for the boundary matrix $S\overline{B}_2$ [B & C, 1992] of the symmetric hyperbolic subsystem; the matrix $M_2 = \sqrt{(S\overline{B}_2)^2}$ generates in each case respectively zero, two and three semi-admissible boundary conditions. With our particular choice of M_2, the boundary conditions for the global system are $T = 0$ for a supersonic outflow, $\mathbf{w} = \mathbf{0}$ for a supersonic inflow and $T = 0$, $n = \frac{\overline{n}}{\overline{c}}\mathbf{u}\cdot\boldsymbol{\nu}$, $u_1\nu_2 - u_2\nu_1 = 0$ for a subsonic inflow. This last condition ($\mathbf{u}\perp\boldsymbol{\nu}$) seems to be in contradiction with the hypothesis $-c < \overline{\mathbf{u}}\cdot\boldsymbol{\nu} < 0$ in the true nonlinear case for which we set $\mathbf{w} = \overline{\mathbf{w}}$. In fact, we have to consider inhomogeneous boundary conditions of the type $(B_2 - M_2)(\mathbf{w} - \mathbf{w}_0) = \mathbf{0}$; these conditions consist in specifying the temperature and both, the normal and tangential components of the velocity, that is the velocity itself.

Remark 4 *The full hydrodynamic model requires to compute the electric field $\mathbf{E} = -\boldsymbol{\nabla}\Phi$ by solving the Poisson equation for the electric potential, which is a second order elliptic equation. Introducing the derivatives of Φ as new unknowns, we can formulate the global linearized system for T, n, u and Φ as a Friedrichs system and generalize our results in the sense that semi-admissible boundary conditions for the global system are obtained from the same conditions as before and including any condition for the potential making the Poisson equation well-posed (Dirichlet, Dirichlet-Neumann,...).*

7 Appendix

To prove Lemma 2.1, we build the matrix $T_{\varepsilon\beta} = \begin{pmatrix} A + \varepsilon I & -\beta B^t \\ -\beta B & 0 \end{pmatrix}$, with real numbers ε and $\beta \geq 1$, and I the unit matrix. One can choose ε near zero, such that $A+\varepsilon I$ is regular. Hence, we can compute the block Crout decomposition $T_{\varepsilon\beta} = LDL^t$, which gives $D = \begin{pmatrix} A + \varepsilon I & 0 \\ 0 & -\beta^2 B(A + \varepsilon I)^{-1}B^t \end{pmatrix}$; we see that the diagonal matrix D is similar to $diag[\lambda_1, \lambda_2, ..., \lambda_n, -\beta\sigma_1, -\beta\sigma_2, ..., -\beta\sigma_m]$ with the λ_i's denoting the eigenvalues of $A+\varepsilon I$ and the σ_i's those of $B(A+\varepsilon I)^{-1}B^t$. Hence, the number of positive eigenvalues of D and consequently those of $T_{\varepsilon\beta}$ and $T_{0\beta}$ does not depend of β. Now, the eigenvalues of $\beta^{-1}T_{0\beta} = \begin{pmatrix} \beta^{-1}A & -B^t \\ -B & 0 \end{pmatrix}$ are analytic functions of β^{-1} when $\beta \to \infty$. This means that the eigenvalues of $T_{0\beta}$ of the form $\lambda(\beta) = \lambda_0\beta + O(1)$ are those of $\pm B_{11}$ and m of them are positive; the eigenvalues of the form $\sigma(\beta) = \sigma_1 + O(\beta^{-1})$ are those of A_{22} and we get the result.

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