Math. Nachr. (1998),

# The Bordered Operator and the Index of a Constrained Critical Point

By LEON GREENBERG of College Park, JOHN H. MADDOCKS of Lausanne, KATHLEEN A. ROGERS of Minneapolis

(Received May 1998)

**Abstract.**
*It is shown that the index of a constrained critical point in the isoperimetric calculus of variations is simply related to the number of negative eigenvalues of a certain bordered operator associated with the second variation. A conjugate point theory for this bordered operator is then established.*

## 1. Introduction

In the context of finite dimensional equality constrained optimization theory, it has recently been observed [7, 20, 21] that the index [8] of a constrained critical point when regarded as a function of the basic unknowns, is simply related to an index of the same critical point when regarded as an unconstrained solution to an extended variational principle in which the Lagrange multipliers are treated as additional independent variables. An alternative proof of this fact can be constructed from various theories of restricted quadratic forms in finite dimensional spaces applied to the Kuhn-Tucker bordered matrix. The bordered matrix arises as the appropriate Hessian, or second variation, matrix with respect to the extended variables (see for example [3, 4, 6, 9, 12]). In this article, we demonstrate that a completely analogous theory holds true for a 'bordered operator' that arises as the 'Hessian' in the infinite dimensional case of an isoperimetrically constrained one-dimensional calculus of variations problem. That is, we show that the index of a critical point (in the sense of Morse [15, 16], but suitably modified to the case of isoperimetric problems) is simply related to the number of negative eigenvalues of an appropriate bordered operator. Related

1991 *Mathematics Subject Classification.* Primary 47A60; Secondary 49R05, 58F10, 58F19, 93D22
*Keywords and phrases.* index, isoperimetric conjugate point, constrained variational principle

results have been applied to classify stability properties of equilibria within elastic rod models of deformed DNA molecules (cf. [14, 18, 19]).

Section 2. contains a summary of the finite dimensional theory. In Section 3. we formulate an isoperimetrically constrained one-dimensional calculus of variations problem, and state our main result, namely the relationship between the index of a constrained critical point and the number of negative eigenvalues of an appropriate bordered operator. The proof of the main result presented in Section 5. follows from a series of lemmas that are described in Section 4. In Section 6., we detail a method for determining the number of negative eigenvalues of the bordered operator via a conjugate point theory, which, by means of the main result, provides one way to compute the index of a constrained critical point. Section 7. contains a discussion of the connection between this article and other works.

## 2. The Finite Dimensional Case

Consider the objective function

$$\mathbf{f} : \mathrm{I\!R}^p \to \mathrm{I\!R}, \tag{2.1}$$

which we would like to minimize subject to $n < p$ equality constraints

$$\mathbf{g}(\mathbf{x}) = \mathbf{0}, \tag{2.2}$$

where $\mathbf{g}(\mathbf{x}) = \big(g^1(\mathbf{x}), g^2(\mathbf{x}), \ldots, g^n(\mathbf{x})\big)^T$. In order to consider conditions that are necessary for a particular solution $\mathbf{x}$ to be a minimum of (2.1) subject to (2.2), we construct the *Lagrangian* as a function of $\mathbf{x} \in \mathrm{I\!R}^p$ and $\boldsymbol{\lambda} \in \mathrm{I\!R}^n$ in a standard way:

$$\mathbf{L}(\mathbf{x}, \boldsymbol{\lambda}) \equiv \mathbf{f}(\mathbf{x}) + \boldsymbol{\lambda} \cdot \mathbf{g}(\mathbf{x}).$$

Then, under comparatively mild assumptions, for example linear independence of the gradient vectors $\nabla_{\mathbf{x}} g^i(\mathbf{x})$ (see for example [11, Chapter 10]), any solution $\mathbf{x}$ of the problem of minimizing (2.1) subject to the constraints (2.2) is part of a pair $(\mathbf{x}, \boldsymbol{\lambda})$ for some multiplier $\boldsymbol{\lambda}$ that provides a simultaneous solution of

$$\nabla_{\mathbf{x}} \mathbf{L} = \mathbf{0}, \quad \nabla_{\boldsymbol{\lambda}} \mathbf{L} = \mathbf{0}. \tag{2.3}$$

Here, $\nabla_{\mathbf{x}}$ denotes the $p$-vector of partial derivatives with respect to $\mathbf{x}$ and $\nabla_{\boldsymbol{\lambda}}$ denotes the $n$-vector of partial derivatives with respect to $\boldsymbol{\lambda}$. Of course, equating $\nabla_{\boldsymbol{\lambda}} L$ to zero merely recovers the constraints (2.2). We call pairs $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}})$ satisfying (2.3) *constrained critical points.*

It is of further interest to classify those critical points that are constrained local minima, that is, to classify those $\hat{\mathbf{x}}$ for which $\mathbf{f}(\hat{\mathbf{x}})$ is a local minimum amongst all nearby vectors $\mathbf{x}$ that satisfy the constraints (2.2). A necessary condition that a constrained critical point $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}})$ corresponds to a local minimum is that the Hessian $\mathbf{S}$ of $\mathbf{L}$ with respect to $\mathbf{x}$ evaluated at $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}})$, namely the $p \times p$ symmetric matrix

$\nabla^2_{\mathbf{x}}\mathbf{L}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}})$, generates a positive semi-definite quadratic form on the tangent space $T\mathcal{C}$ to the constraint manifold at $\hat{\mathbf{x}}$ (cf. [11, 21]):

$$\mathbf{h}\cdot\mathbf{S}\mathbf{h}\geq 0,\quad\forall\,\mathbf{h}\in T\mathcal{C}. \tag{2.4}$$

For our purposes it is convenient to rewrite condition (2.4) in terms of an eigenvalue problem. To this end, we denote the inclusion of $T\mathcal{C}$ in $\mathbb{R}^p$ by $\mathbf{G}:T\mathcal{C}\to\mathbb{R}^p$, i.e. $\mathbf{G}$ is the linear mapping which identifies an element of $T\mathcal{C}$ as an element of $\mathbb{R}^p$. It can be shown that the adjoint $\mathbf{G}^*:\mathbb{R}^p\to T\mathcal{C}$ of $\mathbf{G}$ is the orthogonal projection onto $T\mathcal{C}$. We remark that an explicit form of $\mathbf{G}$ is not required in our analysis. Then the necessary condition (2.4) can be written

$$\mathbf{h}\cdot\mathbf{G}^*\mathbf{S}\mathbf{G}\mathbf{h}\geq 0,\quad\forall\,\mathbf{h}\in T\mathcal{C}. \tag{2.5}$$

Now $\mathbf{G}^*\mathbf{S}\mathbf{G}:T\mathcal{C}\to T\mathcal{C}$ is a self-adjoint linear operator, so that it has only real eigenvalues which can be characterized by a Rayleigh quotient. If we denote the number of negative eigenvalues of $\mathbf{G}^*\mathbf{S}\mathbf{G}$ by $\nu(\mathbf{G}^*\mathbf{S}\mathbf{G})$, then the necessary condition (2.5) is equivalent to the condition $\nu(\mathbf{G}^*\mathbf{S}\mathbf{G})=0$. If we further define the *constrained index* of $(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}})$ to be the unique maximal dimension of a subspace of $T\mathcal{C}$ on which the quadratic form induced by $\mathbf{S}$ can be made negative, then it can also be shown that the constrained index is identical to $\nu(\mathbf{G}^*\mathbf{S}\mathbf{G})$, (see for example [12] and references therein).

A sizable literature exists concerning tests for determining the index of a constrained critical point. Here, we focus on an indirect test using the Kuhn-Tucker bordered matrix $\mathbf{M}$. If $\mathbf{B}$ denotes the transpose of the matrix that is the gradient of the constraints evaluated at $\hat{\mathbf{x}}$, then the bordered matrix $\mathbf{M}\in\mathbb{R}^{(p+n)\times(p+n)}$ is

$$\mathbf{M}\equiv\begin{pmatrix}\mathbf{S} & \mathbf{B}\\ \mathbf{B}^T & \mathbf{0}\end{pmatrix}.$$

The bordered matrix $\mathbf{M}$ acts on the extended variables $(\mathbf{h},\boldsymbol{\beta})^T\in\mathbb{R}^{p+n}$ corresponding to variations in $\mathbf{x}$ and $\boldsymbol{\lambda}$ respectively, and can be regarded as the Hessian $\nabla^2{}_{(\mathbf{x},\boldsymbol{\lambda})}L(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}})$ of the Lagrangian $\mathbf{L}$ with respect to the extended variables evaluated at the critical point $(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}})$. Assuming that the linearized constraints $\nabla_{\mathbf{x}}g^i(\hat{\mathbf{x}})$ are linearly independent, the relationship between the index of a constrained critical point $\nu(\mathbf{G}^*\mathbf{S}\mathbf{G})$ and the number $\nu(\mathbf{M})$ of negative eigenvalues of $\mathbf{M}$ is given by

$$\nu(\mathbf{M})=\nu(\mathbf{G}^*\mathbf{S}\mathbf{G})+n, \tag{2.6}$$

see for example [3, 4, 6, 9, 12, 13]. Thus, the index of a constrained critical point is simply related to the index of an unconstrained critical point of the extended variational principle involving $\mathbf{L}(\mathbf{x},\boldsymbol{\lambda})$ in which the Lagrange multipliers are treated as additional variables. This is also the conclusion reached in [7, 20, 21].

We now proceed to derive analogous relations for certain infinite dimensional calculus of variations problems. It should be noted that in the finite dimensional case it is quite reasonable to develop a slightly different theory. For example instead of the inclusion map $\mathbf{G}:T\mathcal{C}\to\mathbb{R}^p$ one could easily work with orthogonal projection $\mathbf{P}$ onto $T\mathcal{C}$ with $\mathbf{P}:R^p\to R^p$. However in the infinite dimensional context the situation can be more delicate, and the framework that we shall extend to infinite dimensions is precisely the version of the finite dimensional theory with the inclusion map as described above.

## 3. An Analogous Calculus of Variations Problem

Consider the one-dimensional calculus of variations problem of minimizing the functional

$$\int_0^1 W(\boldsymbol{\zeta}, \boldsymbol{\zeta}', s)\, ds\ , \tag{3.1}$$

subject to the isoperimetric constraints

$$\int_0^1 g^i(\boldsymbol{\zeta}, s)\, ds = 0, \quad i = 1 \dots n, \tag{3.2}$$

and the boundary conditions

$$\boldsymbol{\zeta}(0) = \mathbf{f}_0, \quad \boldsymbol{\zeta}(1) = \mathbf{f}_1. \tag{3.3}$$

Here the basic unknown is a function $\boldsymbol{\zeta}(s)$ taking values in $\mathbb{R}^m$. Various generalizations of (3.1), (3.2), and (3.3) are also of interest, but this particular case illustrates our point and encompasses the application that motivated this work [19]. As in the finite dimensional case, we write the associated Lagrangian functional

$$J(\boldsymbol{\zeta}, \boldsymbol{\lambda}) \equiv \int_0^1 \mathbf{F}(\boldsymbol{\zeta}, \boldsymbol{\zeta}', \boldsymbol{\lambda}, s)\, ds = \int_0^1 \left[ W(\boldsymbol{\zeta}, \boldsymbol{\zeta}', s) + \boldsymbol{\lambda} \cdot \mathbf{g}(\boldsymbol{\zeta}, s) \right]\, ds,$$

where $\boldsymbol{\lambda} \in \mathbb{R}^n$ is the Lagrange multiplier. Although issues of existence and smoothness of the minimizers arise, they are not of primary interest here. Instead, we assume that a smooth minimizer exists, and that the $m \times m$ symmetric matrix $W_{\boldsymbol{\zeta}'\boldsymbol{\zeta}'}(\boldsymbol{\zeta}, \boldsymbol{\zeta}', s)$ is positive definite on the interval $[0, 1]$ (i.e. Legendre's strengthened condition holds). At such minimizers the first-order necessary conditions (analogous to the finite dimensional case) are that $(\boldsymbol{\zeta}(s), \boldsymbol{\lambda})$ is a simultaneous solution of the parameter-dependent system of $m$ Euler-Lagrange equations

$$-\frac{d}{ds}\mathbf{F}_{\boldsymbol{\zeta}'}(\boldsymbol{\zeta}, \boldsymbol{\zeta}', \boldsymbol{\lambda}, s) + \mathbf{F}_{\boldsymbol{\zeta}}(\boldsymbol{\zeta}, \boldsymbol{\zeta}', \boldsymbol{\lambda}, s) = \mathbf{0}, \tag{3.4}$$

the constraints (3.2), and the boundary conditions (3.3). A solution pair $(\hat{\boldsymbol{\zeta}}(s), \hat{\boldsymbol{\lambda}})$ which satisfies (3.2), (3.3), and (3.4) will be called a *constrained critical point.*

As in the finite dimensional case, we would like to classify those critical points that are actually constrained local minima. To this end, we consider the second variation of $J$, $\delta^2 J[\mathbf{h}] = \langle \mathbf{h}, \mathcal{S}\mathbf{h} \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard $L^2(0,1)^m$ inner product, and the second-order self-adjoint linear vector differential operator $\mathcal{S} : \mathcal{D}(\mathcal{S}) \to L^2(0,1)^m$ is defined by

$$\mathcal{S}\mathbf{h} \equiv -\frac{d}{ds}\left[\mathbf{P}\mathbf{h}' + \mathbf{C}^T\mathbf{h}\right] + \mathbf{C}\mathbf{h}' + \mathbf{Q}\mathbf{h}, \tag{3.5}$$

with $\mathbf{P} = W_{\boldsymbol{\zeta}'\boldsymbol{\zeta}'} > 0$, $\mathbf{Q} = \sum_{i=1}^n \lambda_i g^i{}_{\boldsymbol{\zeta}\boldsymbol{\zeta}} + W_{\boldsymbol{\zeta}\boldsymbol{\zeta}}$, and $\mathbf{C} = W_{\boldsymbol{\zeta}\boldsymbol{\zeta}'}$ being $m \times m$ matrices evaluated at the constrained critical point. The natural domain of the densely defined operator $\mathcal{S}$ is

$$\mathcal{D}(\mathcal{S}) \equiv \left\{ \mathbf{h} \in \mathcal{H}^2(0,1)^m : \mathbf{h}(0) = \mathbf{0} = \mathbf{h}(1) \right\}. \tag{3.6}$$

The equation $\mathcal{S}\mathbf{h} = \mathbf{0}$ is referred to as Jacobi's accessory equation in the classic calculus of variations literature. As $\mathbf{P} > 0$ by assumption, it is standard that $\mathcal{S}$ is a regular, self-adjoint operator which is bounded below [22]. Moreover, $\mathcal{S}$ has discrete spectrum consisting of a sequence of eigenvalues $\xi_1 \le \xi_2 \le \xi_3 \le \ldots \le \xi_n \le \ldots$ with

$$\lim_{n\to\infty} \xi_n = \infty.$$

As in the finite dimensional case, a necessary condition for a constrained critical point $(\hat{\boldsymbol{\zeta}}, \hat{\boldsymbol{\lambda}})$ to actually be a local minimum is that

$$\langle \mathbf{h}, \mathcal{S}\mathbf{h}\rangle \ge 0 \tag{3.7}$$

for all $\mathbf{h}$ in tangent space to the constraint set. In other words, the admissible set of variations is

$$\mathcal{D}^{cons} \equiv \mathcal{D}(\mathcal{S}) \cap \mathcal{T}^{\perp} = \left\{ \mathbf{h} \in \mathcal{D}(\mathcal{S}) : \int_0^1 \mathbf{h}\cdot\mathbf{T}_i \, ds = 0, \;\; i = 1, \ldots, n \right\}, \tag{3.8}$$

where

$$\mathbf{T}_i(s) \equiv \nabla_{\boldsymbol{\zeta}}\, g^i(\hat{\boldsymbol{\zeta}}, s)$$

and

$$\mathcal{T} \equiv \operatorname{span}(\mathbf{T}_1, \ldots, \mathbf{T}_n). \tag{3.9}$$

We assume the $\mathbf{T}_i(s)$ to be functionally independent on every subinterval $(a, b) \subset [0, 1]$.

In order to relate the necessary condition (3.7) to an eigenvalue problem, we define the inclusion map

$$\mathcal{G} : \overline{\mathcal{D}^{cons}} \hookrightarrow \overline{\mathcal{D}(\mathcal{S})} \equiv L^2(0,1)^m$$

from the closure of the tangent space of the constraint manifold to the closure of the domain of $\mathcal{S}$. Then the necessary condition for a minimum (3.7) can be written as $\delta^2 J[\mathbf{h}] = \langle \mathbf{h}, \mathcal{G}^*\mathcal{S}\mathcal{G}\mathbf{h}\rangle \ge 0$, for all $\mathbf{h} \in \mathcal{D}^{cons}$, where $\mathcal{G}^*$ denotes the adjoint of $\mathcal{G}$. Now $\mathcal{G}^*\mathcal{S}\mathcal{G} : \mathcal{D}^{cons} \to \mathcal{T}^{\perp}$ is self-adjoint, bounded below, and has discrete spectrum (see Lemmas 4.7 and 4.9 below). It can be shown that the eigenvalues of $\mathcal{G}^*\mathcal{S}\mathcal{G}$ can be written in terms of a Rayleigh quotient, thus allowing the non-negativity condition on the second variation to be characterized in terms of the number $\sigma^-(\mathcal{G}^*\mathcal{S}\mathcal{G})$ of negative eigenvalues of $\mathcal{G}^*\mathcal{S}\mathcal{G}$ counted according to multiplicity.* The isoperimetric index is then defined to be the unique maximal dimension of the subspaces of $\mathcal{D}^{cons}$ on which the quadratic form induced by $\mathcal{S}$ can be made negative. It can also be shown that this index equals $\sigma^-(\mathcal{G}^*\mathcal{S}\mathcal{G})$ (see for example [14]).

The main objective of this article is to show that, as in the finite dimensional case, there is an alternate formulation of the necessary test for a minimum in terms of the spectrum of a 'bordered operator' $\mathcal{M} : \mathcal{D}(\mathcal{M}) \to L^2(0,1)^m \times \mathbb{R}^n$, acting on the

*In finite dimensions, the number of negative eigenvalues of a matrix is denoted $\nu(\cdot)$, here and elsewhere. But in calculus of variations problems $\nu(t)$ is typically reserved for another quantity (see Section 6). Accordingly, we switch notation here.

extended variable $\mathbf{z} = (\boldsymbol{\zeta}, \boldsymbol{\lambda})^T \in \mathcal{D}(\mathcal{M}) \equiv \mathcal{D}(\mathcal{S}) \times \mathbb{R}^n$, defined by

$$\mathcal{M} = \begin{pmatrix} \mathcal{S} & \mathbf{T}_1 \; \dots \; \mathbf{T}_n \\ & \\ \mathcal{T}(\cdot) & \mathbf{0} \end{pmatrix},$$

where $\mathcal{T}_i(\boldsymbol{\zeta}) = \int_0^1 \mathbf{T}_i \cdot \boldsymbol{\zeta} \, ds$ and

$$\mathcal{T}(\boldsymbol{\zeta}) = \begin{pmatrix} \mathcal{T}_1(\boldsymbol{\zeta}) \\ \mathcal{T}_2(\boldsymbol{\zeta}) \\ \vdots \\ \mathcal{T}_n(\boldsymbol{\zeta}) \end{pmatrix}.$$

The inner product on the space $L^2(0,1)^m \times \mathbb{R}^n$ is defined in the natural way by $\langle \mathbf{y}, \mathbf{z} \rangle = \int_0^1 \sum_{j=1}^m \eta_i \cdot \zeta_i \, ds + \sum_{i=1}^n \lambda_i \rho_i$, where $\mathbf{y} = (\boldsymbol{\eta}, \boldsymbol{\rho})^T$ and $\mathbf{z} = (\boldsymbol{\zeta}, \boldsymbol{\lambda})^T$. We will show that $\mathcal{M}$ is bounded below with only discrete spectrum, so that $\sigma^-(\mathcal{M})$ is well defined. Our main result is then

**Theorem 3.1.** *If the linear constraints $\mathbf{T}_i$ are linearly independent functions on $[0,1]$, then*

(3.10) $$\sigma^-(\mathcal{M}) = \sigma^-(\mathcal{G}^* \mathcal{S} \mathcal{G}) + n.$$

The necessary condition for a minimum $\langle \mathcal{S}\boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle \geq 0, \quad \forall \boldsymbol{\zeta} \in \mathcal{D}^{cons}$ is equivalent to $\sigma^-(\mathcal{G}^* \mathcal{S} \mathcal{G}) = 0$. A test of this condition is provided as a corollary to Theorem 3.1:

**Corollary 3.2.** $\langle \mathcal{S}\mathbf{h}, \mathbf{h} \rangle \geq 0, \;\; \forall \mathbf{h} \in \mathcal{D}^{cons}(\mathcal{S}) \Longleftrightarrow \sigma^-(\mathcal{M}) = n.$

## 4. Preliminary Lemmas

The proof of Theorem 3.1 is based on the sequence of lemmas that we present in this section. The thrust of Lemmas 4.1-4.9 is that the spectrum of both $\mathcal{M}$ and $\mathcal{G}^* \mathcal{S} \mathcal{G}$ consists only of a sequence of eigenvalues of finite multiplicity, which tend to $\infty$ and which can be characterized by a Rayleigh quotient. Thus, just as in the finite dimensional case, the numbers of negative eigenvalues of both operators is finite.

**Lemma 4.1.** *$\mathcal{M}$ is self-adjoint.*

Proof. The operator $\mathcal{M}$ can be constructed as a bounded, symmetric perturbation of the operator $\mathcal{S} \oplus \mathbf{0}$:

$\mathcal{M} - (\mathcal{S} \oplus \mathbf{0}) =$

$$\begin{pmatrix} \mathcal{S} & \mathbf{T}_1 \ \ldots \ \mathbf{T}_n \\ & \\ \mathcal{T}(\cdot) & \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathcal{S} & \mathbf{0} \\ & \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{T}_1 \ \ldots \ \mathbf{T}_n \\ & \\ \mathcal{T}(\cdot) & \mathbf{0} \end{pmatrix}.$$

Therefore, $\mathcal{M}$ is self-adjoint, [10, Thm. 4.3, p. 287]. □

**Lemma 4.2.** *$\mathcal{M}$ is bounded below.*

Proof. Consider the Rayleigh quotient of $\mathcal{M}$:

$$\frac{\langle \mathcal{M}\mathbf{z}, \mathbf{z}\rangle}{||\mathbf{z}||^2} = \frac{\int_0^1 \mathcal{S}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}\; ds + 2\sum_{i=1}^n \lambda_i \mathcal{T}_i(\boldsymbol{\zeta})}{||\boldsymbol{\zeta}||^2 + ||\boldsymbol{\lambda}||^2}.$$

Since $\mathcal{S}$ is bounded below, we have that for some constant $c$ (possibly negative)

$$\int_0^1 \mathcal{S}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}\; ds \geq c\; ||\boldsymbol{\zeta}||^2.$$

Also, $|\lambda_i \mathcal{T}_i(\boldsymbol{\zeta})| \leq ||\boldsymbol{\zeta}||_{L^2(0,1)^m}\; ||\mathbf{T}_i||_{L^2(0,1)^n}\; |\lambda_i|$, thus

$$\begin{aligned} \left|\sum_{i=1}^n \lambda_i \mathcal{T}_i(\boldsymbol{\zeta})\right| &\leq \sum_{i=1}^n ||\boldsymbol{\zeta}||_{L^2(0,1)^m}\; ||\mathbf{T}_i||_{L^2(0,1)^n}\; |\lambda_i| = ||\boldsymbol{\zeta}||_{L^2(0,1)^m} \sum_{i=1}^n ||\mathbf{T}_i||_{L^2(0,1)^n}\; |\lambda_i| \\ &\leq ||\boldsymbol{\zeta}||_{L^2(0,1)^m} \left(\sum_{i=1}^n ||\mathbf{T}_i||^2_{L^2(0,1)^n}\right)^{\frac{1}{2}} ||\boldsymbol{\lambda}||_{\mathbb{R}^n} = \gamma\; ||\boldsymbol{\zeta}||_{L^2(0,1)^m}\; ||\boldsymbol{\lambda}||_{\mathbb{R}^n}, \end{aligned}$$

where $\gamma = \left(\sum_{i=1}^n ||\mathbf{T}_i||^2_{L^2(0,1)^n}\right)^{\frac{1}{2}}$ is a constant independent of $\boldsymbol{\zeta}$ and $\boldsymbol{\lambda}$. Therefore, $2|\sum_{i=1}^n \lambda_i \mathcal{T}_i(\boldsymbol{\zeta})| \leq 2\gamma ||\boldsymbol{\zeta}||_{L^2(0,1)^m}\; ||\boldsymbol{\lambda}||_{\mathbb{R}^n} \leq \gamma\; (||\boldsymbol{\zeta}||^2_{L^2(0,1)^m} + ||\boldsymbol{\lambda}||^2_{\mathbb{R}^n})$, and

$$\begin{aligned} \frac{\int_0^1 \mathcal{S}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}\; ds + 2\sum_{i=1}^n \lambda_i \mathcal{T}_i(\boldsymbol{\zeta})}{||\boldsymbol{\zeta}||^2 + ||\boldsymbol{\lambda}||^2} &\geq \frac{(c-\gamma)||\boldsymbol{\zeta}||^2 - \gamma||\boldsymbol{\lambda}||^2}{||\boldsymbol{\zeta}||^2 + ||\boldsymbol{\lambda}||^2} \\ &\geq k, \end{aligned}$$

where $k = \min(c-\gamma, -\gamma)$. Thus, $\langle \mathcal{M}\mathbf{z}, \mathbf{z}\rangle \geq k\langle \mathbf{z}, \mathbf{z}\rangle$, and $\mathcal{M}$ is bounded below. □

**Lemma 4.3.** *$\mathcal{M}$ has no essential spectrum.*

Proof. In the proof of Lemma 4.1, we showed that $\mathcal{M}$ was a bounded symmetric perturbation of $\mathcal{S}\oplus\mathbf{0}$. Note that it is also a finite rank, and thus compact, perturbation of $\mathcal{S}\oplus\mathbf{0}$. Trivially, the spectrum of the bordered operator $\mathcal{S}\oplus\mathbf{0} : \mathcal{D}(\mathcal{S}) \times \mathbb{R}^n \to$

$L^2(0,1)^m \times \mathbb{R}^n$ defined by

$$\begin{pmatrix} \mathcal{S} & \mathbf{0} \\ & \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathcal{S}\boldsymbol{\zeta} \\ \mathbf{0} \end{pmatrix}$$

consists of the eigenvalues of $\mathcal{S}$ with $n$ additional zero eigenvalues. Since $\mathcal{S} \oplus \mathbf{0}$ has no essential spectrum, neither does $\mathcal{M}$ [10, Theorem 5.35 p. 244]. □

For self-adjoint operators $\mathcal{L}$, bounded below, the eigenvalues below the essential spectrum can be characterized as follows:

$$\xi_n(\mathcal{L}) = \min_{\mathcal{B} \in \mathcal{B}_n} \max_{\substack{\mathbf{z} \in \mathcal{B} \\ \mathbf{z} \neq \mathbf{0}}} \frac{\langle \mathcal{L}\mathbf{z}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle}, \tag{4.1}$$

where $\mathcal{B}_n$ is the set of all $n$-dimensional subspaces of $\mathcal{D}(\mathcal{L})$.

**Theorem 4.4.** *[1, see Theorem 2.2, p. 527]*

- *The $\xi_n(\mathcal{L})$ form a monotonically increasing sequence.*
- *$\xi_*(\mathcal{L}) = \lim_{n\to\infty} \xi_n(\mathcal{L})$ is the infimum of the essential spectrum $\sigma_e(\mathcal{L})$.*
- *If $\xi_n(\mathcal{L}) < \xi_*(\mathcal{L})$, then $\mathcal{L}$ has at least $n$ eigenvalues below $\xi_*(\mathcal{L})$, and $\xi_n(\mathcal{L})$ is the $n^{th}$ eigenvalue.*

Since $\mathcal{M}$ is self-adjoint, bounded below, and $\sigma_e(\mathcal{M}) = \emptyset$, Theorem 4.4 implies that $\mathcal{M}$ has discrete spectrum consisting of a sequence of eigenvalues $\xi_1(\mathcal{M}) \leq \xi_2(\mathcal{M}) \leq \cdots \leq \xi_n(\mathcal{M}) \leq \cdots$ with $\lim_{n\to\infty} \xi_n(\mathcal{M}) = \infty$.

**Lemma 4.5.** *Let $\xi_n$ be the $n^{th}$ eigenvalue of $\mathcal{M}$. If there is a subspace $\mathcal{V} \subset \mathcal{D}(\mathcal{M})$, with $\dim \mathcal{V} = n$, and*

$$\xi_n = \max_{\substack{v \in \mathcal{V} \\ \mathbf{v} \neq \mathbf{0}}} \frac{\langle \mathcal{M}\mathbf{v}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}, \tag{4.2}$$

*then $\mathcal{V}$ contains an eigenvector for $\xi_n$.*

Proof. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\}$ be an orthonormal basis of eigenvectors of $\mathcal{M}$ where $\mathbf{e}_j$ is an eigenvector for $\xi_j$. Using completeness of the eigenvectors [10, p. 360], every $\mathbf{v} \in \mathcal{V}$ can be expressed $\mathbf{v} = \sum_{i=1}^{\infty} x_i \mathbf{e}_i$, such that $\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^{\infty} x_i^2$, [10]. Let

$$\mathcal{W} = \left\{ \mathbf{v} = \sum_{i=1}^{\infty} x_i \mathbf{e}_i \in \mathcal{V} : x_1 = x_2 = \ldots = x_{n-1} = 0 \right\}.$$

Since $\dim \mathcal{V} = n$ and $\mathcal{W}$ is a subspace defined by $n-1$ linear homogeneous equations, $\dim \mathcal{W} \geq 1$. Let $\mathbf{0} \neq \mathbf{w} \in \mathcal{W}$, $\mathbf{w} = x_n \mathbf{e}_n + x_{n+1} \mathbf{e}_{n+1} + \ldots$. Then $\langle \mathcal{M}\mathbf{w}, \mathbf{w} \rangle =$

$\xi_n x_n^2 + \xi_{n+1} x_{n+1}^2 + \ldots$. Suppose that $\xi_n = \xi_{n+1} = \ldots = \xi_{n+k} < \xi_{n+k+1}$. If there exists $j \geq n+k+1$ such that $x_j \neq 0$, then $\langle \mathcal{M}\mathbf{w}, \mathbf{w} \rangle > \xi_n(x_n^2 + x_{n+1}^2 + \ldots) = \xi_n \langle \mathbf{w}, \mathbf{w} \rangle$. This however contradicts (4.2). Therefore, $x_j = 0$ for $j \geq n+k+1$, and $\mathcal{M}\mathbf{w} = \xi_n \mathbf{w}$. $\square$

**Lemma 4.6.** $\langle \mathbf{z}, \mathcal{M}\mathbf{z} \rangle = \langle \boldsymbol{\zeta}, \mathcal{S}\boldsymbol{\zeta} \rangle = \langle \boldsymbol{\zeta}, \mathcal{G}^*\mathcal{S}\mathcal{G}\boldsymbol{\zeta} \rangle$ *whenever* $\mathbf{z} = (\boldsymbol{\zeta}, \boldsymbol{\lambda})^T$, *with* $\boldsymbol{\zeta} \in \mathcal{D}^{cons}$ *and* $\boldsymbol{\lambda} \in \mathbb{R}^n$.

Proof. Consider

$$\langle \mathbf{z}, \mathcal{M}\mathbf{z} \rangle \quad = \quad \langle \boldsymbol{\zeta}, \mathcal{S}\boldsymbol{\zeta} \rangle + \sum_{i=1}^{n} 2\lambda_i \int_0^1 \boldsymbol{\zeta} \cdot \mathbf{T}_i ds.$$

Since for $\boldsymbol{\zeta} \in \mathcal{D}^{cons}$, $\int_0^1 \boldsymbol{\zeta} \cdot \mathbf{T}_i \; ds = 0$, so $\langle \mathbf{z}, \mathcal{M}\mathbf{z} \rangle = \langle \boldsymbol{\zeta}, \mathcal{S}\boldsymbol{\zeta} \rangle$. Also since $\mathcal{G}\boldsymbol{\zeta} = \boldsymbol{\zeta}$ for $\boldsymbol{\zeta} \in \mathcal{D}^{cons}$, $\langle \boldsymbol{\zeta}, \mathcal{G}^*\mathcal{S}\mathcal{G}\boldsymbol{\zeta} \rangle = \langle \mathcal{G}\boldsymbol{\zeta}, \mathcal{S}\mathcal{G}\boldsymbol{\zeta} \rangle = \langle \boldsymbol{\zeta}, \mathcal{S}\boldsymbol{\zeta} \rangle$. $\square$

The next two lemmas ensure that $\mathcal{G}^*\mathcal{S}\mathcal{G}$ is self-adjoint and has no essential spectrum. Lemma 4.7 is contained in [23, Thm 2 & 3, p185 & p187], but we include a self-contained proof here.

**Lemma 4.7.** *The domain* $\mathcal{D}(\mathcal{G}^*\mathcal{S}\mathcal{G}) \equiv \mathcal{D}^{cons}$ *is dense in* $\mathcal{T}^\perp$ *and* $\mathcal{G}^*\mathcal{S}\mathcal{G}$ *is self-adjoint.*

Proof. For notational convenience we will use the following notation: $\mathcal{H} = L^2(0,1)^m$, $\mathcal{K} = \mathcal{T}^\perp$, $\mathcal{K}_\mathcal{S} = \mathcal{D}^{cons} = \mathcal{D}(\mathcal{S}) \cap \mathcal{K}$.

We show that there is a finite-dimensional subspace $\mathcal{U} \subset \mathcal{D}(\mathcal{S})$ such that $\mathcal{H} = \mathcal{K} \oplus \mathcal{U}$ and $\mathcal{D}(\mathcal{S}) = \mathcal{K}_\mathcal{S} \oplus \mathcal{U}$. Consider $\mathcal{H}/\mathcal{K}$ with its quotient norm ($||\mathbf{h} + \mathcal{K}|| = dist(\mathbf{h}, \mathcal{K})$) [17, Ch. III p.78-79]. The quotient map $\phi : \mathcal{H} \to \mathcal{H}/\mathcal{K}$ is continuous. Since $\mathcal{D}(\mathcal{S})$ is dense in $\mathcal{H}$, $\phi(\mathcal{D}(\mathcal{S}))$ is dense in $\mathcal{H}/\mathcal{K}$. Since $\dim(\mathcal{H}/\mathcal{K}) < \infty$, $\phi(\mathcal{D}(\mathcal{S})) = \mathcal{H}/\mathcal{K}$. By pulling back a basis of $\mathcal{H}/\mathcal{K}$ to $\mathcal{D}(\mathcal{S})$, we obtain a subspace $\mathcal{U} \subset \mathcal{D}(\mathcal{S})$ such that $\mathcal{H} = \mathcal{K} \oplus \mathcal{U}$ and $\mathcal{D}(\mathcal{S}) = \mathcal{K}_\mathcal{S} \oplus \mathcal{U}$.

We show that $\mathcal{K}_\mathcal{S}$ is dense in $\mathcal{K}$. Let $k \in \mathcal{K}$. Since $\mathcal{D}(\mathcal{S})$ is dense in $\mathcal{H}$, there is a sequence $\mathbf{x}_n = \mathbf{k}_n + \mathbf{u}_n \in \mathcal{D}(\mathcal{S})$ (where $\mathbf{k}_n \in \mathcal{K}_\mathcal{S}$ and $\mathbf{u}_n \in \mathcal{U}$) such that $\mathbf{k}_n + \mathbf{u}_n \to \mathbf{k}$ as $n \to \infty$. The restricted map $\phi|_\mathcal{U} : \mathcal{U} \to \mathcal{H}/\mathcal{K}$ is a bicontinuous bijection, and $\phi(\mathbf{u}_n) = \phi(\mathbf{k}_n + \mathbf{u}_n) \to \phi(\mathbf{k}) = \mathbf{0}$. Therefore $\mathbf{u}_n \to \mathbf{0}$. $||\mathbf{k}_n - \mathbf{k}|| \leq ||\mathbf{k}_n + \mathbf{u}_n - \mathbf{k}|| + ||\mathbf{u}_n||$, and therefore $\mathbf{k}_n \to \mathbf{k}$.

We now show that $\mathcal{G}^*\mathcal{S}\mathcal{G}$ is self-adjoint. Suppose that $\langle \mathcal{G}^*\mathcal{S}\mathcal{G}\mathbf{k}, \mathbf{p} \rangle = \langle \mathbf{k}, \mathbf{q} \rangle$ for all $\mathbf{k} \in \mathcal{K}_\mathcal{S}$, and some $\mathbf{p}, \mathbf{q} \in \mathcal{K}$. We will show that $\mathbf{p} \in \mathcal{K}_\mathcal{S}$ and $\mathcal{G}^*\mathcal{S}\mathcal{G}\mathbf{p} = \mathbf{q}$. It is easily seen that for $\mathbf{k} \in \mathcal{K}_\mathcal{S}$ and $\mathbf{u} \in \mathcal{U}$: $\langle \mathcal{G}^*\mathcal{S}\mathcal{G}\mathbf{k}, \mathbf{p} \rangle = \langle \mathcal{S}\mathbf{k}, \mathbf{p} \rangle = \langle \mathbf{k}, \mathbf{q} \rangle$, and $\langle \mathcal{S}(\mathbf{k} + \mathbf{u}), \mathbf{p} \rangle = \langle \mathbf{k}, \mathbf{q} \rangle + \langle \mathcal{S}\mathbf{u}, \mathbf{p} \rangle$. We claim that there exists an element $\mathbf{t}_0 \in \mathcal{T}$ such that $\langle \mathcal{S}\mathbf{u}, \mathbf{p} \rangle = \langle \mathbf{u}, \mathbf{q} + \mathbf{t}_0 \rangle$, or equivalently $\langle \mathcal{S}\mathbf{u}, \mathbf{p} \rangle - \langle \mathbf{u}, \mathbf{q} \rangle = \langle \mathbf{u}, \mathbf{t}_0 \rangle$, for all $\mathbf{u} \in \mathcal{U}$. This follows from the fact that $\psi : \mathcal{T} \to \mathcal{U}'$ is a bijection (see Lemma 4.8 below). Therefore $\langle \mathcal{S}(\mathbf{k} + \mathbf{u}), \mathbf{p} \rangle = \langle \mathbf{k}, \mathbf{q} \rangle + \langle \mathbf{u}, \mathbf{q} + \mathbf{t}_0 \rangle = \langle \mathbf{k} + \mathbf{u}, \mathbf{q} + \mathbf{t}_0 \rangle$, for all $\mathbf{k} \in \mathcal{K}_\mathcal{S}$ and $\mathbf{u} \in \mathcal{U}$. Thus $\langle \mathcal{S}\zeta, \mathbf{p} \rangle = \langle \zeta, \mathbf{q} + \mathbf{t}_0 \rangle$, for all $\zeta \in \mathcal{D}(\mathcal{S})$. Since $S$ is self-adjoint, this implies that $\mathbf{p} \in D(\mathcal{S})$ and $\mathcal{S}\mathbf{p} = \mathbf{q} + \mathbf{t}_0$. Therefore $\mathbf{p} \in \mathcal{K}_\mathcal{S}$ and $\mathcal{G}^*\mathcal{S}\mathcal{G}\mathbf{p} = \mathbf{q}$. $\square$

**Lemma 4.8.** *Let $\mathcal{U}'$ denote the dual space of $\mathcal{U}$, and consider the map $\psi : \mathcal{T} \to \mathcal{U}'$, $\psi(t)(u) = \langle \mathbf{t}, \mathbf{u} \rangle$. The map $\psi : \mathcal{T} \to \mathcal{U}'$ is a bijection.*

Proof. Since dim $\mathcal{T}$ = dim $\mathcal{U}'$, it suffices to show that Ker $(\psi) = 0$. Let $\mathcal{T}'$ denote the dual space of $\mathcal{T}$, and consider the map $\theta : \mathcal{U} \to \mathcal{T}'$, $\theta(\mathbf{u})(\mathbf{t}) = \langle \mathbf{u}, \mathbf{t} \rangle$. Let $\{u_1, ..., u_n\}$ be a basis for $\mathcal{U}$, and let $\{t_1, ..., t_n\}$ be a basis for $\mathcal{T}$. Let $\mathbf{A}$ be the $n \times n$ matrix $\mathbf{A} = (a_{jk})$, where $a_{jk} = \langle u_j, t_k \rangle$. Then dim Ker $(\psi)$ = Nullity $(\mathbf{A}) = n - Rank(\mathbf{A})$, and dim Ker $(\theta)$ = Nullity $(\mathbf{A}^T) = n - Rank(\mathbf{A}^T)$. Therefore dim Ker $(\psi)$ = dim Ker $(\theta)$. But Ker $(\theta) = 0$, since $\mathcal{U} \cup \mathcal{K} = 0$. Therefore Ker $(\psi) = 0$. □

**Lemma 4.9.** *$\mathcal{G}^*\mathcal{S}\mathcal{G}$ has no essential spectrum.*

Proof. According to Lemma 4.6, the Rayleigh quotients for $\mathcal{G}^*\mathcal{S}\mathcal{G}$ and $\mathcal{S}$ are equal provided $\boldsymbol{\zeta} \in \mathcal{D}^{cons}$. Therefore, if $\mathcal{W}$ is a $k$-dimensional subspace of $\mathcal{D}^{cons}$, then

$$\max_{\boldsymbol{\zeta} \in \mathcal{W}} \frac{\langle \mathcal{G}^*\mathcal{S}\mathcal{G}\boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle}{\langle \boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle} = \max_{\boldsymbol{\zeta} \in \mathcal{W}} \frac{\langle \mathcal{S}\boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle}{\langle \boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle} \geq \xi_k(\mathcal{S}),$$

which implies that

$$\xi_k(\mathcal{G}^*\mathcal{S}\mathcal{G}) \equiv \min_{\substack{\mathcal{W} \subset \mathcal{D}^{cons} \\ \dim \mathcal{W} = k}} \max_{\boldsymbol{\zeta} \in \mathcal{W}} \frac{\langle \mathcal{G}^*\mathcal{S}\mathcal{G}\boldsymbol{\zeta}\boldsymbol{\zeta} \rangle}{\langle \boldsymbol{\zeta}, \boldsymbol{\zeta} \rangle} \geq \xi_k(\mathcal{S}).$$

Since

$$\lim_{k \to \infty} \xi_k(\mathcal{S}) = \infty,$$

it follows that

$$\lim_{k \to \infty} \xi_k(\mathcal{G}^*\mathcal{S}\mathcal{G}) = \infty.$$

Therefore, by Theorem 4.4, the spectrum of $\mathcal{G}^*\mathcal{S}\mathcal{G}$ consists only of finite multiplicity eigenvalues $\xi_k(\mathcal{G}^*\mathcal{S}\mathcal{G})$ with $\xi_1 \leq \xi_2 \leq \cdots \leq \xi_n \leq \cdots$, and $\lim_{n \to \infty} \xi_n = \infty$. □

## 5. Proof of Theorem 3.1

With Lemmas 4.1-4.9 in hand, we can now turn to a proof of Theorem 3.1. In particular, $\sigma^-(\mathcal{M})$ and $\sigma^-(\mathcal{G}^*\mathcal{S}\mathcal{G})$ are now known to be well-defined finite numbers and the proof of Theorem 3.1 is essentially a completely finite dimensional argument.

First we show that $\sigma^-(\mathcal{M}) \geq \sigma^-(\mathcal{G}^*\mathcal{S}\mathcal{G}) + n$. Suppose that $\mathcal{G}^*\mathcal{S}\mathcal{G}$ has $k$ negative eigenvalues $\xi_1, \ldots, \xi_k$ with eigenvectors $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_k \in \mathcal{D}(\mathcal{G}^*\mathcal{S}\mathcal{G}) = \mathcal{D}^{cons} \equiv \{\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n\}^\perp \cap \mathcal{D}(\mathcal{S})$. Let

$$\mathcal{W} = \mathcal{V} \oplus \text{ span } \left\{ \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{0} \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{f}_k \\ \mathbf{0} \end{pmatrix} \right\},$$

where

$$\mathcal{V} \equiv \left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} : \mathbf{v} \in \mathbb{R}^n \right\}.$$

If $\mathbf{f} \in$ span $\{\mathbf{f}_1, \dots, \mathbf{f}_k\}$, then (by Lemma 4.6)

$$\left\langle \mathcal{M} \begin{pmatrix} \mathbf{f} \\ \mathbf{v} \end{pmatrix}, \begin{pmatrix} \mathbf{f} \\ \mathbf{v} \end{pmatrix} \right\rangle = \langle \mathcal{G}^* \mathcal{S} \mathcal{G} \mathbf{f}, \mathbf{f} \rangle.$$

And because $\mathbf{f} = \sum_{i=1}^k c_i f_i$,

$$\langle \mathcal{G}^* \mathcal{S} \mathcal{G} \mathbf{f}, \mathbf{f} \rangle = \sum_{i=1}^k \xi_i c_i^2 \le 0.$$

This implies that the eigenvalue $\xi_{n+k}(\mathcal{M}) \le 0$. If $\xi_{n+k}(\mathcal{M}) = 0$, then by Lemma 4.5, $\mathcal{W}$ contains an eigenvector for $\xi_{n+k} = 0$. Since

$$\left\langle \mathcal{M} \begin{pmatrix} \mathbf{f} \\ \mathbf{v} \end{pmatrix}, \begin{pmatrix} \mathbf{f} \\ \mathbf{v} \end{pmatrix} \right\rangle < 0,$$

unless $\mathbf{f} = 0$, the eigenvector must be of the form

$$\mathbf{w} = \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix}.$$

But

$$\mathcal{M} \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n \mathbf{v}_i \mathbf{T}_i \\ \mathbf{0} \end{pmatrix} = \mathbf{0}$$

only if $\mathbf{v} = \mathbf{0}$, by the linear independence of the $\mathbf{T}_i$. This contradiction shows that $\xi_{n+k}(\mathcal{M}) < 0$, and $\sigma^-(\mathcal{M}) \ge n + k = n + \sigma^-(\mathcal{G}^* \mathcal{S} \mathcal{G})$.

Now we show that $\sigma^-(\mathcal{M}) \le \sigma^-(\mathcal{G}^* \mathcal{S} \mathcal{G}) + n$. Suppose that $\mathcal{M}$ has $n + k$ negative eigenvalues. Then there is an $(n + k)$-dimensional subspace $\mathcal{W} \subset \mathcal{D}(\mathcal{M})$ such that $\langle \mathcal{M} \mathbf{w}, \mathbf{w} \rangle < 0$ for $\mathbf{w} \in \mathcal{W}$, $\mathbf{w} \ne \mathbf{0}$. Consider the linear functionals $\mathcal{L}_i : \mathcal{W} \to \mathbb{R}$:

$$\mathcal{L}_i \begin{pmatrix} \mathbf{f} \\ \mathbf{v} \end{pmatrix} = \langle \mathbf{f}, \mathbf{T}_i \rangle, \ 1 \le i \le n.$$

Let

$$\mathcal{W}_0 \equiv \left\{ \begin{pmatrix} \mathbf{f} \\ \mathbf{v} \end{pmatrix} \in \mathcal{W} : \mathcal{L}_i \begin{pmatrix} \mathbf{f} \\ \mathbf{v} \end{pmatrix} = 0, \ 1 \le i \le n \right\},$$

Then because $W$ has dimension $n + k$ and there are only $n$ functionals, $\dim \mathcal{W}_0 \ge k$. Define $\pi : \mathcal{W}_0 \to \mathcal{D}(\mathcal{G}^* \mathcal{S} \mathcal{G})$ by

$$\pi \begin{pmatrix} \mathbf{f} \\ \mathbf{v} \end{pmatrix} = \mathbf{f},$$

and let $\mathcal{W}_1 = \pi(\mathcal{W}_0)$. For $\mathbf{f} \in \mathcal{W}_1$, $\mathbf{f} \ne \mathbf{0}$,

$$\langle \mathcal{G}^* \mathcal{S} \mathcal{G} \mathbf{f}, \mathbf{f} \rangle = \left\langle \mathcal{M} \begin{pmatrix} \mathbf{f} \\ \mathbf{v} \end{pmatrix}, \begin{pmatrix} \mathbf{f} \\ \mathbf{v} \end{pmatrix} \right\rangle < 0.$$

Because

$$\left\langle \mathcal{M}\begin{pmatrix}\mathbf{0}\\ \mathbf{v}\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{v}\end{pmatrix}\right\rangle = 0,$$

we conclude that $(\mathbf{0},\mathbf{v})^T \notin \mathcal{W}$, so $\dim \mathcal{W}_1 = \dim \mathcal{W}_0 \geq k$. Thus, $\langle \mathcal{G}^*\mathcal{S}\mathcal{G}\mathbf{f},\mathbf{f}\rangle < 0$ on the space $\mathcal{W}_1 \subset \mathcal{D}(\mathcal{G}^*\mathcal{S}\mathcal{G})$ of dimension at least $k$. Therefore, $\sigma^-(\mathcal{G}^*\mathcal{S}\mathcal{G}) \geq k = \sigma^-(\mathcal{M})-n$. □

## 6. Calculation of $\sigma^-(\mathcal{M})$ via Conjugate Points

We now develop a conjugate point test for computing $\sigma^-(\mathcal{M})$. For $0 < t \leq 1$, let $\mathcal{M}(t)$ be the restriction of $\mathcal{M}$ to the interval $[0,t]$ with Dirichlet boundary conditions. Thus

(6.1) $$\mathcal{D}(\mathcal{M}(t)) \equiv \left\{ \mathbf{z} = \begin{pmatrix}\boldsymbol{\zeta}\\ \mathbf{v}\end{pmatrix} \in \mathcal{H}^2(0,t)^m \times \mathbb{R}^n : \boldsymbol{\zeta}(0) = \mathbf{0} = \boldsymbol{\zeta}(t)\right\},$$

and $\mathcal{M}(t)$ is represented by the same bordered operator as $\mathcal{M}$, except that all integrals are taken over $[0,t]$ instead of $[0,1]$.

Just as for $\mathcal{M} = \mathcal{M}(1)$, it is straightforwardly shown that the spectrum of $\mathcal{M}(t)$ consists of a sequence of eigenvalues

$$\xi_1(t) \leq \xi_2(t) \leq \cdots \leq \xi_n(t) \leq \cdots$$

with $\lim_{n\to\infty} \xi_n(t) = \infty$.

**Lemma 6.1.** *The eigenvalues $\xi_i(t)$ are nonincreasing functions of $t$. If $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n$ are linearly independent on every subinterval $(a,b) \subset [0,1]$, then the $\xi_i(t)$ are strictly decreasing.*

Proof. Let $0 < t_1 < t_2 < 1$, and let $\mathbf{z} = (\boldsymbol{\zeta},\mathbf{v})^T \in \mathcal{D}(\mathcal{M}(t_1))$. We may extend $\mathbf{z}$ to a function $\hat{\mathbf{z}} = (\hat{\boldsymbol{\zeta}},\hat{\mathbf{v}})^T$, where $\hat{\mathbf{v}} = \mathbf{v}$, and

$$\hat{\boldsymbol{\zeta}}(s) = \begin{cases} \boldsymbol{\zeta}(s) & \text{for } 0 \leq s \leq t_1, \\ \mathbf{0} & \text{for } t_1 \leq s \leq t_2. \end{cases}$$

Unfortunately, $\hat{\boldsymbol{\zeta}}$ need not be in $\mathcal{H}^2(0,t_2)^m$ but after an integration by parts, we may calculate eigenvalues via Rayleigh quotients in $\mathcal{H}^1$ instead of $\mathcal{H}^2$. If $\boldsymbol{\zeta} \in \mathcal{H}^1(0,t_1)^m$ and $\boldsymbol{\zeta}(t_1) = 0$, then the extended function $\hat{\boldsymbol{\zeta}} \in \mathcal{H}^1(0,t_2)^m$. We shall use the notation

$$\mathcal{D}_t = \left\{ \begin{pmatrix}\boldsymbol{\zeta}\\ \mathbf{v}\end{pmatrix} \in \mathcal{H}^1(0,t)^m \times \mathbb{R}^n : \boldsymbol{\zeta}(0) = \mathbf{0} = \boldsymbol{\zeta}(t)\right\}.$$

Note that trivially $\int_0^{t_2} \mathcal{S}\hat{\boldsymbol{\zeta}}\cdot\hat{\boldsymbol{\zeta}}\,ds = \int_0^{t_1} \mathcal{S}\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}\,ds$, $\int_0^{t_2} \mathbf{T}_i\cdot\hat{\boldsymbol{\zeta}}\,ds = \int_0^{t_1} \mathbf{T}_i\cdot\boldsymbol{\zeta}\,ds$, and $\int_0^{t_2} \hat{\boldsymbol{\zeta}}\cdot\hat{\boldsymbol{\zeta}}\,ds = \int_0^{t_1} \boldsymbol{\zeta}\cdot\boldsymbol{\zeta}\,ds$, so it follows that the Rayleigh quotients satisfy:

$$\frac{\langle \mathcal{M}(t_2)\hat{\mathbf{z}},\hat{\mathbf{z}}\rangle}{\langle \hat{\mathbf{z}},\hat{\mathbf{z}}\rangle} = \frac{\langle \mathcal{M}(t_1)\mathbf{z},\mathbf{z}\rangle}{\langle \mathbf{z},\mathbf{z}\rangle}.$$

Thus we may regard $\mathcal{D}_{t_1}$ as a subset of $\mathcal{D}_{t_2}$, and we see that

$$\begin{aligned}
\xi_i(t_1) &= \min_{\substack{\dim \mathcal{B}=i \\ \mathcal{B}\subset\mathcal{D}_{t_1}}} \max_{\substack{\mathbf{z}\neq\mathbf{0} \\ \mathbf{z}\in\mathcal{B}}} \frac{\langle \mathcal{M}(t_1)\mathbf{z},\mathbf{z}\rangle}{\langle \mathbf{z},\mathbf{z}\rangle} \\
&\geq \min_{\substack{\dim \mathcal{B}=i \\ \mathcal{B}\subset\mathcal{D}_{t_2}}} \max_{\substack{\mathbf{z}\neq\mathbf{0} \\ \mathbf{z}\in\mathcal{B}}} \frac{\langle \mathcal{M}(t_2)\hat{\mathbf{z}},\hat{\mathbf{z}}\rangle}{\langle \hat{\mathbf{z}},\hat{\mathbf{z}}\rangle} \\
&= \xi_i(t_2)
\end{aligned}$$

This shows that $\xi_i(t_1) \geq \xi_i(t_2)$. If $\xi_i(t_1) = \xi_i(t_2)$, then (by Lemma 4.5) $\mathcal{M}(t_2)$ has an eigenvector $\hat{\mathbf{z}} = (\hat{\boldsymbol{\zeta}},\mathbf{v})^T \in \mathcal{D}(\mathcal{M}(t_2))$ which is an extension of an element $\mathbf{z} = (\boldsymbol{\zeta},\mathbf{v})^T \in \mathcal{D}(\mathcal{M}(t_1))$. This means that

$$\begin{aligned}
\mathcal{S}\hat{\boldsymbol{\zeta}} + \sum_{k=1}^{n} v_k \mathbf{T}_k &= \xi_i \hat{\boldsymbol{\zeta}}, \qquad 0 < s < t_2, \\
\int_0^{t_2} \hat{\boldsymbol{\zeta}} \cdot \mathbf{T}_k ds &= \xi_i v_k, \qquad 1 \leq k \leq n, \\
\hat{\boldsymbol{\zeta}}(s) = \mathbf{0} \ \text{ for } & \qquad t_1 \leq s \leq t_2,
\end{aligned}$$

which implies that $\sum_{k=1}^{n} v_k \mathbf{T}_k(s) = 0$ for $t_1 \leq s \leq t_2$. If the $\mathbf{T}_k(s)$ are linearly independent on the interval $(t_1,t_2)$, then $v_k = 0$ $(1 \leq k \leq n)$, and $\mathcal{S}\hat{\boldsymbol{\zeta}} = \xi_i\hat{\boldsymbol{\zeta}}$. Since $\hat{\boldsymbol{\zeta}}(s) = \mathbf{0}$ for $t_1 \leq s \leq t_2$, the uniqueness theorem for ordinary differential equations implies that $\hat{\boldsymbol{\zeta}}(s) = \mathbf{0}$ for $0 \leq s \leq t_2$. Thus $\hat{\mathbf{z}} = (\hat{\boldsymbol{\zeta}},\mathbf{v})^T = \mathbf{0}$. □

**Definition:** A point $t \in (0,1)$ is a *conjugate point* for $\mathcal{M}$ if zero is an eigenvalue of $\mathcal{M}(t)$. This means that there is a nontrivial element $(\boldsymbol{\zeta},\mathbf{v})^T \in \mathcal{D}(\mathcal{M}(t))$ such that

$$\begin{aligned}
\mathcal{S}\boldsymbol{\zeta} + \sum_{k=1}^{n} v_k \mathbf{T}_k &= \mathbf{0}, \qquad 0 < s < t \\
\int_0^{t} \boldsymbol{\zeta} \cdot \mathbf{T}_k \, ds &= \mathbf{0}, \qquad 1 \leq k \leq n, \\
\boldsymbol{\zeta}(0) = \mathbf{0} &= \boldsymbol{\zeta}(t).
\end{aligned}$$

We define $\nu(t)$ to be the multiplicity of the zero eigenvalue so that $\nu(t) > 0$ if and only if $t$ is a conjugate point.

**Theorem 6.2.** *Suppose that the functions $\mathbf{T}_1, \mathbf{T}_2, \cdots, \mathbf{T}_n$ are linearly independent on every subinterval of* $[0,1]$*. Then $\mathcal{M}$ has at most finitely many conjugate points in* $(0,1)$ *and*

$$\sigma^-(\mathcal{M}) = n + \sum_{0<t<1} \nu(t).$$

Proof. For $0 < t \leq 1$, let $\mathcal{S}(t)$ denote the restriction of the operator $\mathcal{S}$ to the interval $[0,t]$ with Dirichlet boundary conditions. Let $\eta_1(t) \leq \eta_2(t) \leq \cdots$ denote the eigenvalues of $\mathcal{S}(t)$. Greenberg [5, Theorem 8.1, p. 28] has shown that $\lim_{t\to 0^+}\eta_1(t) = \infty$. This implies that for sufficiently small $t$ all eigenvalues $\eta_i(t) > 0$, and $\sigma^-(\mathcal{S}(t)) = 0$. Theorem 3.1 then implies that for small $t$, $\sigma^-(\mathcal{M}(t)) = n$. Lemma 6.1 implies that at each conjugate point $t \in (0,1)$, $\nu(t)$ eigenvalues $\xi_i(t)$ decrease from positive to negative values. Therefore, the number of negative eigenvalues $\xi_i(1)$ of $\mathcal{M} = \mathcal{M}(1)$ equals $n + \sum_{0<t<1}\nu(t)$. Since $\mathcal{M}$ has only a finite number of negative eigenvalues, the sum must be finite, and therefore $\mathcal{M}$ has only finitely many conjugate points. □

Theorems 3.1 and 6.2 have the following immediate consequence.

**Corollary 6.3.** *If the functions* $\mathbf{T}_1, \dots, \mathbf{T}_n$ *are linearly independent on every subinterval of* $[0,1]$, *then* $\sigma^-(\mathcal{G}^*\mathcal{S}\mathcal{G}) = \sum_{0<t<1}\nu(t)$.

To calculate $\sum_{0<t<1}\nu(t)$, we proceed as follows. Let $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_m$ be a basis for the space of solutions of the initial value problem

$$\mathcal{S}\boldsymbol{\theta} = \mathbf{0},\ \boldsymbol{\theta}(0) = \mathbf{0}.$$

For each $\mathbf{T}_i$, let $\boldsymbol{\phi}_i$ be a particular solution of the initial value problem

$$\mathcal{S}\boldsymbol{\phi}_i = -\mathbf{T}_i,\ \boldsymbol{\phi}_i(0) = \mathbf{0}.$$

Let $\mathbf{v} = (v_1, v_2, \dots, v_n)^T$ and

$$\boldsymbol{\zeta}(s) = \sum_{i=1}^{m} c_i\boldsymbol{\theta}_i(s) + \sum_{i=1}^{n} v_i\boldsymbol{\phi}_i(s).$$

Then $\boldsymbol{\zeta}$ is the general solution of

$$\begin{aligned} \mathcal{S}\boldsymbol{\zeta} \quad + \quad & \sum_{j=1}^{n} v_j\mathbf{T}_j = \mathbf{0}, \\ & \boldsymbol{\zeta}(0) = \mathbf{0}. \end{aligned}$$

$\mathcal{M}(t)$ has a zero eigenvalue if and only if there exist constants $c_1, c_2, \cdots, c_m, v_1, v_2, \cdots, v_n$ (not all zero) such that

$$\begin{aligned} \boldsymbol{\zeta}(t) = \sum_{j=1}^{m} c_j\boldsymbol{\theta}_j(t) + \sum_{k=1}^{n} v_k\boldsymbol{\phi}_k(t) &= \mathbf{0}, \\ \int_0^{\mathbf{t}} \left(\sum_{j=1}^{m} c_j\boldsymbol{\theta}_j(s) + \sum_{k=1}^{n} v_k\boldsymbol{\phi}_k(s)\right)\cdot \mathbf{T}_i(s)\,ds &= 0,\ \ i = 1, \dots, n \end{aligned}$$

The system can be written in the compact form:

$$A(t) \equiv \begin{bmatrix} \boldsymbol{\Theta}(t) & \boldsymbol{\Phi}(t) \\ \mathbf{F}_{\boldsymbol{\theta}}(t) & \mathbf{F}_{\boldsymbol{\phi}}(t) \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{v} \end{bmatrix} = \mathbf{0}, \tag{6.2}$$

where

$$
\begin{aligned}
\mathbf{c} &= [c_1 \dots c_m]^T \in \mathbb{R}^m, \\
\mathbf{v} &= [v_1 \dots v_n]^T \in \mathbb{R}^n, \\
\mathbf{\Theta}(t) &= \left[\begin{array}{ccc} \boldsymbol{\theta}_1(t) & \dots & \boldsymbol{\theta}_m(t) \end{array}\right] \in \mathbb{R}^{m \times m}, \\
\mathbf{\Phi}(t) &= \left[\begin{array}{ccc} \boldsymbol{\phi}_1(t) & \dots & \boldsymbol{\phi}_n(t) \end{array}\right] \in \mathbb{R}^{m \times n}, \\
\mathbf{F}_{\boldsymbol{\theta}}(t)_{ij} &= \int_0^t \boldsymbol{\theta}_j(\tau) \cdot \mathbf{T}_i(\tau) d\tau, \ \mathbf{F}_{\boldsymbol{\theta}}(t) \in \mathbb{R}^{n \times m}, \\
\mathbf{F}_{\boldsymbol{\phi}}(t)_{ij} &= \int_0^t \boldsymbol{\phi}_j(\tau) \cdot \mathbf{T}_i(\tau) d\tau, \ \mathbf{F}_{\boldsymbol{\phi}} \in \mathbb{R}^{n \times n}.
\end{aligned}
$$

The above discussion implies the following:

**Theorem 6.4.** *A point $t \in (0,1)$ is a conjugate point for $\mathcal{M}$ if and only if* $\det(A(t)) = 0$. *Furthermore, $\nu(t) = nullity(A(t)) = m + n - rank(A(t))$.*

The matrix $A(t)$ was previously described in [14] (see the discussion below) and by Bolza [2] in regards to necessary and sufficient conditions for positivity of the second variation arising in isoperimetric calculus of variations problems.

## 7. Discussion

We have shown that, just as in the finite dimensional case, the index of a constrained critical point in the isoperimetric calculus of variations is simply related to the index of an unconstrained critical point of an extended system in which the Lagrange multipliers are treated as additional dependent variables. The relationship is encompassed in the formula from Theorem 3.1:

$$
\sigma^-(\mathcal{M}) = \sigma^-(\mathcal{G}^*\mathcal{S}\mathcal{G}) + n, \tag{7.1}
$$

where $n$ is the number of linearly independent constraints.

It can be awkward to explicitly compute the projection $\mathcal{G}^*$, so even the finite dimensional version of the general relation (7.1) provides a potentially useful, indirect way to determine the constrained index via computation of the number of negative eigenvalues of the explicitly known bordered matrix $\mathbf{M}$. Similarly, in the infinite dimensional case, the conjugate point theory developed in Section 6 combined with Equation (7.1) provides one computational approach to determine the constrained index. In point of fact, for one-dimensional calculus of variations problems a conjugate point theory for $\mathcal{G}^*\mathcal{S}\mathcal{G}$ has previously been constructed to determine the constrained index directly [14]. It can be shown that the conjugate points and multiplicities $\nu(t)$ for the two operators $\mathcal{M}$ and $\mathcal{G}^*\mathcal{S}\mathcal{G}$ are identical, and are determined by the roots of the determinant of the matrix $A(t)$ given in Equation (6.2). But in the limit $t \to 0$, $\mathcal{G}^*\mathcal{S}\mathcal{G}$ is positive definite while $\mathcal{M}(t)$ has $n$ negative eigenvalues. This observation actually gives an alternate proof of Theorem 3.1. The proof of Theorem 3.1 presented in Section 4 is of interest because it is not intrinsically tied to one-dimensional calculus of variations problems. It not only provides a bridge to the finite dimensional case, but it can also be extended to multi-dimensional isoperimetric calculus of variations problems, provided that spectral properties analogous to Lemmas 4.1-4.9 can be proven.

Our motivation for developing tests to determine the index of a constrained critical point in isoperimetrically constrained calculus of variations problems was to determine index changes in an elastic rod model of DNA [14]. The isoperimetric calculus of variations problem arising in that example has the additional feature that the boundary conditions contain an additional scalar parameter, and consequently critical points arise in one-dimensional families or branches. For such *bifurcation* problems it is typically the case that the index of the critical point can be shown to change only at special isolated solutions such as fold points where the parameter goes through a local extreme value. The identity (7.1) is exploited in [19] as part of the development of the appropriate theory for such *stability exchanges* in parameter dependent isoperimetric problems.

## References

[1] J. Berkowitz. On the discreteness of spectra of singular Sturm-Liouville problems. *Communications on Pure and Applied Mathematics*, XII:523–542, 1959.

[2] O. Bolza. *Lectures on the Calculus of Variations.* Chelsea Publishing Company, 1973.

[3] Y. Chabrillac, J. P. Crouzeix. Definiteness and semidefiniteness of quadratic forms revisited. *Linear Algebra Appl.*, 63:283–292, 1984.

[4] R. W. Cottle. Manifestations of the Schur complement. *Linear Algebra Appl.*, 8:189–211, 1974.

[5] L. Greenberg. A Prüfer method for calculating eigenvalues of self-adjoint systems of ordinary differential equations, Part 1. Technical Report TR91-24, University of Maryland, 1991.

[6] S. P. Han. On the Hessian of Lagrangian and second-order optimality conditions. *SIAM J. Control Optim.*, 24:339–345, 1986.

[7] C. Hassell, E. Rees. The index of a constrained critical point. *Amer. Math. Monthly*, 100(8):772–778, 1993.

[8] H. Th. Jongen, P. Jonker, and F. Twilt. *Nonlinear Optimization in $R^n$, I. Morse Theory, Chebyshev Approximation.* Peter Lang Verlag, 1983.

[9] H. Th. Jongen, T. Möbert, J.-J. Rückmann, K. Tammer. On inertia and Schur complement in optimization. *Lin. Alg. Appl.*, 95:97–109, 1987.

[10] T. Kato. *Perturbation Theory for Linear Operators.* Springer-Verlag, Berlin, 1984.

[11] D. G. Luenberger. *Linear and Nonlinear Programming.* Addison-Wesley, 1989.

[12] J. H. Maddocks. Restricted quadratic forms, inertia theorems, and the Schur complement. *Linear Algebra and Appl.*, 108:1–36, 1988.

[13] H. B. Mann. Quadratic forms with linear constraints. *Amer. Math. Monthly*, 50:430–433, 1943.

[14] R. S. Manning, K. A. Rogers, J. H. Maddocks. Isoperimetric conjugate points and stability of DNA minicircles. *Proc. R. Soc. Lond. A*, 454:3047-3074, 1998.

[15] J. Milnor. *Morse Theory*, volume 51 of *Ann. Math. Stud.* Princeton University Press, 1963.

[16] M. Morse. *The Calculus of Variations in the Large*, volume 18 of *Colloquium Publications.* American Mathematical Society, 1932.

[17] M. Reed and B. Simon. Methods of Modern Mathematical Physics, I: Functional Analysis. Academic Press, 1972.

[18] K. A. Rogers. *Stability Exchange in Parameter-Dependent Constrained Variational Principles with Applications to Elastic Rod Models of DNA Minicircles.* PhD thesis, University of Maryland, 1997.

[19] K. A. Rogers, J. H. Maddocks. Distinguished bifurcation diagrams and the stability of a twisted elastic loop. *preprint*, 1998.

[20] P. Shutler. Constrained critical points. *American Math. Monthly*, 102(1):49–52, 1995.

[21] D. Spring. On the second derivative test for constrained local extrema. *Amer. Math. Monthly*, 92(9):631–643, 1985.

[22] J. Weidmann. Spectral theory of ordinary differential operators. In *Lecture Notes in Mathematics*, volume 1258. Springer-Verlag, Berlin-New York, 1987.

[23] A. Weinstein and W. Stenger. *Methods of Intermediate Problems for Eigenvalues: Theory and Ramifications.* Academic Press, 1972.

*Department of Mathematics*
*University of Maryland*
*College Park, MD 20742*

*Département de Mathématiques*
*Ecole Polytechnique Fédérale de Lausanne*
*CH-1015 Lausanne Switzerland*

*Institute for Mathematics and its Applications*
*University of Minnesota*
*207 Church Street SE*
*400 Lind Hall*
*Minneapolis, MN 55455-0436 USA*