Journal of Elasticity (2004) **77:** 221–247
DOI: 10.1007/s10659-005-0931-z

Kirchhoff's Problem of Helical Equilibria of Uniform Rods

NADIA CHOUAIEB[1] and JOHN H. MADDOCKS[2]
[1]*Institut Préparatoire aux Etudes d'Ingénieurs, El Manar, Tunis, Tunisia. E-mail: nadia.chouaieb@ipeiem.rnu.tn*
[2]*Institute for Mathematics B, Swiss Federal Institute of Technology, Lausanne, CH-1015, Switzerland. E-mail: john.maddocks@epfl.ch*

Received 8 June 2004; in revised form 28 December 2004

Abstract. It is demonstrated that a uniform and hyperelastic, but otherwise arbitrary, nonlinear Cosserat rod subject to appropriate end loadings has equilibria whose center lines form two-parameter families of helices. The absolute energy minimizer that arises in the absence of any end loading is a helical equilibrium by the assumption of uniformity, but more generally the helical equilibria arise for non-vanishing end loads. For inextensible, unshearable rods the two parameters correspond to arbitrary values of the curvature and torsion of the helix. For non-isotropic rods, each member of the two-dimensional family of helical center lines has at least two possible equilibrium orientations of the director frame. The possible orientations are characterized by a pair of finite-dimensional, dual variational principles involving pointwise values of the strain-energy density and its conjugate function. For isotropic rods, the characterization of possible equilibrium configurations degenerates, and in place of a discrete number of two-parameter families of helical equilibria, typically a single four-parameter family arises. The four continuous parameters correspond to the two of the helical center lines, a one-parameter family of possible angular phases, and a one-parameter family of imposed excess twists.

Mathematics Subject Classifications (2000): 74K10.

Key words: elastic rods, helical equilibria, relative equilibria of Hamiltonian systems.

1. Introduction

Helices are ubiquitous both in nature and in man-made objects. They can be found on many different scales ranging from nanostructures, such as α-helices in proteins, the DNA double helix, the collagen triple helix, and carbon nano-tubes, to macroscopic structures such as horns of sheep, tendrils of plants, springs and helical staircases. Pauling (see [7]) argued that the only configurations that can be adopted by molecular chains compatible with equivalence are helical ones, while Crane [5] wrote "any structure that is straight or rod like is probably a structure having a repetition along a screw axis."

Helices are merely three-dimensional curves, and their importance in all of the above mentioned applications is as center lines of slender, three-dimensional objects that might reasonably be approximated by a rod theory of some type. Kirchhoff [12, 13, 2] started the classification of rod equilibria with helical center lines. He showed that an intrinsically straight, inextensible, unshearable, isotropic, and uniform rod admits helical solutions under the action of appropriate forces and torques applied at its ends. Kirchhoff's theory rests upon the constitutive assumption that the stress couple depends linearly upon the curvature and the twist. Using the director rod theory of the Cosserats, Antman [1] and Whitman and DeSilva [16] generalized Kirchhoff's analysis to the case of a uniform, extensible and shearable, but still intrinsically straight and isotropic elastic rod. While Whitman and DeSilva adopted linear constitutive relations, Antman considered nonlinear constitutive laws. Both studies found that the stretch is constant along the rod, and that, as in the classic Kirchhoff problem, the end forces and moments needed to maintain the helical deformation are statically equivalent to a wrench acting along the helical axis. Both studies exploited an Euler angle parametrization of the director frame, and relied on the fact that for a uniform and isotropic rod the resulting system of equilibrium conditions is explicitly integrable. For a more general class of non-isotropic hyperelastic materials, Ericksen [6] showed that certain invariance requirements characterizing a "uniform state" imply that the solutions of the rod problem must be helical, but, as he observes, "it is impossible to say much about the existence or multiplicity of these solutions without introducing some assumptions concerning the form of the strain-energy density function" [6, p. 376].

There have of course been prior analyses of helical equilibria of non-isotropic rods in various special cases of particular constitutive relations; we cite in particular the recent work of Goriely and co-workers [8, 9]. However we are unaware of a prior classification of helical equilibria of the general class of uniform, hyperelastic, non-isotropic rods, even in the case of linear constitutive relations. In the isotropic case the conclusions described below could be extracted from the coordinate-based approach exploited in [1, 16]. However the analysis that will be presented here emphasizes the connections between the isotropic and non-isotropic cases, along with the role of the continuous symmetry of isotropy in forcing degeneracy and associated non-isolation of critical points within a finite-dimensional variational characterization of helical equilibria.

In this article we construct, within the Cosserat rod theory and subject to pure end loading, equilibria with helical center lines as *relative equilibria* of the balance laws written in a Hamiltonian form (in which arc length is the time-like variable). Our analysis applies to uniform, hyperelastic rods, including nonquadratic energies and uniform (and therefore helical) intrinsic shape. The results encompass the four cases of rods that are either isotropic or non-isotropic and either extensible and shearable or inextensible and unshearable. Our conclusions are simplest to state for the two inextensible, unshearable cases. In the non-isotropic subcase we find that relative equilibria with helical center lines arise in two-parameter families corresponding to arbitrarily prescribed radius and pitch (or equivalently curvature and torsion) of the helical center line, and that for each helical center line there are at least two possible orientations of the associated director frame that correspond to equilibria. For a prescribed helical center line, the possible equilibrium orientations of the directors are (generically) finite in number. In contrast, for the isotropic subcase, each helical center line has an associated continuous, two-dimensional family of equilibrium director frames corresponding to an arbitrary angular phase and an arbitrary imposed additional twist around the $\boldsymbol{d}_3$ director. Our conclusions for the case of extensible, shearable rods are essentially the same, but with the exception that we are unable to conclude globally that the two-parameter family of helical center lines corresponds to arbitrary radius and pitch. The pitch can be prescribed, but the constitutive dependence of arc length in the deformed configuration prevents us from concluding immediately that an arbitrary radius can be achieved.

As already mentioned, the main tool we adopt is the observation that the relative equilibria of a Hamiltonian formulation of the equations governing equilibria of elastic rods are helices. In the theory of Hamiltonian systems there is a standard variational characterization of relative equilibria that involves the integrals of the system. In the context of Cosserat rods this standard variational characterization is manifested as a finite-dimensional constrained minimization in the space of stress variables. However, we obtain our existence and multiplicity results from a non-standard, dual variational formulation in terms of the six strains. This dual variational principle relies upon the fact that the integrals of the Hamiltonian system are quadratic in the phase variables, and involves the inversion of a matrix of Lagrange multipliers. While both primal and dual characterizations yield information, the dual formulation in terms of the strains permits a direct analysis of existence and multiplicity that is particularly simple in the inextensible and unshearable case.

In Section 2 we present the necessary background material and we set notation. In Section 3 we classify families of helical equilibria of non-isotropic rods. The isotropic case is described in Section 4, and in Section 5 we discuss the divers results. Throughout the presentation we adopt the convention that when one of the Latin letters i, j, k appears as an index, it takes values in the set $\{1, 2, 3\}$. We do not address minimal regularity assumptions on the various functions that appear; rather we assume sufficient smoothness to justify all the derivatives that are written down.

2. Background Material

2.1. FRENET–SERRET FRAME OF A SPACE CURVE

A curve in space can be defined by a vector-valued function

$$\boldsymbol{r}\colon\ I \subset \mathbb{R} \to \mathbb{E}^3,$$

which maps some open interval $I \in \mathbb{R}$ into Euclidean 3-space $\mathbb{E}^3$. At each $\sigma \in I$, the vector $\boldsymbol{r}(\sigma)$ gives the position vector from the origin O of the Euclidean space $\mathbb{E}^3$ to the point of the curve specified by σ. We assume that the curve $\boldsymbol{r}$ is a regular curve, i.e., $\boldsymbol{r}'(\sigma) \neq \mathbf{0}$ for every $\sigma \in I$. Furthermore, we assume, after reparametrization if necessary, that the parameter σ is an arc length along the curve, so that at any $\sigma \in I$ the tangent to the curve is the unit vector $\boldsymbol{\tau} = \boldsymbol{r}'(\sigma)$. The curvature of the curve $\boldsymbol{r}(\sigma)$ is the non-negative scalar-valued function $\kappa(\sigma)$ defined by

$$\boldsymbol{\tau}' = \boldsymbol{r}''(\sigma) = \kappa \boldsymbol{\nu}, \tag{1}$$

where $\boldsymbol{\nu}$ is a unit vector perpendicular to the tangent $\boldsymbol{\tau}$ called the *principal normal* to the curve. (Note that when $\boldsymbol{\tau}'(\sigma) = \mathbf{0}$, as for straight lines, the curvature is necessarily zero, and the principal normal is not well defined.) The *binormal* vector $\boldsymbol{\beta}$ is defined so that $\{\boldsymbol{\nu}, \boldsymbol{\beta}, \boldsymbol{\tau}\}$ is a right-handed orthogonal triad of unit vectors, i.e., $\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}$. This *Frenet* triad

$$\boldsymbol{F} := \{\boldsymbol{\nu}, \boldsymbol{\beta}, \boldsymbol{\tau}\}^{\star} \tag{2}$$

is an orthonormal frame at every point $\sigma \in I$ along the curve.

We have the relations

$$\boldsymbol{\beta}' = -\tau \boldsymbol{\nu}, \tag{3}$$

$$\boldsymbol{\nu}' = -\kappa \boldsymbol{\tau} + \tau \boldsymbol{\beta}, \tag{4}$$

where the scalar-valued function $\tau(\sigma)$ is called the (geometric) *torsion* of the curve. Equations (1), (3), and (4) giving the evolution in arc length σ of $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ along the curve are called the Frenet–Serret equations. They can be written in compact form as

$$\begin{aligned} [\,\boldsymbol{\nu} \quad \boldsymbol{\beta} \quad \boldsymbol{\tau}\,]' &= [\,\boldsymbol{\nu} \quad \boldsymbol{\beta} \quad \boldsymbol{\tau}\,] \begin{bmatrix} 0 & -\tau & \kappa \\ \tau & 0 & 0 \\ -\kappa & 0 & 0 \end{bmatrix}, \quad \text{or} \\ \boldsymbol{F}' &= \boldsymbol{F} \begin{bmatrix} 0 & -\tau & \kappa \\ \tau & 0 & 0 \\ -\kappa & 0 & 0 \end{bmatrix}. \end{aligned} \tag{5}$$

2.2. FRAMED CURVES

In order to discuss the (physical) twist of a rod, it is necessary to add to the space curve $\boldsymbol{r}$ an additional structure that can model the orientation of the material points in the cross-section. To do so we introduce the notion of a *framed curve*. A framed curve is a space curve $\boldsymbol{r}(s)$ together with a right-handed orthonormal frame $\{\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s)\}$ defined at each s. In order to account for possible effects of extension we do not assume that the parameter s is necessarily an arc length. Often, the vector $\boldsymbol{d}_3(s)$ is taken to be the unit tangent vector $\boldsymbol{\tau}$ to the (oriented) curve, in which case the vector $\boldsymbol{d}_1(s)$ specifies a particular direction in the normal plane, and the

⋆ We here use the Greek $\{\boldsymbol{\nu}, \boldsymbol{\beta}, \boldsymbol{\tau}\}$ to denote the Frenet frame instead of the more standard notation $\{\boldsymbol{n}, \boldsymbol{b}, \boldsymbol{t}\}$ because we reserve the symbol $\boldsymbol{n}$ to denote the net force acting across a cross-section of a Cosserat rod.

third vector $\boldsymbol{d}_2(s)$ is defined such that $\{\boldsymbol{d}_i(s)\}$ is orthonormal and right handed. These special cases will be described as *adapted framings*; both the adapted and non-adapted cases will be important here.

We stress that in general a framing of a curve, whether adapted or not, contains information beyond that contained in the curve $\boldsymbol{r}(s)$ itself, and for that reason we will sometimes refer to *extrinsically* defined framings. For example, in the context of rod theories the extrinsically defined framing can be chosen to contain information about the deformed configuration of the cross-section of material points that composed the normal plane in the undeformed reference configuration. In contrast the Frenet frame is defined entirely in terms of the curve $\boldsymbol{r}$ and its derivatives so that we refer to it as an *intrinsic* framing of the curve (which also happens to be an adapted framing).

As s varies, the orientation of the frame $\{\boldsymbol{d}_i(s)\}$ is assumed to change smoothly relative to a fixed frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, and the change can be expressed as a three-dimensional rotation, or direction cosine matrix $\boldsymbol{d}(s)$. The rate of rotation can be represented by a vector-valued function $\boldsymbol{u}(s)$ called the *Darboux vector*. The evolution of the frame along the curve is governed by the following differential equations

$$\boldsymbol{d}_i' = \boldsymbol{u} \times \boldsymbol{d}_i,$$

or

$$[\boldsymbol{d}_1 \quad \boldsymbol{d}_2 \quad \boldsymbol{d}_3]' = [\boldsymbol{d}_1 \quad \boldsymbol{d}_2 \quad \boldsymbol{d}_3] \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}, \quad \text{or} \quad \boldsymbol{d}' = \boldsymbol{d}\mathbf{u}^{\times}, \tag{6}$$

where, in general, for any triple $\mathbf{u}$ of real numbers we adopt the notation

$$\mathbf{u}^{\times} \equiv \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}, \tag{7}$$

and specifically here $u_i = \boldsymbol{u} \cdot \boldsymbol{d}_i$. In fact,

$$u_i = \epsilon_{ijk} \boldsymbol{d}_j' \cdot \boldsymbol{d}_k, \tag{8}$$

where ϵ_{ijk} denotes the alternating tensor, or

$$\mathbf{u}^{\times} = \boldsymbol{d}^{\mathrm{T}} \boldsymbol{d}'. \tag{9}$$

The component u_3 is called the *twist* of the frame about $\boldsymbol{d}_3$, which is, in general, different from the geometric torsion τ, even for adapted framings. Comparison between (6) and (5) reveals that the Frenet triad is a special adapted framing for which $u_1 \equiv 0$, $u_2 \equiv \lambda\kappa$, and $u_3 \equiv \lambda\tau$, where the factor $\lambda = \sigma' = \mathrm{d}\sigma/\mathrm{d}s$ accounts for the fact that s need not be an arc-length parametrization. In the next section we will compute the general relations between Darboux vectors of two different framings.

2.3. RELATIONS BETWEEN DARBOUX VECTORS

Given a curve $\boldsymbol{r}(s)$ and two orthonormal framings $\{\boldsymbol{d}_i(s)\}$ and $\{\boldsymbol{D}_i(s)\}$, their two direction cosine matrices $\boldsymbol{d}$ and $\boldsymbol{D}$ are related by a simple rotation

$$\boldsymbol{D} = \boldsymbol{d}\mathbf{Q}, \tag{10}$$

where each matrix can depend upon s. Relation (10) implies that $\mathbf{Q}_{ji} = \boldsymbol{D}_i \cdot \boldsymbol{d}_j$.

Let $\boldsymbol{u}(s)$ be the Darboux vector defined in (6) associated with the frame $\{\boldsymbol{d}_i(s)\}$ with components $\mathbf{u}$ satisfying (9), and let $\boldsymbol{U}(s)$ be the Darboux vector associated with the frame $\{\boldsymbol{D}_i(s)\}$, with components $\mathbf{U}$ in the basis $\boldsymbol{D}_i$, satisfying

$$\mathbf{U}^{\times} = \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D}'. \tag{11}$$

Trivially we have that

$$\boldsymbol{u}(s) = \boldsymbol{U}(s) + \boldsymbol{p}(s) \tag{12}$$

for some vector field $\boldsymbol{p}(s)$ (which is merely an appropriate expression of the fact that vectors can be added). We next demonstrate that the components $\mathbf{p}$ of $\boldsymbol{p}$ in the $\{\boldsymbol{d}_i(s)\}$ frame satisfy

$$\mathbf{p}^{\times} = -\mathbf{Q}'\mathbf{Q}^{\mathrm{T}}, \tag{13}$$

or equivalently

$$\mathbf{u}^{\times} = (\mathbf{Q}\mathbf{U})^{\times} - \mathbf{Q}'\mathbf{Q}^{\mathrm{T}}. \tag{14}$$

In fact, differentiation of (10) yields

$$\boldsymbol{D}' = \boldsymbol{d}'\mathbf{Q} + \boldsymbol{d}\mathbf{Q}', \tag{15}$$

which is equivalent to

$$\boldsymbol{D}\mathbf{U}^{\times} = \boldsymbol{d}\mathbf{Q}\mathbf{U}^{\times} = \boldsymbol{d}\mathbf{u}^{\times}\mathbf{Q} + \boldsymbol{d}\mathbf{Q}', \tag{16}$$

which implies

$$\mathbf{u}^{\times} = \mathbf{Q}\mathbf{U}^{\times}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}'\mathbf{Q}^{\mathrm{T}}. \tag{17}$$

A simple computation then yields (14).

We remark that $\mathbf{Q}'\mathbf{Q}^{\mathrm{T}}$ is a skew matrix, so that there is a set of three numbers $\mathbf{p}$ such that $-\mathbf{Q}'\mathbf{Q}^{\mathrm{T}} = \mathbf{p}^{\times}$. The expression (14) then becomes equivalent to

$$\mathbf{u} = \mathbf{Q}\mathbf{U} + \mathbf{p}, \tag{18}$$

which is equivalent to the vector identity (12) expressed in the $\{\boldsymbol{d}_i\}$ basis.

In the particular case that the two frames $\{\boldsymbol{D}_i(s)\}$ and $\{\boldsymbol{d}_i(s)\}$ share a common vector, say $\boldsymbol{D}_3(s) \equiv \boldsymbol{d}_3(s)$, as would arise in the case of any two adapted framings, the rotation matrix $\mathbf{Q}$ in (10) takes the particular form

$$\mathbf{Q}(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

and (13) implies that (18) reduces to

$$\mathbf{u} = \mathbf{QU} - \varphi'\mathbf{d}_3 = \mathbf{QU} + \varphi'[0, 0, 1]^{\mathrm{T}}, \tag{20}$$

so that $\boldsymbol{p} = -\varphi'\boldsymbol{d}_3$. In particular, $u_1^2 + u_2^2 = U_1^2 + U_2^2$ and $u_3 - U_3 = \varphi'$.

In the further case that $\{\boldsymbol{D}_i\}$ is the Frenet frame, so that $\boldsymbol{D}_3 \equiv \boldsymbol{\tau} \equiv \boldsymbol{d}_3$, and $\boldsymbol{U} = \lambda\kappa\boldsymbol{\beta} + \lambda\tau\boldsymbol{\tau}$, we recover the relations

$$u_1 = -\lambda\kappa\sin\varphi, \qquad u_2 = \lambda\kappa\cos\varphi, \qquad u_3 = \lambda\tau + \varphi', \tag{21}$$

where as before, the scalar factor $\lambda = \mathrm{d}\sigma/\mathrm{d}s$ accounts for the fact that s need not be an arc-length parametrization. In particular, when s is an arc-length parametrization then $\lambda = 1$ and relations (21) are all classic results (cf. [13, p. 383]).

2.4. CIRCULAR HELICES

A circular helix is a curve in space given by parametric equations of the form

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = p\theta, \tag{22}$$

where r is the radius of the helix, and p is a constant giving the pitch, so that the vertical separation between loops along a generator of the helix equals $2\pi p$. When the pitch tends to zero at constant radius the helix degenerates to a circle, and when the radius r tends to zero at constant pitch, the helix degenerates to a straight line. A circular helix has constant curvature κ and torsion τ

$$\kappa = \frac{r}{r^2 + p^2}, \qquad \tau = \frac{p}{r^2 + p^2}, \tag{23}$$

which relations can be inverted to parametrize the helix by κ and τ:

$$r = \frac{\kappa}{\kappa^2 + \tau^2}, \qquad p = \frac{\tau}{\kappa^2 + \tau^2}. \tag{24}$$

2.5. COSSERAT THEORY OF ELASTIC RODS

In this section we review, rather briefly, the theory of elastic rods introduced by the Cosserat brothers [2, 4]. The configuration of a special Cosserat rod is a smooth vector function $\boldsymbol{r}$ and a pair of orthonormal vector functions $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ of the single variable s with $0 \leqslant s \leqslant L$. The vector $\boldsymbol{r}(s)$ gives the position in Euclidean 3-space

$\mathbb{E}^3$ of a material point s on the rod. The curve $\{\boldsymbol{r}(s),\ s \in [0, L]\}$ is called the center line of the rod. The unit vectors $\boldsymbol{d}_1(s)$ and $\boldsymbol{d}_2(s)$ give information about the orientation of the material cross-section at s in the deformed configuration. If we define the additional vector function $\boldsymbol{d}_3$ by $\boldsymbol{d}_3(s) = \boldsymbol{d}_1(s) \times \boldsymbol{d}_2(s)$, we obtain at each s an orthonormal frame $\{\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s)\}$. The unit vectors $\boldsymbol{d}_i(s)$ are called directors.

The configuration of the rod can alternatively be described by the pair $(\boldsymbol{r}, \boldsymbol{d})$, where $\boldsymbol{d}$ is the direction cosine matrix of the $\boldsymbol{d}_i$ frame. The kinematics of the rod are described by two strain vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ through the relations

$$\boldsymbol{r}'(s) = \boldsymbol{v}(s), \tag{25a}$$

$$\boldsymbol{d}_i'(s) = \boldsymbol{u}(s) \times \boldsymbol{d}_i(s), \tag{25b}$$

where $'$ denotes differentiation with respect to the independent variable s.

The components $v_k = \boldsymbol{v} \cdot \boldsymbol{d}_k$ of the vector $\boldsymbol{v}$ with respect to the orthonormal basis $\{\boldsymbol{d}_k\}$ are the strain variables of the axial curve: v_1 and v_2 are associated with shear, while v_3 is associated with stretch or compression. The components $u_k = \boldsymbol{u} \cdot \boldsymbol{d}_k$ of the Darboux vector $\boldsymbol{u}$ are the strain variables of the directors: u_1 and u_2 are associated with flexure, and u_3 is associated with twist.

The stresses acting across the material cross-section at s are equivalent to a resultant force $\boldsymbol{n}(s)$ and a resultant moment $\boldsymbol{m}(s)$ applied at $\boldsymbol{r}(s)$. When the only external loads are couples and forces applied at the ends of the rod, which is the case of interest here, balance of forces and moments yields the equilibrium equations

$$\boldsymbol{n}'(s) = \boldsymbol{0}, \qquad \boldsymbol{m}'(s) + \boldsymbol{r}'(s) \times \boldsymbol{n}(s) = \boldsymbol{0}. \tag{26}$$

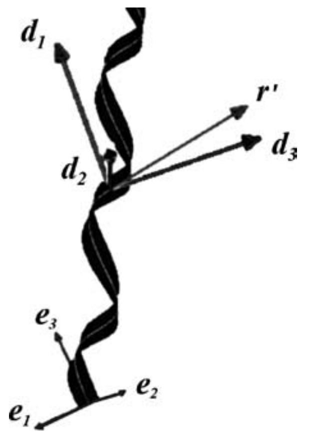

Figure 1. A ribbon representation of the configuration of an extensible, shearable Cosserat rod. The center curve is parametrized by undeformed arc length s. The location of the material point at s is given by its position $\boldsymbol{r}(s)$ with respect to the origin. The orthonormal directors $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ and $\boldsymbol{d}_3$ describe the orientation of the material cross-section of the rod at s, with $\boldsymbol{d}_1$ determining the ribbon. The components $\mathbf{v}$ of $\boldsymbol{r}'$ and $\mathbf{u}$ of the Darboux vector $\boldsymbol{u}$ (in the $\boldsymbol{d}_i$ basis) determine the strains of shear and extension and of flexure and twist.

Here we shall make the assumption that the rod is uniform and hyperelastic.

DEFINITION 2.1. For a uniform, extensible, shearable, hyperelastic rod, there is a convex, coercive strain-energy density function W defined on a domain

$$\mathcal{V} = \mathbb{R}^3 \times \mathbb{R}^2 \times (-1, \infty) \subset \mathbb{R}^3 \times \mathbb{R}^3 \tag{27}$$

such that (cf. [2, p. 277])

$$W\colon\ \mathcal{V} \to \mathbb{R}^+, \quad (\mathbf{z}, \mathbf{w}) \mapsto W(\mathbf{z}, \mathbf{w})$$

and

$$\frac{\partial W}{\partial \mathbf{w}}(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \qquad \frac{\partial W}{\partial \mathbf{z}}(\mathbf{0}, \mathbf{0}) = \mathbf{0},$$

with the property that the triples $\mathbf{n}$ and $\mathbf{m}$ of components of the resultant force $\boldsymbol{n}$ and moment $\boldsymbol{m}$ in the director frame $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ are related to the triples $\mathbf{u}$ and $\mathbf{v}$ of components of the strains in the same basis via the constitutive relations

$$\mathbf{m} = \frac{\partial W}{\partial \mathbf{u}}(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{v} - \hat{\mathbf{v}}), \qquad \mathbf{n} = \frac{\partial W}{\partial \mathbf{v}}(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{v} - \hat{\mathbf{v}}). \tag{28}$$

The triples $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are the strains in the reference configuration and there is no loss of generality in assuming that

$$\hat{\mathbf{v}} = [0, 0, 1]^{\mathrm{T}}, \tag{29}$$

so that the parameter s is an arc length in the undeformed configuration. Uniformity implies that the strain-energy density has no explicit dependence on s and that $\hat{\mathbf{u}}$ is constant, so that the center line of the reference state is helical.

By coercive we mean that the function W tends to infinity as its argument approaches the boundary (including points at infinity) of the domain $\mathcal{V}$ defined in (27). In this definition the asymmetry in the last argument is related to the particular choice (29) for $\hat{\mathbf{v}}$, and the requirement that configurations preserve orientation, that is we require that

$$v_3 = \boldsymbol{v} \cdot \boldsymbol{d}_3 > 0, \tag{30}$$

in any finite energy configuration of the rod, and with the shifted argument of W, consequently arrive at the choice of domain (27). More sophisticated choices of domain could also be made to ensure (30) and other physically appropriate kinematic restrictions; see [2] for a more detailed discussion.

In the case of linear constitutive relations, the function W appearing in Hypothesis 2.1 is quadratic in the strains, and the constitutive relations (28) take the particular form

$$\begin{pmatrix} \mathbf{m} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^{\mathrm{T}} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{u} - \hat{\mathbf{u}} \\ \mathbf{v} - \hat{\mathbf{v}} \end{pmatrix}, \tag{31}$$

where $\mathbf{K}$ and $\mathbf{A}$ are symmetric positive-definite 3×3 matrices, and the 3×3 matrix $\mathbf{G}$ is such that the 6×6 partitioned matrix appearing in (31) is also positive-definite and symmetric. In many applications this linear case is assumed, but of course the associated quadratic energy does not, *a priori*, ensure the constraint (30). Rather it must be verified, *a posteriori*, that the solutions that are constructed are physically reasonable.

Because W is convex and coercive, the constitutive relations (28) can be inverted to yield

$$\mathbf{u} = \frac{\partial W^*}{\partial \mathbf{m}}(\mathbf{m}, \mathbf{n}) + \hat{\mathbf{u}}, \qquad \mathbf{v} = \frac{\partial W^*}{\partial \mathbf{n}}(\mathbf{m}, \mathbf{n}) + \hat{\mathbf{v}}, \tag{32}$$

where $W^*(\mathbf{y})$ is the Legendre transform of $W(\mathbf{x})$, i.e.,

$$W^*(\mathbf{y}) = \sup_{\mathbf{x}\in\mathbb{R}^6} \{\mathbf{y}\cdot\mathbf{x} - W(\mathbf{x})\}. \tag{33}$$

Then the Legendre transform of the function $W(\mathbf{u}-\hat{\mathbf{u}}, \mathbf{v}-\hat{\mathbf{v}})$ of shifted arguments is the shifted Legendre transform

$$W^*(\mathbf{m}, \mathbf{n}) + \mathbf{m}\cdot\hat{\mathbf{u}} + \mathbf{n}\cdot\hat{\mathbf{v}}. \tag{34}$$

2.6. INEXTENSIBLE AND UNSHEARABLE RODS

The rod is said to be inextensible if in any configuration $|\boldsymbol{r}'| \equiv 1$, and to be unshearable if $v_1 := \boldsymbol{v}\cdot\boldsymbol{d}_1 \equiv 0$ and $v_2 := \boldsymbol{v}\cdot\boldsymbol{d}_2 \equiv 0$. Thus, for inextensible and unshearable rods

$$\mathbf{v} \equiv \hat{\mathbf{v}} := [0, 0, 1]^{\mathrm{T}}, \tag{35}$$

in which case, equation (25a) may be rewritten simply as

$$\boldsymbol{r}'(s) = \boldsymbol{d}_3(s), \tag{36}$$

and the parameter s can be interpreted as an arc length in any configuration.

DEFINITION 2.2. For a uniform, inextensible, unshearable, hyperelastic rod, there is a convex, coercive strain-energy density function W

$$W\colon \mathbb{R}^3 \to \mathbb{R}^+, \quad \mathbf{z} \mapsto W(\mathbf{z})$$

with

$$\frac{\partial W}{\partial \mathbf{z}}(\mathbf{0}) = \mathbf{0},$$

such that the moment $\mathbf{m}$ is given by a constitutive relation

$$\mathbf{m} = \frac{\partial W}{\partial \mathbf{u}}(\mathbf{u} - \hat{\mathbf{u}}). \tag{37}$$

Uniformity implies that the strain-energy density has no explicit dependence on s and that $\hat{\mathbf{u}}$ is constant, so that the unstressed state is helical.

For inextensible, unshearable rods the strain-energy density W is a function of the strains $\mathbf{u}$ only, only the moment $\mathbf{m}$ is given via a constitutive relation, and the force $\boldsymbol{n}$ becomes a reactive parameter that is to be determined from the equilibrium equations. In the case where the strain-energy density W is quadratic in the strains $\mathbf{u}$, we obtain a linear constitutive relation of the form

$$\mathbf{m} = \mathbf{K}(\mathbf{u} - \hat{\mathbf{u}}), \tag{38}$$

where $\mathbf{K}$ is a symmetric 3×3 positive-definite matrix. When $\mathbf{K}$ is a constant diagonal matrix with equal bending rigidities, we recover the classic Kirchhoff theory of elastic rods.

Just as in the extensible, shearable case, the Legendre transform of the function $W(\mathbf{u} - \hat{\mathbf{u}})$ of shifted arguments can be computed to be the shifted Legendre transform

$$W^*(\mathbf{m}) + \mathbf{m} \cdot \hat{\mathbf{u}}, \tag{39}$$

$$\mathbf{u} = \frac{\partial W^*}{\partial \mathbf{m}}(\mathbf{m}) + \hat{\mathbf{u}}. \tag{40}$$

2.7. ISOTROPIC RODS

In continuum mechanics, a material is called *isotropic* if it has no preferred material direction. If the response of the material is the same for all directions orthogonal to a given one, then the material is called *transversely isotropic*. In the theory of Cosserat rods, the direction defined by the strain $\hat{\mathbf{v}}$ in the reference configuration is distinct from others, and only transverse isotropy to this direction is a pertinent notion. Thus in the following, we suppress the adjective transversally and discuss isotropic or non-isotropic rods.

DEFINITION 2.3. For isotropic, hyperelastic rods, the strain-energy function W is invariant under rotations of its arguments about $\hat{\mathbf{v}} = [0, 0, 1]^{\mathrm{T}}$. For extensible, shearable rods this means

$$\frac{\partial}{\partial \alpha} W(\mathbf{Q}(\alpha)\mathbf{w}, \mathbf{Q}(\alpha)\mathbf{z}) = 0, \tag{41}$$

for $\mathbf{Q}(\alpha)$ defined in (19) with $0 \leqslant \alpha \leqslant 2\pi$, and $\mathbf{w}, \mathbf{z} \in \mathbb{R}^3$, while for inextensible, unshearable rods we require

$$\frac{\partial}{\partial \alpha} W(\mathbf{Q}(\alpha)\mathbf{w}) = 0. \tag{42}$$

A rod with $\hat{\mathbf{v}} = [0, 0, 1]^{\mathrm{T}}$, is isotropic if (41), or (42), holds and if, in addition, the bending strains $\hat{u}_1$ and $\hat{u}_2$ in the reference configuration vanish [2].

We remark that in his work on symmetries of rods, Healey [10] defines isotropic strain-energy functions to be ones that are invariant under both proper and improper rotations. He uses the terminology *hemitropy* to refer to what is called isotropy here. It is shown by Healey that for inextensible and unshearable rods with linear constitutive relations, hemitropy and isotropy coincide, but that they can differ in the case of extensible and shearable rods.

2.8. NONCANONICAL HAMILTONIAN FORMULATION

The equations governing the equilibrium conditions of elastic rods written entirely in terms of components in the director frame are

$$\mathbf{m}' = \mathbf{m} \times \mathbf{u} + \mathbf{n} \times \mathbf{v}, \qquad \mathbf{n}' = \mathbf{n} \times \mathbf{u}, \tag{43}$$

which can be recognized as the equations of a Hamiltonian system with associated Hamiltonian (cf., e.g., [11])

$$H(\mathbf{m}, \mathbf{n}) := W^*(\mathbf{m}, \mathbf{n}) + \mathbf{m} \cdot \hat{\mathbf{u}} + \mathbf{n} \cdot \hat{\mathbf{v}} \tag{44}$$

for extensible, shearable rods, or

$$H(\mathbf{m}, \mathbf{n}) := W^*(\mathbf{m}) + \mathbf{m} \cdot \hat{\mathbf{u}} + \mathbf{n} \cdot \hat{\mathbf{v}} \tag{45}$$

for inextensible, unshearable rods. Note that for either Hamiltonian the relations (32), (40) and (35) imply that

$$\frac{\partial H}{\partial \mathbf{m}}(\mathbf{m}, \mathbf{n}) = \mathbf{u}, \qquad \frac{\partial H}{\partial \mathbf{n}}(\mathbf{m}, \mathbf{n}) = \mathbf{v}. \tag{46}$$

The associated Hamiltonian structure is given by

$$\begin{pmatrix} \mathbf{m} \\ \mathbf{n} \end{pmatrix}' = \mathbf{J}(\mathbf{m}, \mathbf{n}) \nabla H(\mathbf{m}, \mathbf{n}), \tag{47}$$

where $\mathbf{J}$ is the skew-symmetric operator defined by

$$\mathbf{J}(\mathbf{m}, \mathbf{n}) = \begin{pmatrix} \mathbf{m}^{\times} & \mathbf{n}^{\times} \\ \mathbf{n}^{\times} & \mathbf{0} \end{pmatrix}. \tag{48}$$

We remark that when $|\mathbf{n}| \neq 0$, the null space of $\mathbf{J}$ is the two-dimensional space spanned by

$$\begin{pmatrix} \mathbf{n} \\ \mathbf{m} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{0} \\ \mathbf{n} \end{pmatrix}.$$

2.9. ANALYSIS OF INTEGRALS

The integrals of the governing equations can be classified as follows:

– *Constitutive-independent integrals*. Equations (26) have the integrals

$$\boldsymbol{n}(s) = \boldsymbol{n}(0), \qquad \boldsymbol{m}(s) + \boldsymbol{r}(s) \times \boldsymbol{n}(s) = \boldsymbol{c}. \tag{49}$$

That is, the three components of force $\boldsymbol{n}$ in the fixed basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are constant, due to translational symmetry in space, and the three components of $\boldsymbol{m} + \boldsymbol{r} \times \boldsymbol{n}$ in the fixed basis are constant, as a consequence of rotational symmetry about the three fixed axes. For the reduced variables of the noncanonical Hamiltonian formulation (47), the only integrals that persist are $I_1 = \mathbf{n} \cdot \mathbf{n}$ and $I_2 = \mathbf{m} \cdot \mathbf{n}$, which are functions of the director components $\mathbf{m}$ and $\mathbf{n}$ only. The integrals I_1 and I_2 are sometimes called Casimir functions because their gradients lie in the null space of the structure matrix (48).

– *Constitutive-dependent integrals*. Further integrals arise in each of the cases of a uniform rod, and an isotropic rod. The appropriate Hamiltonian (44) or (45) is an integral for uniform rods. And if the rod is isotropic then one can show that the twisting moment $m_3 = \mathbf{m} \cdot \mathbf{d}_3 = I_3$ is another independent integral.

For our analysis the reduced noncanonical Hamiltonian system (47) will prove to be convenient. In particular the integrals other than the Hamiltonian are quadratic in the reduced phase variables, and the Hamiltonian itself is also quadratic in the special case of linear elasticity. As discussed in [11], for example, the remaining integrals can be used to reconstruct the center line.

3. Relative Equilibria of Non-isotropic Rods

3.1. VARIATIONAL CHARACTERIZATION OF RELATIVE EQUILIBRIA

In general, relative equilibria of an autonomous Hamiltonian system are characterized by points $\mathbf{z}$ in phase space satisfying the equation

$$\nabla H(\mathbf{z}) = \sum_{i=1}^{N} \lambda_i \nabla I_i(\mathbf{z}), \tag{50}$$

where $H(\mathbf{z})$ is the Hamiltonian and $I_i(\mathbf{z})$ are other integrals of the system. For a uniform, but non-isotropic rod, we have only the two integrals I_1 and I_2 with

$$\nabla I_1 = \begin{pmatrix} \mathbf{0} \\ \mathbf{n} \end{pmatrix}, \qquad \nabla I_2 = \begin{pmatrix} \mathbf{n} \\ \mathbf{m} \end{pmatrix}.$$

Solutions of equation (50) therefore satisfy

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \lambda_2 \mathbf{I} & \mathbf{0} \\ \lambda_1 \mathbf{I} & \lambda_2 \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{n} \\ \mathbf{m} \end{pmatrix}. \tag{51}$$

Here the expression for the gradient of the Hamiltonian follows from (46), and $\mathbf{I}$ is the 3×3 identity matrix. The relation (51) is merely a particular case of the general and standard form (50). However, because the integrals I_1 and I_2 are quadratic in the Hamiltonian variables $\mathbf{m}$ and $\mathbf{n}$, we have the rather special feature that the right-hand side of equation (51) has the form of a constant-coefficient linear system. In the following, we suppose that $\lambda_2 \neq 0$; the case $\lambda_2 = 0$ can be analyzed as a limit problem in which the strains are $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = \lambda_1 \mathbf{n}$, which correspond to configurations with straight center lines, constant director frame, force parallel to the center line, and constant moment. For $\lambda_2 \neq 0$, equation (51) is equivalent to

$$\begin{pmatrix} \mathbf{n} \\ \mathbf{m} \end{pmatrix} = \begin{pmatrix} \mu_2 \mathbf{I} & \mathbf{0} \\ \mu_1 \mathbf{I} & \mu_2 \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}, \tag{52}$$

or

$$\frac{\partial W}{\partial \mathbf{v}}(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{v} - \hat{\mathbf{v}}) = \mu_2 \mathbf{u}, \tag{53a}$$

$$\frac{\partial W}{\partial \mathbf{u}}(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{v} - \hat{\mathbf{v}}) = \mu_1 \mathbf{u} + \mu_2 \mathbf{v}. \tag{53b}$$

The system (52) can be re-inverted to recover (51) provided that $\mu_2 \neq 0$. For $\mu_2 \neq 0 \neq \lambda_2$, the constants μ_1 and μ_2 are related to λ_1 and λ_2 via the symmetrically invertible relations

$$\mu_1 = -\frac{\lambda_1}{\lambda_2^2}, \qquad \mu_2 = \frac{1}{\lambda_2}, \tag{54}$$

$$\lambda_1 = -\frac{\mu_1}{\mu_2^2}, \qquad \lambda_2 = \frac{1}{\mu_2}. \tag{55}$$

And $\mu_2 = 0$ corresponds to $\mathbf{n} = \mathbf{0}$, in which case the system (47) is completely integrable, and easily analyzed to conclude that solutions of (52) correspond to helical equilibria with vanishing force.

Accordingly, for given constitutive relations, the variables characterizing relative equilibria can be identified either as $\mathbf{m}$ and $\mathbf{n} \neq \mathbf{0}$, or as $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v}$, with the limit cases $\mu_2 = 0$ and $\lambda_2 = 0$ treated separately. If we introduce the constants C_1 and C_2

$$\tfrac{1}{2}\mathbf{n} \cdot \mathbf{n} = \tfrac{1}{2}C_1^2, \qquad \mathbf{n} \cdot \mathbf{m} = C_2 C_1, \tag{56}$$

we deduce from equations (52) and (56) that

$$\tfrac{1}{2}\mathbf{u} \cdot \mathbf{u} = \tfrac{1}{2}\eta_1^2, \qquad \mathbf{u} \cdot \mathbf{v} = \eta_2 \eta_1, \tag{57}$$

for some constants η_1, η_2, with

$$\begin{pmatrix} \lambda_2 & 0 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} C_1^2 \\ C_1 C_2 \end{pmatrix} = \mu_2 \begin{pmatrix} \eta_1^2 \\ \eta_1 \eta_2 \end{pmatrix}, \tag{58}$$

or, equivalently

$$\lambda_2 \begin{pmatrix} C_1^2 \\ C_1 C_2 \end{pmatrix} = \begin{pmatrix} \mu_2 & 0 \\ \mu_1 & \mu_2 \end{pmatrix} \begin{pmatrix} \eta_1^2 \\ \eta_1 \eta_2 \end{pmatrix}. \tag{59}$$

Equations (51) are the first-order necessary conditions associated with the variational problem

$$\text{Minimize } H(\mathbf{m}, \mathbf{n}) \tag{60}$$

subject to the constraints (56). On the other hand, with constitutive relations (28), equations (52) can be recognized as the first-order necessary conditions associated with the dual variational problem

$$\text{Minimize } W(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{v} - \hat{\mathbf{v}}) \tag{61}$$

subject to the constraints (57). It is this dual formulation that will be exploited in our analysis of existence and multiplicity of relative equilibria.

The analysis presented above is for the general case of an extensible, shearable rod, but the dual formulation is particularly simple in the inextensible, unshearable case. Then $\mathbf{v}$ satisfies the constraint (35), and the characterization (61) reduces to

$$\text{Minimize } W(\mathbf{u} - \hat{\mathbf{u}}) \tag{62}$$

subject to

$$\tfrac{1}{2}\mathbf{u} \cdot \mathbf{u} = \tfrac{1}{2}\eta_1^2, \qquad \mathbf{u} \cdot \hat{\mathbf{v}} = \eta_1 \eta_2, \tag{63}$$

whose critical points satisfy (52b), while at relative equilibrium the internal force $\mathbf{n}$ is determined by (52a).

3.2. EXISTENCE AND MULTIPLICITY OF RELATIVE EQUILIBRIA

For any explicitly given strain energy function, locally two-dimensional families of relative equilibria, i.e., solutions of either the primal or dual first-order necessary conditions, can (in principle) be found either analytically or numerically in terms of the two associated Lagrange multipliers. For example, the case of linear, diagonal constitutive relations is completely treated in [3]. Here we prefer to use the dual form of the variational principles described in the previous section to prove existence and to analyze multiplicity of relative equilibria for general strain energy functions. The analysis is completely straightforward in the inextensible, unshearable case, but is slightly more delicate for extensible, shearable rods.

LEMMA 3.1. (i) *For any strain energy density function W satisfying Definition* 2.2 *of an inextensible and unshearable rod, any* $\eta_1 \in (0, \infty)$*, and any* $\eta_2 \in (-1, 1)$*, there exist at least two critical points of the variational principle* (62) *subject to the constraints* (63)*.*

(ii) *For any strain energy density function W satisfying Definition* 2.1 *of an extensible and shearable rod, any* $\eta_1 \in (0, \infty)$ *and any* η_2 *there exists a critical point of the variational principle* (61) *subject to the constraints* (57)*, and for* $|\eta_2|$ *sufficiently small there exist at least two such critical points.*

Proof. (i) With constraint (35) equation (63b) is equivalent to $u_3 = \eta_1\eta_2$, so that for any $\eta_2 \in (-1, 1)$, u_3 takes a prescribed, admissible value. Then the remaining constraint (63a) can be satisfied via parametrization with a scalar $\xi \in [0, 2\pi]$ by

$$u_1 = \eta_1\sqrt{1-\eta_2^2}\cos\xi, \qquad u_2 = \eta_1\sqrt{1-\eta_2^2}\sin\xi. \tag{64}$$

The constrained variational principle is then reduced to finding unconstrained extrema of the continuous, real-valued, periodic function

$$\widetilde{W}(\xi) := W\Big(\eta_1\sqrt{1-\eta_2^2}\cos\xi - \hat{u}_1, \eta_1\sqrt{1-\eta_2^2}\sin\xi - \hat{u}_2, \eta_1\eta_2 - \hat{u}_3\Big) \tag{65}$$

of the real variable ξ. The function $\widetilde{W}$ achieves its maximum and minimum so that there are at least two critical points for each allowable choice of η_1 and η_2. There can of course be more than two critical points. Figure 2 illustrates various possibilities for the particular quadratic, diagonal strain energy density function

$$W = \tfrac{1}{2}\sum_1^3 K_i(u_i - \hat{u}_i)^2. \tag{66}$$

(ii) In the extensible, shearable case we prove existence of solutions for the first-order necessary conditions (53) from consideration of two successive variational principles. The first variational problem comprises for each $\eta_1 > 0$ and each η_2, minimization of the functional

$$W(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{v} - \hat{\mathbf{v}})$$

regarded as a function of $\mathbf{v}$ only, and subject to the linear constraints

$$\hat{\mathbf{v}}\cdot\mathbf{v} > 0, \qquad \frac{\mathbf{u}}{\eta_1}\cdot\left(\mathbf{v} - \eta_2\frac{\mathbf{u}}{\eta_1}\right) = 0, \tag{67}$$

where the first constraint is (30), and the second is the second constraint in (57) re-expressed using the first. Dependent upon $\mathbf{u}$ and the sign of η_2, the set of $\mathbf{v}$ satisfying constraints (67) is either the empty set or an affine half-space. Because W is a convex and coercive function of $\mathbf{v}$, and the constraint set is affine and therefore convex, for each $\mathbf{u}$ and η_2 such that the constraint is non-empty, there exists a unique minimizer $\mathbf{V}(\mathbf{u}; \eta_1, \eta_2)$, which lies in the interior of the domain (27). Moreover, there is a unique Lagrange multiplier $\mu_2(\mathbf{u}; \eta_1, \eta_2)$ such that

$$\frac{\partial W}{\partial \mathbf{v}}(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{V}(\mathbf{u}) - \hat{\mathbf{v}}) = \mu_2\mathbf{u}. \tag{68}$$

Here we suppress the dependence on η_1 and η_2 because they will be held fixed throughout, whereas in the second stage $\mathbf{u}$ will be allowed to vary. For $\mathbf{u}$ for which it is defined, the function $\mathbf{V}(\mathbf{u})$ satisfies both (68) and

$$\mathbf{u}\cdot\mathbf{V}(\mathbf{u}) = \eta_1\eta_2. \tag{69}$$

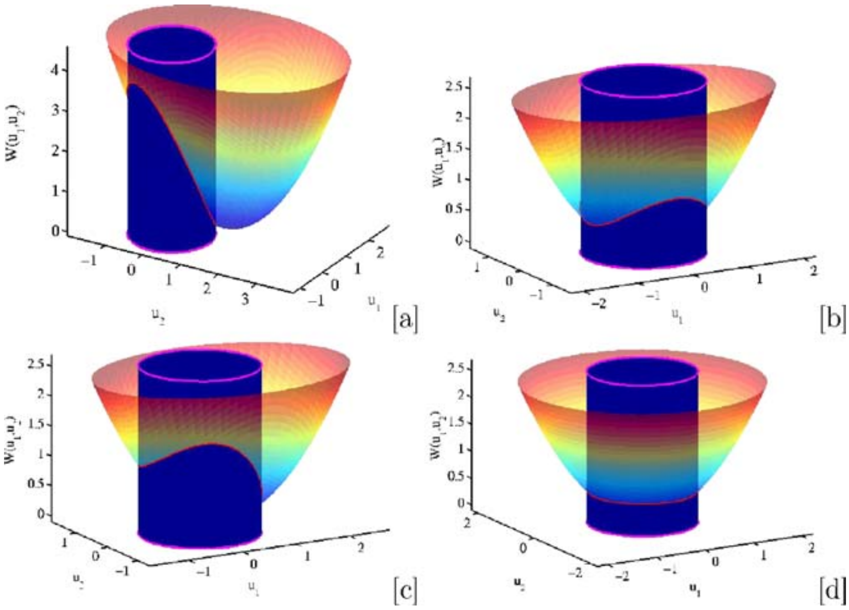

Figure 2. Illustration of the numbers of critical points of the function $\widetilde{W}(\xi)$ defined in (65) in the particular case of the quadratic, diagonal strain energy density function (66). Each critical point corresponds to a relative equilibrium, or helical configuration, of the associated inextensible, unshearable rod. [a]: nonsymmetric case ($K_1 = 1$, $K_2 = 2$, $\hat{u}_1 = 1$, $\hat{u}_2 = 0.8$, $\hat{u}_3 = 0$) with two critical points. [b]: case of discrete symmetry ($K_1 = 1$, $K_2 = 2$, $\hat{u}_1 = 0$, $\hat{u}_2 = 0$, $\hat{u}_3 = 0$) with four critical points. [c]: nonsymmetric case ($K_1 = 1$, $K_2 = 2$, $\hat{u}_1 = 0$, $\hat{u}_2 = 0.2$, $\hat{u}_3 = 0$) with four critical points. [d]: case of continuous symmetry of isotropy ($K_1 = 1$, $K_2 = 1$, $\hat{u}_1 = 0$, $\hat{u}_2 = 0$, $\hat{u}_3 = 0$) with an orbit of critical points.

The identity (69) can be differentiated with respect to $\mathbf{u}$, to yield

$$\frac{\partial \mathbf{V}^{\mathrm{T}}(\mathbf{u})}{\partial \mathbf{u}}\mathbf{u} = -\mathbf{V}(\mathbf{u}). \tag{70}$$

The second variational principle then comprises finding extrema over the unknown $\mathbf{u}$ of the composite function

$$W(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{V}(\mathbf{u}) - \hat{\mathbf{v}}) \tag{71}$$

subject to the constraint

$$\mathbf{u} \cdot \mathbf{u} = \eta_1^2.$$

At any critical point we have the first-order necessary condition

$$\frac{\partial W}{\partial \mathbf{u}}(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{V}(\mathbf{u}) - \hat{\mathbf{v}}) + \frac{\partial \mathbf{V}^{\mathrm{T}}(\mathbf{u})}{\partial \mathbf{u}}\frac{\partial W}{\partial \mathbf{v}} = \mu_1 \mathbf{u},$$

which, with the two properties (68) and (70) of $\mathbf{V}$, is equivalent to

$$\frac{\partial W}{\partial \mathbf{u}}(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{V}(\mathbf{u}) - \hat{\mathbf{v}}) = \mu_1 \mathbf{u} - \mu_2 \frac{\partial \mathbf{V}^{\mathrm{T}}(\mathbf{u})}{\partial \mathbf{u}}\mathbf{u} = \mu_1 \mathbf{u} + \mu_2 \mathbf{V}.$$

In other words, the two successive variational principles generate the desired critical points of the original constrained variational principle.

It therefore only remains to obtain a better understanding of the number of critical points as a function of η_1 and η_2. Note that the constant $\eta_1 > 0$ merely serves to normalize $\mathbf{u}$. We first consider the case $\eta_2 = 0$. Then the constraint set (67) is non-empty provided $\mathbf{u} \neq \pm\eta_1\hat{\mathbf{v}}$. Moreover, as $\mathbf{u} \to \pm\eta_1\hat{\mathbf{v}}$, the composite function (71) approaches infinity, because the entire constraint set satisfying (67), and therefore $\mathbf{V}(\mathbf{u})$ in particular, approaches the boundary of the domain (27) and W is by definition (2.1) coercive. Thus the second variational principle involves a potential defined on a sphere except for two poles at antipodal points. Consequently there is at least one valley separating the two infinities, and so at least two critical points along the valley, a global minimum and a saddle point.

For $0 < \eta_2 < 1$ the constraint set (67) is non-empty provided $\mathbf{u} \neq -\eta_1\hat{\mathbf{v}}$, and as $\mathbf{u} \to -\eta_1\hat{\mathbf{v}}$, the composite function (71) approaches infinity. Now the potential remains finite as $\mathbf{u} \to +\eta_1\hat{\mathbf{v}}$, but by continuity the potential will be large there, and so for η_2 sufficiently small there will be at least three critical points, a global minimum, a saddle along a valley, and a local maximum close to $\mathbf{u} = \eta_1\hat{\mathbf{v}}$. However as η_2 increases, it is possible that the saddle and local maximum merge to leave a potential function on the sphere with a single pole and a single critical point, namely, the global minimum. The behavior for $\eta_2 < 0$ is analogous, but with the roles of $\hat{\mathbf{v}}$ and $-\hat{\mathbf{v}}$ reversed. □

It is instructive to consider the form of the function (71) in the particular case of a quadratic strain energy with coefficient matrix as in (31). As already remarked in Section 2.5, in the case of a quadratic energy the constraint (30) is not implied by finiteness of the energy, and must instead be verified *a posteriori*. Nevertheless in the case of a quadratic energy, or equivalently linear constitutive relations, the function (71) can be calculated explicitly. A short computation yields the nonquadratic function

$$\tfrac{1}{2}(\mathbf{u}-\hat{\mathbf{u}})\cdot\left[\mathbf{K}-\mathbf{G}\mathbf{A}^{-1}\mathbf{G}^{\mathrm{T}}\right](\mathbf{u}-\hat{\mathbf{u}}) + \frac{(\eta_1\eta_2 - \mathbf{u}\cdot\hat{\mathbf{v}} + \mathbf{u}\cdot\mathbf{A}^{-1}\mathbf{G}^{\mathrm{T}}(\mathbf{u}-\hat{\mathbf{u}}))^2}{\mathbf{u}\cdot\mathbf{A}^{-1}\mathbf{u}}. \tag{72}$$

The matrix $[\mathbf{K}-\mathbf{G}\mathbf{A}^{-1}\mathbf{G}^{\mathrm{T}}]$ is a Schur complement and it is a standard result that it is positive-definite because the matrix appearing in (31) is positive-definite. For rods that are close to inextensible and unshearable, i.e., rods for which the matrix $\mathbf{A}^{-1}$ has small norm, the term $(\eta_1\eta_2 - \mathbf{u}\cdot\hat{\mathbf{v}})^2/\mathbf{u}\cdot\mathbf{A}^{-1}\mathbf{u}$ will dominate, and for $|\eta_2| < 1$ the low energy regions will be close to the circle $(\eta_1\eta_2 - \mathbf{u}\cdot\hat{\mathbf{v}}) = 0$. In particular, there will be a nearby valley along which there are two critical points. It may also be observed, at least formally, that in limits $\mathbf{A}^{-1} \to \mathbf{0}$, only the finite-energy critical points will satisfy the constraint (63b), and the inextensible, unshearable variational principle of minimizing (62) subject to constraints (63) is recovered.

3.3. PROPERTIES OF RELATIVE EQUILIBRIA

With the existence of relative equilibria in hand, we now turn to a description of the properties of such configurations.

LEMMA 3.2. *At relative equilibrium, the director components* $\mathsf{m}(s)$, $\mathsf{n}(s)$, $\mathsf{u}(s)$, *and* $\mathsf{v}(s)$ *of the vectors* $\boldsymbol{m}(s)$, $\boldsymbol{n}(s)$, $\boldsymbol{u}(s)$ *and* $\boldsymbol{v}(s)$ *are constant,*

$$\mathsf{m}(s) = \mathsf{m}_0, \qquad \mathsf{n}(s) = \mathsf{n}_0, \tag{73a}$$

$$\mathsf{u}(s) = \mathsf{u}_0, \qquad \mathsf{v}(s) = \mathsf{v}_0. \tag{73b}$$

Proof. Relative equilibria are solutions of the equation

$$\nabla H(\mathsf{z}) = \lambda_1 \begin{pmatrix} \mathbf{0} \\ \mathsf{n} \end{pmatrix} + \lambda_2 \begin{pmatrix} \mathsf{n} \\ \mathsf{m} \end{pmatrix}, \tag{74}$$

where $\mathsf{z} = [\mathsf{m}, \mathsf{n}]^{\mathrm{T}}$. Multiplication of both sides of equation (74) by the 6×6 matrix J defined in (48), and use of the fact that $\frac{1}{2}\mathsf{n} \cdot \mathsf{n}$ and $\mathsf{m} \cdot \mathsf{n}$ are Casimir functions, implies

$$\mathsf{z}'(s) = \mathbf{0}, \tag{75}$$

i.e., the director-frame components $\mathsf{m}(s)$ and $\mathsf{n}(s)$ are constant. And (73b) follow directly from the (uniform) constitutive relations (32). □

Lemma 3.2 gives an explicit characterization of relative equilibria for uniform, hyperelastic rods, as being equilibria in which the director-components of both the strains and the stresses are constant. We shall refer to such solutions as ***d****-uniform.* We next characterize all ***d***-uniform equilibria. Note that because the rod is assumed to be uniform it is easy to use the constitutive relations to verify that a configuration is ***d***-uniform in m and n if and only if it is ***d***-uniform in u and v.

LEMMA 3.3. (i) *There is a family of* ***d****-uniform equilibria with* $\mathsf{u} = \mathbf{0}$. *For inextensible, unshearable rods there is no remaining freedom in the strains. For extensible, shearable rods the strain* v *is a critical point of the constrained variational principle*

$$\textit{Minimize } W(-\hat{\mathsf{u}}, \mathsf{v} - \hat{\mathsf{v}}), \tag{76}$$

for v *in the domain* (27) *and subject to the constraint*

$$|\mathsf{v}| = \alpha, \quad \alpha > 0.$$

These equilibria have the force $\boldsymbol{n}$ *parallel to a straight center line, and a uniformly translating director frame.*

(ii) *There are (at least) two families of* $\boldsymbol{d}$*-uniform equilibria with* $\mathbf{n} = \mathbf{0}$*, which are characterized by the moment* $\mathbf{m}$ *being a critical point of the constrained variational principle*

$$\text{Minimize } H(\mathbf{m}, \mathbf{0}), \tag{77}$$

subject to the constraint

$$|\mathbf{m}| = \beta, \quad \beta > 0.$$

These families have Darboux vector $\boldsymbol{u}$ *parallel to the moment* $\boldsymbol{m}$*, and include the unstressed configuration* $\mathbf{u} = \hat{\mathbf{u}}$*,* $\mathbf{v} = \hat{\mathbf{v}}$ *in the limit* $\beta \to \mathbf{0}$*.*

(iii) *The minimizing families of parts* (i) *and* (ii) *intersect at a unique* $\boldsymbol{d}$*-uniform configuration (with* $\mathbf{n} = \mathbf{0} = \mathbf{u}$*) characterized by* $\mathbf{m}$ *satisfying* (77) *(with no constraint), and, for extensible, shearable rods,* $\mathbf{v}$ *satisfying* (76) *(with no constraint).*

(iv) *All other* $\boldsymbol{d}$*-uniform equilibria are relative equilibria that satisfy equations* (51) *and* (52) *with* $\lambda_2 \neq 0 \neq \mu_2$*.*

Proof. (i) In the inextensible, unshearable case the condition $\mathbf{u} = \mathbf{0}$ completely specifies the configuration of the rod. Its center line is necessarily straight, and the balance laws (43) are satisfied for any uniform $\mathbf{n}$ (possibly vanishing) that is parallel to the tangent $\hat{\mathbf{v}}$.

In the extensible, shearable case, for each $\alpha > 0$ the function W achieves its minimum on the intersection of the constraint ball and the domain (27). The first-order necessary conditions for the constrained variational principle taken with the constitutive relations generate a pair $(\mathbf{n}, \mathbf{v})$ with $\mathbf{n}$ (possibly vanishing) parallel to $\mathbf{v}$. Given this $\mathbf{v}$ and $\mathbf{u} = \mathbf{0}$ the constitutive relations can be used again to find $\mathbf{m}$. It can be verified directly that such configurations also satisfy the balance laws (43). Thus they are $\boldsymbol{d}$-uniform equilibria.

(ii) For each $\beta > 0$ the Hamiltonian H achieves its maximum and minimum on the ball. The first-order necessary conditions for the constrained variational principle therefore generate at least two $\boldsymbol{d}$-uniform configurations satisfying the constitutive relations, of the form $(\mathbf{m}, \mathbf{0}, \mathbf{u}, \mathbf{v})$ with $\mathbf{u}$ (possibly vanishing) parallel to $\mathbf{m}$. It can be verified directly that such configurations also satisfy the balance laws (43). Thus they are $\boldsymbol{d}$-uniform equilibria.

For $\beta = 0$ the variational principle collapses, as there is only one possible competitor $\mathbf{m} = \mathbf{0}$. However, the unstressed configuration $(\mathbf{0}, \mathbf{0}, \hat{\mathbf{u}}, \hat{\mathbf{v}})$ is an equilibrium, and it is approached smoothly in the limit $\beta \to 0$ along each family because as the moment $\mathbf{m} \to \mathbf{0}$, the strain $\mathbf{u} \to \hat{\mathbf{u}}$, with the associated Lagrange multiplier tending to $\pm|\hat{\mathbf{u}}|$.

(iii) The unconstrained minimization principles (76) and (77) generate the unique strain $\mathbf{v}$ and moment $\mathbf{m}$ corresponding, via the constitutive relations, to $\mathbf{n} = \mathbf{0} = \mathbf{u}$. We remark that the associated norms $|\mathbf{v}|$ and $|\mathbf{m}|$ are the values of α and β for which the Lagrange multipliers in the corresponding constrained variational principles vanish. If $\hat{\mathbf{u}} = \mathbf{0}$, then $\alpha = 1$, $\beta = 0$, and it is the unstressed configuration that arises at the intersection of the two families.

(iv) By definition $\boldsymbol{d}$-uniform equilibria satisfy (75). It has already been argued that if $\mathbf{n} \neq \mathbf{0}$, then all solutions of (75) satisfy (74), i.e., they all are relative equilibria. The conditions $\mathbf{n} \neq \mathbf{0} \neq \mathbf{u}$ rule out the cases $\lambda_2 = 0$ or $\mu_2 = 0$ in equations (51) and (52). □

We describe relative equilibria with $\lambda_2 \neq 0 \neq \mu_2$ as *nondegenerate*. On nondegenerate relative equilibria $\mathbf{u}$ vanishes precisely if $\mathbf{n}$ vanishes. We next demonstrate that $\boldsymbol{d}$-uniform configurations have either straight or helical center lines.

LEMMA 3.4. *Framed curves satisfying the kinematics* (25) *with* $\mathbf{u}(s) \equiv \mathbf{u}_0$ *and* $\mathbf{v}(s) \equiv \mathbf{v}_0 \neq \mathbf{0}$ *have either* (i) *straight center lines when* $\mathbf{u}_0 \times \mathbf{v}_0 = \mathbf{0}$, *or* (ii) *circular helices as center lines when* $\mathbf{u}_0 \times \mathbf{v}_0 \neq \mathbf{0}$.

The helices are parametrized by $\lambda = |\mathbf{v}_0|$ *and the constants* $\eta_1 > 0$ *and* η_2, *with* $|\eta_2| < |\mathbf{v}_0|$, *appearing in the constraints* (57). *Specifically the curvature and torsion are*

$$\kappa = \frac{\eta_1}{\lambda}\sqrt{\left(1 - \frac{\eta_2^2}{\lambda^2}\right)}, \qquad \tau = \frac{\eta_1 \eta_2}{\lambda^2}. \tag{78}$$

Moreover the direction cosine matrices of the director frame $\boldsymbol{d}$ *and the Frenet frame* $\boldsymbol{F}$ *defined in* (2) *of the helix are related by*

$$\boldsymbol{d}(s) \equiv \boldsymbol{F}(s)\mathbf{R}^{\mathrm{T}}, \tag{79}$$

with $\mathbf{R}$ *a constant rotation matrix.*

Proof. (i) Trivially the case $\mathbf{u}_0 \times \mathbf{v}_0 = \mathbf{0}$ with $\mathbf{v}_0 \neq \mathbf{0}$ corresponds to a framed curve with a straight center line, with the director frame uniformly twisting about the constant tangent direction. We note that by definition the constant η_2 satisfies $|\eta_2| \leqslant |\mathbf{v}_0|$, with equality only possible when $\mathbf{u}_0$ is parallel to $\mathbf{v}_0$, i.e., when the center line is straight.

(ii) When $\mathbf{u}_0 \times \mathbf{v}_0 \neq \mathbf{0}$ the vectors with components

$$\left[\frac{\mathbf{u}_0 \times \mathbf{v}_0}{|\mathbf{u}_0 \times \mathbf{v}_0|} \quad \frac{(\mathbf{u}_0 - (\mathbf{u}_0 \cdot \mathbf{v}_0/(\mathbf{v}_0 \cdot \mathbf{v}_0))\mathbf{v}_0)}{|(\mathbf{u}_0 - (\mathbf{u}_0 \cdot \mathbf{v}_0/(\mathbf{v}_0 \cdot \mathbf{v}_0))\mathbf{v}_0)|} \quad \frac{\mathbf{v}_0}{|\mathbf{v}_0|}\right] \tag{80}$$

in the director frame $\boldsymbol{d}(s)$ form an orthonormal triad $\boldsymbol{f}(s)$. We now demonstrate that $\boldsymbol{f}$ is, in fact, the Frenet frame $\boldsymbol{F}$ of $\boldsymbol{r}$. We first observe that the third vector of $\boldsymbol{f}$ is by construction parallel to $\boldsymbol{r}'$, and the normalization by $\lambda = |\mathbf{v}_0|$ corresponds to switching to arc length in the deformed configuration. Then, since $\boldsymbol{f}(s)$ is a constant rotation $\mathbf{R}$ from $\boldsymbol{d}(s)$, we remark that the two frames share the common Darboux vector $\boldsymbol{u}(s)$, and by construction $\boldsymbol{u}$ has vanishing $\boldsymbol{f}_1$-component. This suffices to prove that $\boldsymbol{f}(s)$ is in fact the Frenet frame, and computation of the $\boldsymbol{f}_2$ and $\boldsymbol{f}_3$ components of $\boldsymbol{u}$ scaled by $\lambda = |\mathbf{v}_0|$ gives the expressions for curvature and torsion detailed in (78). Since the curvature and torsion are constant, it is a standard result of differential geometry (or indeed just uniqueness for linear, constant-coefficient systems of ordinary differential equations) that the center line is a circular helix. □

LEMMA 3.5. *Relative equilibria of uniform, hyperelastic rods have either straight or helical center lines, and in the helical case are both* $\boldsymbol{d}$*-uniform and* $\boldsymbol{F}$*-uniform* (*where* $\boldsymbol{F}$ *is the Frenet frame defined in* (2) *of the helix*). *The force vector* $\boldsymbol{n}$ *is parallel to the axis of the helix, while the moment vector* $\boldsymbol{m}(s)$ *is orthogonal to the principal normal* $\boldsymbol{\nu}(s)$ *of the helix.*

Proof. By Lemma 3.2 relative equilibria are $\boldsymbol{d}$-uniform, and so by Lemma 3.4 have either straight or helical center lines. And in the helical case the Frenet frame is a constant rotation of the director frame. The components of any vector in the Frenet frame are therefore a constant rotation of the components in the director frame, and so relative equilibria are also $\boldsymbol{F}$-uniform. For a helix the Darboux vector of the Frenet frame is parallel to the axis of the helix, and for relative equilibria the Darboux vector of the Frenet frame coincides with the Darboux vector $\boldsymbol{u}$ of the director frame. The first-order necessary condition (52a) is the statement, expressed in director components, that the force and Darboux vectors $\boldsymbol{n}$ and $\boldsymbol{u}$ are parallel, so that for a helix, $\boldsymbol{n}$ is also parallel to the axis. Similarly, the first-order necessary condition (52b) implies that the moment vector $\boldsymbol{m}(s)$ is a (constant) linear combination of the Darboux vector and the tangent vector, and is therefore orthogonal to the principal normal of the helix at each s. □

Lemmas 3.1–3.5 provide a characterization of relative equilibria that is particularly simple in the case $\lambda = 1$ of an inextensible, unshearable rod. Then each pair (η_1, η_2) with $\eta_1 > 0$ and $|\eta_2| < 1$ prescribes a particular helical center line with curvature and torsion given by (78). And for each helical center line there are at least two equilibrium configurations of the director frame $\boldsymbol{d}$ with differing angular phases around the tangent $\boldsymbol{d}_3$ to the helix. For $\eta_2 > 0$ the helix is right-handed, for $\eta_2 < 0$ the helix is left-handed, and $\eta_2 = 0$ is the degenerate case of a circular center line. In the limit $\eta_2 = \pm 1$ with $\eta_1 > 0$ the degenerate case of a straight center line is obtained. Relations (78) can be inverted to yield

$$\eta_1 = \lambda\sqrt{\kappa^2 + \tau^2}, \qquad \eta_2 = \frac{\lambda\tau}{\sqrt{\kappa^2 + \tau^2}}. \tag{81}$$

Thus we see that in the $\lambda = 1$ case there is a unique $\eta_1 > 0$ and $\eta_2 \in (-1, 1)$ for each value of curvature $\kappa > 0$ and torsion τ.

The case of extensible, shearable rods is slightly more complicated. According to Lemma 3.1 each pair (η_1, η_2) with $\eta_1 > 0$ and $|\eta_2|$ sufficiently small, generate at least two helical equilibria, but the curvature and torsion given by (78) are not set until $\lambda = |\mathbf{v}|$ is known, and λ depends upon the particular constitutive relation. (Note that constraint (57b) implies that $\lambda \geqslant |\eta_2|$ so that the square roots appearing in (78) are real-valued, and moreover the relative equilibria involve extension whenever $|\eta_2| > 1$.) Indeed, it seems that for two different, relative equilibria corresponding to the same (η_1, η_2) pair, λ could take two different values, so that the pair have different helical center lines. Somewhat curiously one can observe from (81) and (24) that the ratio η_2/η_1 uniquely determines the pitch of all helical center lines corresponding to the pair(η_1, η_2), but that the corresponding radius depends upon λ. As before, $\eta_2 > 0$ implies that the helix is right-handed, $\eta_2 < 0$ implies that the helix is left-handed, and for $\eta_2 = 0$ the center line degenerates to a circle. And when $\eta_2 = \pm\lambda$ with $\eta_1 > 0$ the degenerate cases of twisted rods with straight center lines are obtained.

Finally, formulas (78) imply the identities

$$\kappa^2 + \tau^2 = \frac{\eta_1^2}{\lambda^2}, \tag{82}$$

$$\kappa^2 + \left(\tau - \frac{\eta_1}{2\eta_2}\right)^2 = \frac{\eta_1^2}{4\eta_2^2}. \tag{83}$$

The two circles (82) and (83) in (κ, τ)-space intersect by the inequality $\lambda \geqslant |\eta_2|$, with the unique intersection point in the half-plane $\kappa > 0$ being given by (78). From (83) it may be observed that for given η_1 and η_2, the curvature and torsion are bounded $0 < \kappa \leqslant \eta_1/(2\eta_2)$ and $0 \leqslant |\tau| \leqslant \eta_1/|\eta_2|$, with τ and η_2 taking the same sign.

4. Relative Equilibria of Isotropic Rods

The analysis of the previous section remains valid if the strain energy density function W is isotropic in the sense of Definition 2.3. There are, however, two important differences. First, in all three of the variational principles (60) subject to (56), (61) subject to (57), and (62) subject to (63) that were introduced in section 3, the critical points now arise in non-isolated one-dimensional families whenever either $\mathbf{u}$ or $\mathbf{v}$ is not parallel to the symmetry axis $\hat{\mathbf{v}}$. This can be seen because both the function W and all the constraints (56), (57) and (63) are invariant under the isotropy transformation introduced in Definition 2.3. Thus critical points typically arise in non-isolated families generated by the action of the symmetry. One example is illustrated in Figure 2(d). The exceptional cases, namely twisted straight lines, are fixed points of the symmetry.

Second, the same symmetry of isotropy generates an additional integral $I_3(\mathbf{m}, \mathbf{n}) \equiv \mathbf{m} \cdot \mathbf{d}_3$ of the balance laws that is constant along any equilibrium configuration. In this case the general definition (50) of relative equilibria involves a sum over $N = 3$ integrals. In our context the specific form of these first-order necessary conditions becomes

$$\begin{pmatrix} \bar{\mathbf{u}} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \lambda_2 \mathbf{I} & \mathbf{0} \\ \lambda_1 \mathbf{I} & \lambda_2 \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{n} \\ \mathbf{m} \end{pmatrix}, \tag{84}$$

where

$$\bar{\mathbf{u}} = [u_1, u_2, u_3 - \lambda_3]^{\mathrm{T}}. \tag{85}$$

The introduction of the variable $\bar{\mathbf{u}}$ is useful because the gradient

$$\nabla I_3(\mathbf{m}, \mathbf{n}) = \mathbf{d}_3 = [0, 0, 1]^{\mathrm{T}}$$

is constant.

The computation of the dual form of the variational principles is unaltered. The case $\lambda_2 = 0$ again corresponds to special configurations with straight center lines, while for $\lambda_2 \neq 0$ we obtain the dual first-order conditions

$$\begin{pmatrix} \mathbf{n} \\ \mathbf{m} \end{pmatrix} = \begin{pmatrix} \mu_2 \mathbf{I} & \mathbf{0} \\ \mu_1 \mathbf{I} & \mu_2 \mathbf{I} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{u}} \\ \mathbf{v} \end{pmatrix}, \tag{86}$$

associated with the dual variational characterization

$$\text{Minimize } W(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{v} - \hat{\mathbf{v}}) \tag{87}$$

subject to

$$\tfrac{1}{2}\bar{\mathbf{u}} \cdot \bar{\mathbf{u}} = \tfrac{1}{2}\eta_1^2, \qquad \bar{\mathbf{u}} \cdot \mathbf{v} = \eta_1 \eta_2. \tag{88}$$

The final step is to introduce a rotated frame $\overline{\boldsymbol{D}}(s)$ via a particular case of the transformation (10)

$$\overline{\boldsymbol{D}}(s) = \boldsymbol{d}(s)\mathbf{Q}^{\mathrm{T}}(\lambda_3 s + \delta), \tag{89}$$

with $\mathbf{Q}$ defined in (19), and δ being a constant phase shift. According to the kinematics (20) and the Definition 2.3 of isotropy , the variational principle (87), (88) then transforms to

$$\text{Minimize } W(\bar{\mathbf{U}} - \hat{\mathbf{u}}, \bar{\mathbf{V}} - \hat{\mathbf{v}}) \tag{90}$$

subject to

$$\tfrac{1}{2}\bar{\mathbf{U}} \cdot \bar{\mathbf{U}} = \tfrac{1}{2}\eta_1^2, \qquad \bar{\mathbf{U}} \cdot \bar{\mathbf{V}} = \eta_1 \eta_2, \tag{91}$$

where $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ are respectively the components of the Darboux vector of the frame $\overline{\boldsymbol{D}}$ and the tangent vector $\boldsymbol{r}'$ expressed in the frame $\overline{\boldsymbol{D}}$. For inextensible, unshearable rods the analogous computations lead to the variational principle

$$\text{Minimize } W(\bar{\mathbf{U}} - \hat{\mathbf{u}}) \tag{92}$$

subject to

$$\tfrac{1}{2}\bar{\mathbf{U}} \cdot \bar{\mathbf{U}} = \tfrac{1}{2}\eta_1^2, \qquad \bar{\mathbf{U}} \cdot \hat{\mathbf{v}} = \eta_1 \eta_2. \tag{93}$$

The two variational principles (90) subject to constraints (91), or (92) subject to constraints (93), corresponding respectively to the cases of extensible, shearable or inextensible, unshearable rods, are of precisely the same form as the original (in general non-isotropic) variational principles (61) subject to (57), and (62) subject to (63) that were introduced in Section 3, but now applied to the framing $\overline{\boldsymbol{D}}$ of the center line. Lemmas 3.1–3.5 therefore all carry over, but with two provisos. First, the conclusions apply to the framing $\overline{\boldsymbol{D}}$, so that, for example, the relative equilibria are $\overline{\boldsymbol{D}}$-uniform and not necessarily $\boldsymbol{d}$-uniform. Second, as explained at the beginning of this section, the critical points will typically arise in non-isolated families due to the symmetry of isotropy. More specifically we have:

LEMMA 4.1. *A hyperelastic uniform rod satisfying Definition* 2.1 *or* 2.2 *and the appropriate associated version of Definition* 2.3 *of isotropy, has equilibria with helical center lines in which the director components* $\mathsf{m}(s)$, $\mathsf{n}(s)$, $\mathsf{u}(s)$ *and* $\mathsf{v}(s)$ *of the vectors* $\boldsymbol{m}(s)$, $\boldsymbol{n}(s)$, $\boldsymbol{u}(s)$ *and* $\boldsymbol{v}(s)$ *are simply related to their values at* $s = 0$,

$$\mathsf{m}(s) = \mathsf{Q}^{\mathrm{T}}(\lambda_3 s)\mathsf{m}_0, \qquad \mathsf{n}(s) = \mathsf{Q}^{\mathrm{T}}(\lambda_3 s)\mathsf{n}_0, \tag{94a}$$

$$\mathsf{u}(s) = \mathsf{Q}^{\mathrm{T}}(\lambda_3 s)\mathsf{u}_0, \qquad \mathsf{v}(s) = \mathsf{Q}^{\mathrm{T}}(\lambda_3 s)\mathsf{v}_0, \tag{94b}$$

where Q *is defined in* (19)*, and* λ_3 *is an arbitrary constant. Moreover*

$$\boldsymbol{d}(s) = \overline{\boldsymbol{D}}(s)\mathsf{Q}(\lambda_3 s + \delta) = \boldsymbol{F}(s)\boldsymbol{R}^{\mathrm{T}}\mathsf{Q}(\lambda_3 s + \delta), \tag{95}$$

where δ *is an arbitrary phase,* $\boldsymbol{R}$ *is a constant matrix, and the equilibrium is uniform in both the* $\overline{\boldsymbol{D}}$ *frame and Frenet frame* $\boldsymbol{F}$.

Proof. The conclusions follow from the kinematic relation (20) between the frames $\boldsymbol{d}$ and $\overline{\boldsymbol{D}}$. For isotropic rods there are always straight, twisted equilibria with m, n, u and v all parallel to the symmetry axis $\hat{\mathsf{v}}$. However for $|\eta_2|$ sufficiently small ($|\eta_2| < 1$ for inextensible, unshearable rods), there are relative equilibria with u and v not parallel. These configurations have helical center lines, and they arise in four parameter families corresponding to η_1, η_2, the excess twist λ_3 and a phase δ along the family of non-isolated critical points. □

5. Conclusion and Discussion

We have exploited the noncanonical Hamiltonian formulation of the equations governing the equilibria of a uniform, hyperelastic rod to construct families of solutions with helical center lines. For inextensible, unshearable rods the conclusions are rather explicit. For non-isotropic rods, and for each prescribed (positive) curvature and torsion, there are two or more helical configurations, in which the physical twist of the rod equals the geometric torsion of the center line. On these helical solutions the components of the moment, force and strains in both the director and Frenet frames are all constant. For isotropic rods, and again for each curvature and torsion, there is a single two-dimensional family corresponding to an arbitrary prescribed difference between physical twist and geometric torsion, and an arbitrary angular phase. The Frenet components of moment, force and strain are again constant, but when there is an excess twist the director components vary, albeit in a simple, specific way.

For extensible, shearable rods we obtain similar results, but with the caveats that the conclusions hold only for the constant $|\eta_2|$ sufficiently small, or equivalently for helices of sufficiently small pitch, and that curvature and torsion cannot be controlled independently. This weakening of our conclusions is a direct consequence of the constitutive dependence on deformed arc length.

Our analysis characterizes all equilibria of a uniform rod that are $\boldsymbol{d}$-uniform, i.e., having constant components of stress and strain in the director frame, as having straight or helical center lines. Moreover we presented finite-dimensional variational principles to be able to construct all such solutions explicitly for any given strain-energy density function. In this sense we can claim to have completed the program initiated by Kirchhoff. We have not, however, classified all constitutive relations of a uniform, non-isotropic, hyperelastic rod with the property that all equilibria with a helical segment of center line are in fact $\boldsymbol{d}$-uniform. In simple cases of strain-energy density functions this converse conclusion is valid, but a general result remains elusive. Such a result would certainly have to exploit both uniformity and a strong notion of non-isotropy, because if a rod were isotropic over even a limited range of angles, then counter examples could be constructed using the analysis developed here.

It would also be natural to pose the question of which amongst the solutions we have constructed are stable. This question certainly depends upon the specific fashion in which the end loadings are applied, and there does not appear to be one particularly appropriate choice. One available conclusion is that for complete displacement, or hard, loading in which the center line and director frame are held fixed at both extremities, a sufficiently short segment of all of the equilibria that we have constructed will realize a local minimum of the total elastic energy. A further discussion of such issues can be found in [3].

Amongst the uniform solutions that we classify, there are some particularly simple ones. First, there is a family of helical equilibria with vanishing force $\boldsymbol{n} = \boldsymbol{0}$. Starting, for example, from the unstressed shape, pure end moments can be applied in such a way as to evolve along a one-parameter family of helical equilibria. This family includes a unique member with a straight, untwisted center line. And this equilibrium is itself a member of another one-dimensional family of 'trivial' solutions with straight center lines, and applied force parallel to the axis of the rod. Thus buckling analyses and computations such as [15] could be generalized to a boundary value problem involving a hyperelastic rod with a helical unstressed state. Similarly, the solutions that we construct with $\eta_2 = 0$ are circular helices with vanishing torsion, i.e., arcs of circles, with the Darboux vector $\boldsymbol{u}$ perpendicular to the plane of the circle. Thus our analysis reveals that a hyperelastic rod with a helical unstressed state of arbitrary undeformed arc length, can be closed on itself, and will have two or more equilibrium configurations with circular center lines and vanishing twist. Thus buckling and stability analysis for rings of the type developed in [14] could be extended to rods with helical unstressed states.

References

1. S.S. Antman, Kirchhoff's problem for nonlinearly elastic rods. *Quart. Appl. Math.* **32** (1974) 221–240.
2. S.S. Antman, *Nonlinear Problems of Elasticity*. Springer-Verlag, New York (1995).
3. N. Chouaieb, Kirchhoff's problem of helical solutions of uniform rods and their stability properties. Thèse 2717, Ecole Polytechnique Fédérale de Lausanne (2003).
4. E. Cosserat and F. Cosserat, *Théorie des Corps Déformables*. Hermann, Paris (1909).
5. H.R. Crane, Principles and problems of biological growth. *The Scientific Monthly* **6** (1950) 376–389.
6. J.L. Ericksen, Simpler static problems in nonlinear elastic rods. *Internat. J. Solids Struct.* **6** (1970) 371–377.
7. J. Galloway, Helical imperative: paradigm of form and function. In: *Encyclopedia of Life Sciences* (2001) pp. 1–7; http://www.els.net.
8. A. Goriely, M. Nizette and M. Tabor, On the dynamics of elastic strips. *J. Nonlinear Sci.* **11** (2001) 3–45.
9. A. Goriely and P. Shipman, Dynamics of helical strips. *Phys. Rev. E* **61** (2000) 4508–4517.
10. T.J. Healey, Material symmetry and chirality in nonlinearly elastic rods. *Math. Mech. Solids* **7** (2002) 405–420.
11. S. Kehrbaum and J.H. Maddocks, Elastic rods, rigid bodies, quaternions and the last quadrature. *Phil. Trans. Roy. Soc. London* **355** (1997) 2117–2136.
12. G. Kirchhoff, Über das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes. *J. Reine Angew. Math. (Crelle)* **56** (1859) 285–313.
13. A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity*. Dover, New York (1944).
14. R.S. Manning and K.A. Hoffman, Stability of n-covered circles for elastic rods with constant planar intrinsic curvature. *J. Elasticity* **62** (2001) 1–23.
15. S. Neukirch and M.E. Henderson, Classification of the spatial equilibria of the clamped elastica: Symmetries and zoology of solutions. *J. Elasticity* **68** (2002) 95–121.
16. A.B. Whitman and DeSilva C.N., An exact solution in a nonlinear theory of rods. *J. Elasticity* **4** (1974) 265–280.