Generalized Load-Displacement Relations for Helical Structures

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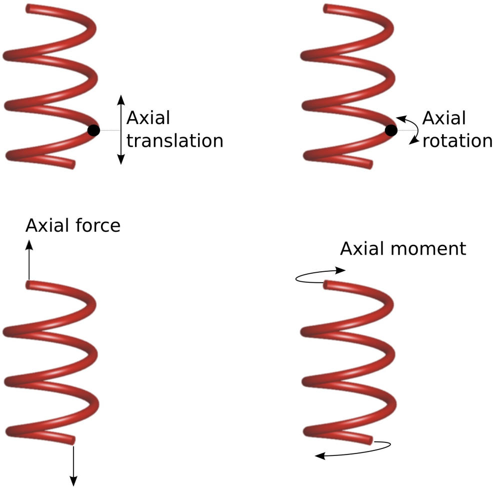
Load-Displacement Problem for Helical Structures



Helical Configurations of an Unshearable, Inextensible, Uniform Rod

- Each helical configuration of an unshearable, inextensible, uniform rod corresponds to a constant value of a strain $\mathsf{u} \in \mathbb{R}^3$.

Helical Configurations of an Unshearable, Inextensible, Uniform Rod

- Each helical configuration of an unshearable, inextensible, uniform rod corresponds to a constant value of a strain $\mathsf{u} \in \mathbb{R}^3$.
- Energy: $E(\mathsf{u}) = (\mathsf{u} - \hat{\mathsf{u}})^T \mathbf{K} (\mathsf{u} - \hat{\mathsf{u}})$
- Stiffness matrix $\mathbf{K} = \begin{bmatrix} K_1 & 0 & K_{13} \\ 0 & K_2 & K_{23} \\ K_{13} & K_{23} & K_3 \end{bmatrix}$.
- Reference state $\hat{\mathsf{u}}$

K

Helical Equilibria Equilibria

The helical equilibria of a rod lie on the surface $\mathcal{H}$ given by

$$0 = Xu_1u_2 + Yu_1u_3 + Zu_2u_3 - Vu_1 - Wu_2$$

Axial Translation, Rotation, Force and Moment in Terms of u

- Axial translation: $R\left(\mathsf{u}\right) = \dfrac{u_3}{\left\|\mathsf{u}\right\|}$
- Axial rotation: $\theta\left(\mathsf{u}\right) = \left\|\mathsf{u}\right\|$
- Axial moment: $M\left(\mathsf{u}\right) = \dfrac{\mathsf{u}^T\mathbf{K}\left(\mathsf{u} - \hat{\mathsf{u}}\right)}{\left\|\mathsf{u}\right\|}$
- Axial force: $N\left(\mathsf{u}\right) = \mu\left(\mathsf{u}\right)\left\|\mathsf{u}\right\|$

Level Curves and Level Surfaces

$$f \in \{R, \theta, M, N\}$$

- f_α: level surface
- $\mathcal{H} \cap f_\alpha$: level curve

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- Self-intersections = saddle points
- Isolated points = extrema

Equation of the Surface Of Helical Equilibria

$\mathcal{H}$ is given by

$$H\left(\mathsf{u}\right) = Xu_1u_2 + Yu_1u_3 + Zu_2u_3 - Vu_1 - Wu_2.$$

$VZ - WY$	X	Y	Z	V	W	$\mathcal{H}$
$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$			1-sheeted hyperboloid
$\neq 0$	$\neq 0$	$\neq 0$	0			Hyperbolic paraboloid
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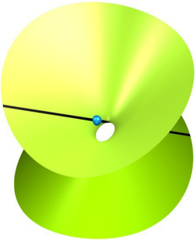
Illustration of the Shapes $\mathcal{H}$ Can Have



Some statements About these Surfaces

Intersecting planes are always perpendicular.

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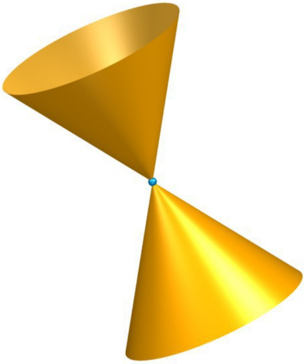
Some statements About these Surfaces

For a circular cone, the opening angle is $> \frac{\pi}{2}$.

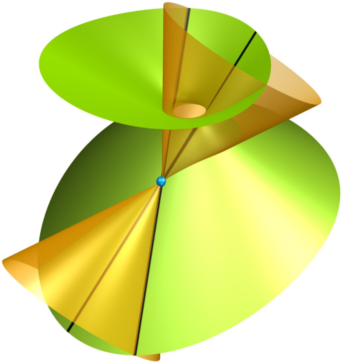
Axial Translation or Rise



$$R(\mathsf{u}) = \frac{u_3}{\|\mathsf{u}\|}$$

$$\mathcal{R}_\alpha := \{\mathsf{u} : R(\mathsf{u}) = \pm\alpha\} \cup \mathbf{0}$$

Straight Lines Contained in the Level Sets of R

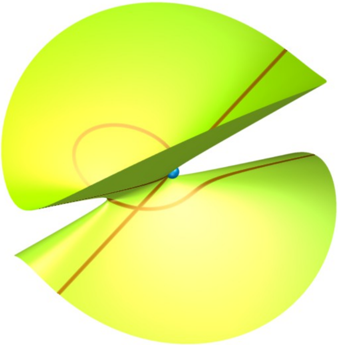


$|\alpha_0| < 1$: height such that $\mathcal{H} \cap \mathcal{R}_{\alpha_0}$ contains a straight line.

Parametrization of the Level Curve $\mathcal{H} \cap \mathcal{R}_\alpha$

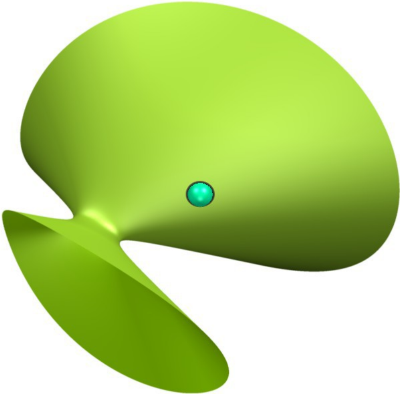
- Every level curve $\mathcal{H} \cap \mathcal{R}_\alpha$ can be parametrized by $\gamma_\alpha(\phi)$ with $\phi \in [0, 2\pi[$.
- γ_{α_0} is not well defined on the straight line contained in $\mathcal{H} \cap \mathcal{R}_{\alpha_0}$.
- γ_{α_0} has a limit on the straight line contained in $\mathcal{H} \cap \mathcal{R}_{\alpha_0}$ if $X \neq 0$ and $V^3 Z + W^3 Y \neq 0$.

Level Curve $\mathcal{H} \cap \mathcal{R}_{\alpha_0}$



$\mathcal{H} \cap \mathcal{R}_{\alpha_0}$ is the only level curve that can have a non zero self-intersection.

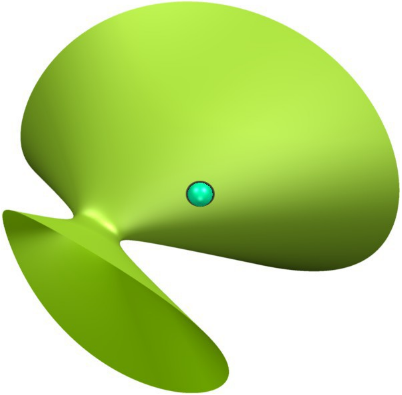
Axial Rotation or Angle



$$\theta\left(\mathsf{u}\right) = \left\|\mathsf{u}\right\|$$

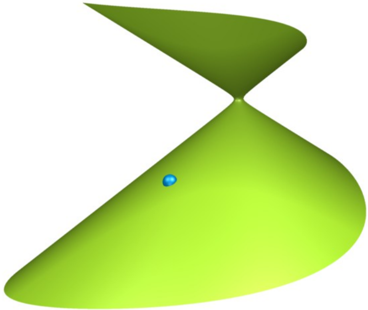
$$\theta_{\alpha} = \left\{\mathsf{u} : \theta\left(\mathsf{u}\right) = 0\right\}$$

Level Sets for Small Heights: Comparison with a Plane



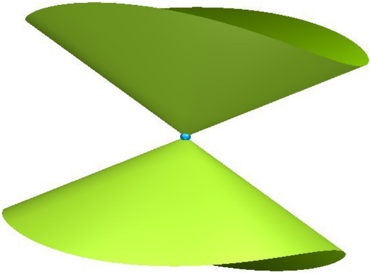
For α small enough, $\mathcal{H} \cap \theta_\alpha$ is a simple closed curve.

Level Sets for Large Heights: Comparison with a Cone



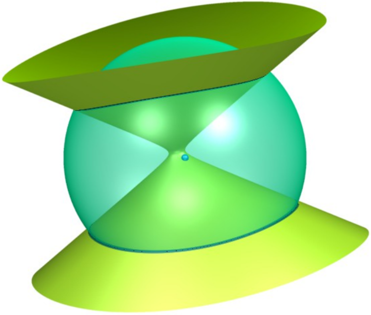
For α large enough, θ has the same behavior as on a cone.

Cone Having its Apex at **0**



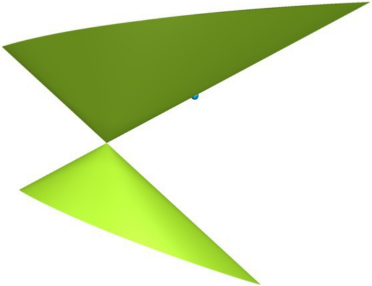
$\mathcal{H} \cap \theta_\alpha$ is composed of two simple closed curves: one on each nappe.

Comparison Between a Cone Having its Apex at $\mathbf{0}$ and a Hyperboloid



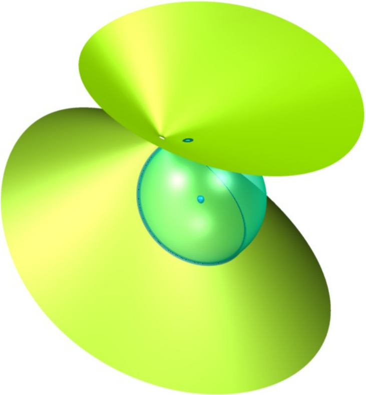
For α large enough, $\mathcal{H} \cap \theta_\alpha$ is composed of two distinct closed curves: one on each nappe.

Circular Cone



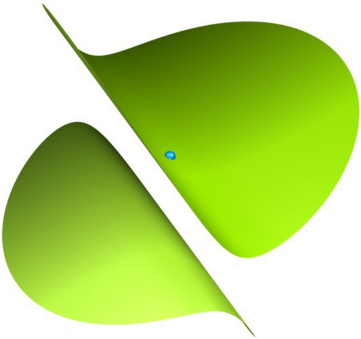
θ has a local minimum and a saddle point.

Comparison Between a Cone Having its Apex not at $\mathbf{0}$ and a Hyperboloid



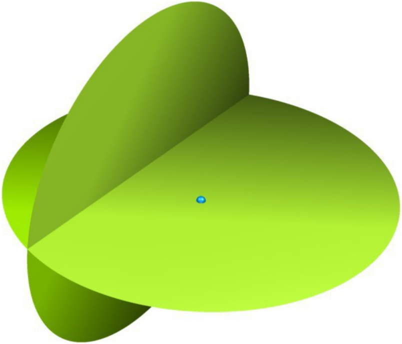
Special case: θ has a local minimum on the nappe that doesn't contain $\mathbf{0}$.

Hyperbolic Cylinder



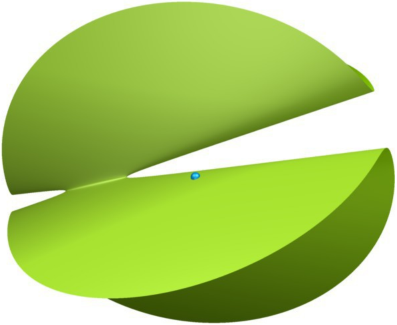
Hyperbolic cylinder: θ has a local minimum on the nappe that doesn't contain $\mathbf{0}$.

Intersecting Planes



Intersecting planes: every point of the intersection line is a self-intersection of $\mathcal{H} \cap \theta_\alpha$ for some α.

Comparison Between Intersecting Planes and Hyperboloid



$\mathcal{H}$ nearly denegerated to intersecting planes.

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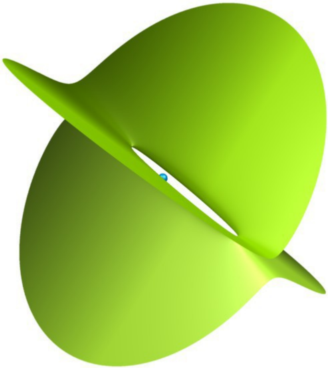

$\mathcal{H}$ nearly denegerated to intersecting planes.

Conjecture on the Critical Points of θ

- θ has a global minimum at $\mathbf{0}$.
- θ can have either 1 or 3 non zero critical points on $\mathcal{H}$.
- If θ has 1 non zero critical point, it is a saddle.
- If θ has 3 non zero critical points, the lowest is a local minimum and the middle and highest are saddles.

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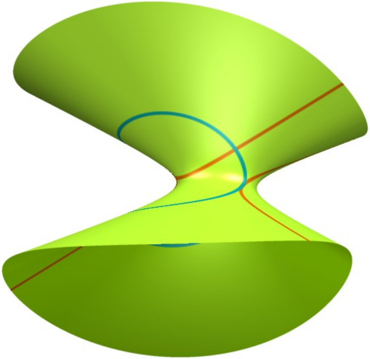
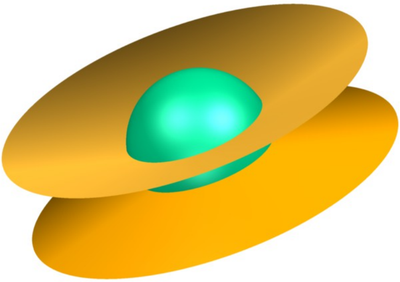
The Displacement Coordinate System



R and θ form a local coordinate system on $\mathcal{H}$ unless

- R or θ has a critical point at u.
- The coordinate line of R and θ are tangent at u.

The Displacement Coordinate System



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Equation for the set of Points where (R, θ) is not a Coordinate System

$$0 = (\nabla H \times \nabla R) \times (\nabla H \times \nabla \theta)$$

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0 &= (\nabla H \times \nabla R) \times (\nabla H \times \nabla \theta) \\
&= (\nabla H \cdot (\nabla R \times \nabla \theta))\, \nabla H
\end{aligned}
$$

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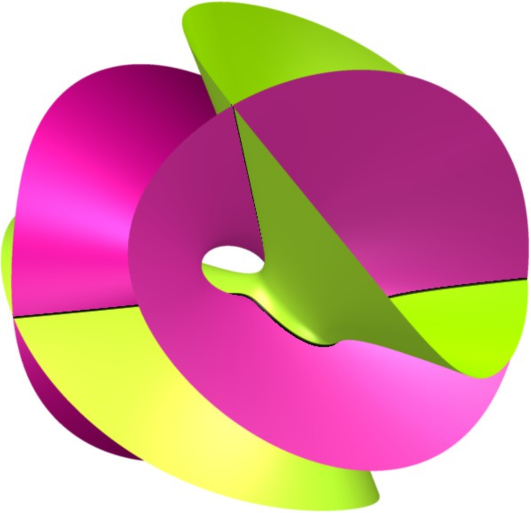
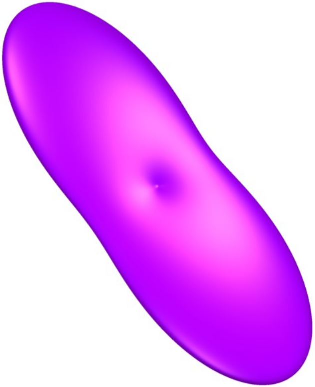
$$\begin{aligned} 0 &= (\nabla H \times \nabla R) \times (\nabla H \times \nabla \theta) \\ &= (\nabla H \cdot (\nabla R \times \nabla \theta)) \nabla H \end{aligned}$$

$$0 = X\left(u_1^2 - u_2^2\right) + Z u_1 u_3 - Y u_2 u_3 - W u_1 + V u_2$$

The set where (R, θ) is not a coordinate system



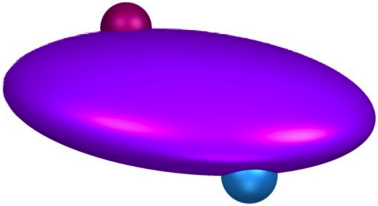
Axial Moment



$$M\left(\mathsf{u}\right) = \frac{\mathsf{u}^T \mathbf{K}\left(\mathsf{u} - \hat{\mathsf{u}}\right)}{\left\|\mathsf{u}\right\|}$$

$$M_\alpha = \left\{\mathsf{u} : M\left(\mathsf{u}\right) = \alpha\right\} \cup \mathbf{0}$$

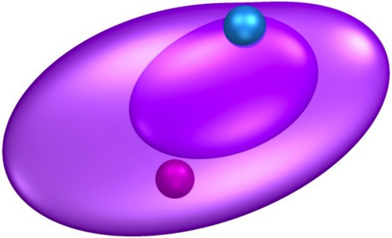
0-Level Surface of M



0-level surface: ellipsoid M_0

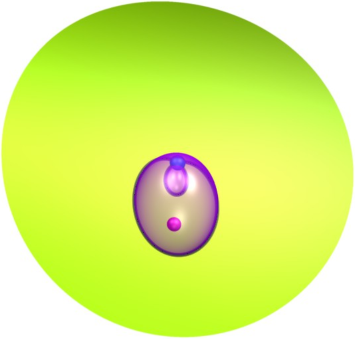
- centered at $\frac{1}{2}\hat{\mathsf{u}}$
- semi-principal axes = orthogonal eigenvectors of $\mathbf{K}$
- length of semi-principal axis $= \dfrac{\|\hat{\mathsf{u}}\|_{\mathbf{K}}}{2\sqrt{\lambda_i}}$

Positive and Negative Level Surfaces



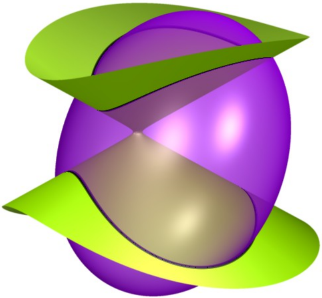
Positive height: the level surface is outside M_0.
Negative height: the level surface is inside M_0.

Small Level Curves of M



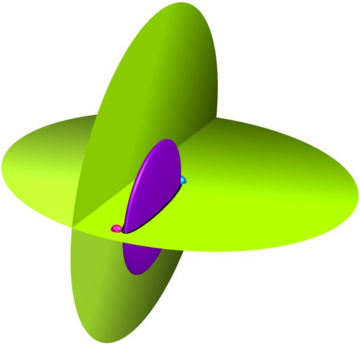
For α small enough, the level curve $\mathcal{H} \cap M_\alpha$ is a simple closed curve.

Large Level Curves of M



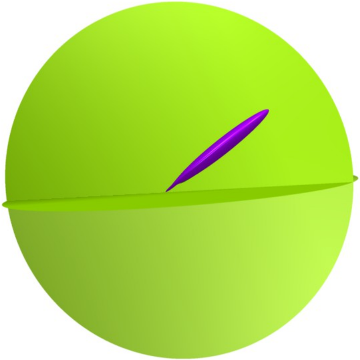
For α large enough, the level curve $\mathcal{H} \cap M_\alpha$ is composed of two simple closed curves.

Level Sets of M on Intersecting Planes



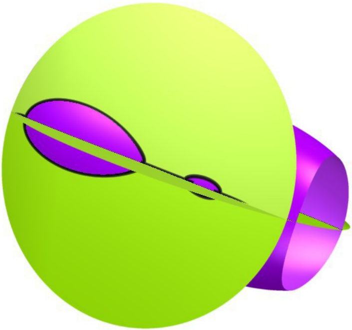
- $\mathbf{0}$ and $\hat{\mathbf{u}}$ are on the same plane.
- One of the principal axes of M_0 is perpendicular to this plane.

Level Sets of M on Intersecting Planes



M may have one minimum on the intersection line.

Level Sets of M on Intersecting Planes

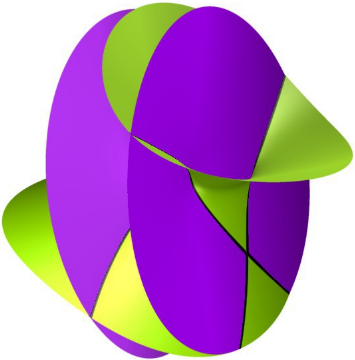


M may have two minima on the intersection line.

Conjecture on M

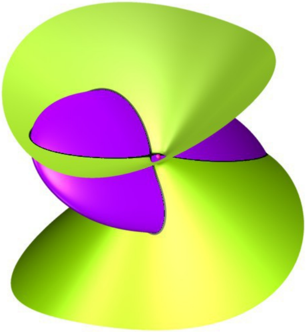
- M has a global minimum on $\mathcal{H}$, at height $\alpha = -\left\|\mathbf{K}\hat{\mathbf{u}}\right\|$.
- It can have either 1, 3 or 5 other critical points.
- If M has 1 non zero critical point, it is a saddle.
- If M has 3 non zero critical points, the lowest one is a minimum and the two other are saddle points.
- If M has 5 non zero critical points, two of them are minima and the three other saddles. The lowest one is a minimum and the highest one a saddle.

M with 1 Non Zero Critical Point



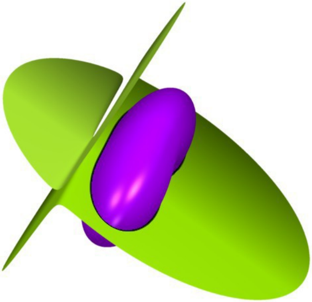
M may have two minima and three saddle points.

M with 3 Non Zero Critical Points



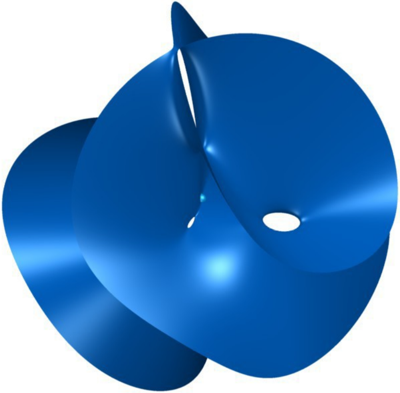
M may have two minima and three saddle points.

M With 5 Non Zero Critical Points



M may have two minima and three saddle points.

Axial Force N



$$N(\mathsf{u}) = \mu(\mathsf{u})\,\|\mathsf{u}\|$$

$$\mu(\mathsf{u}) = \frac{P^3(\mathsf{u})}{u_1 u_2}$$

Conclusion

- We understand well the displacement coordinate system.
- The load coordinate system needs further research.

Rod Model

A physical rod is mathematically modeled by

- A space curve $\mathbf{r}(s)$: its *centerline*.
- An orthonormal right-handed frame $\{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$. It represents the *orientation of the material cross-section* at $\mathbf{r}(s)$.

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  $$\mathbf{R}(s) = \begin{bmatrix} | & | & | \\ \mathbf{d}_1(s) & \mathbf{d}_2(s) & \mathbf{d}_3(s) \\ | & | & | \end{bmatrix}.$$

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- $\mathbf{d}_3(s) = \mathbf{r}'(s)$.

Strains

- $\mathsf{u}(s) = \mathbf{R}(s)^T \mathbf{u}$ where $\mathbf{d}_i'(s) = \mathbf{u} \times \mathbf{d}_i(s)$ for $i = 1, 2, 3$.
- $\mathsf{v}(s) = \mathbf{R}(s)^T \mathbf{r}'(s)$.

Go back

Strains

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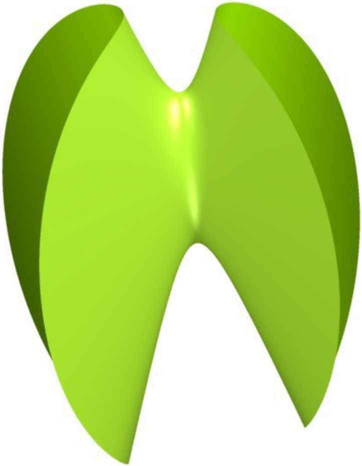
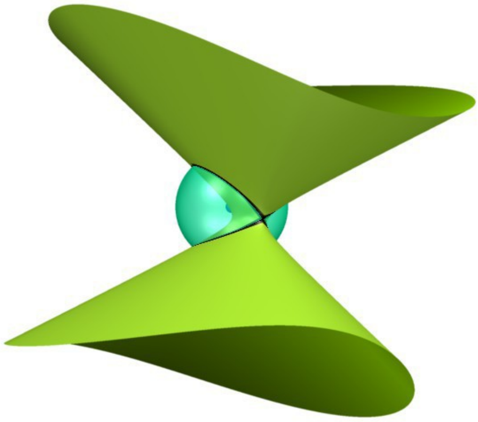
  Unshearable inextensible rod: $\mathsf{v} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Go back

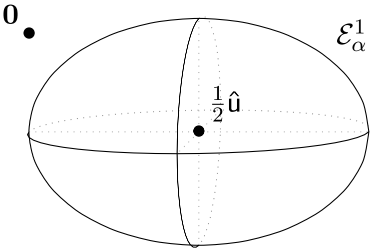
Angles between the Asymptotes of a Hyperboloid



Example Where θ has 1 Non Zero Critical Point

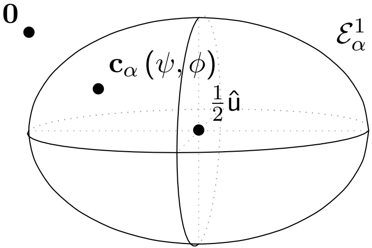


Algorithm to construct the α-level set of M



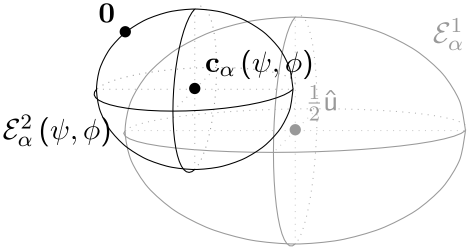
1. Construct the ellipsoid $\mathcal{E}_\alpha^1$
    - centered at $\frac{1}{2}\hat{\mathsf{u}}$
    - semi-principal axes = orthogonal eigenvectors of $\mathbf{K}$
    - length of semi-principal axis = $\dfrac{\alpha}{2\lambda_i}$

Algorithm to construct the α-level set of M



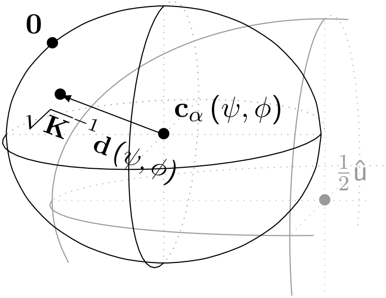
1. Construct the ellipsoid $\mathcal{E}_\alpha^1$
   For all directions $\mathbf{d}(\psi,\phi)$:
2. Take the point $\mathbf{c}_\alpha(\psi,\phi)$
   $\mathbf{c}_\alpha(\psi,\phi) = \frac{1}{2}\left(\hat{\mathsf{u}} + \alpha\mathbf{K}^{-1}\mathbf{d}(\psi,\phi)\right)$

Algorithm to construct the α-level set of M



1. Construct the ellipsoid $\mathcal{E}_\alpha^1$
   For all directions $\mathbf{d}(\psi, \phi)$:
2. Take the point $\mathbf{c}_\alpha(\psi, \phi)$
3. Construct the ellipsoid $\mathcal{E}_\alpha^2(\psi, \phi)$
   - centered at $\mathbf{c}_\alpha(\psi, \phi)$
   - passing through $\mathbf{0}$
   - semi-principal axes = orthogonal eigenvectors of $\mathbf{K}$
   - length of semi-principal axis = $\dfrac{\|\mathbf{c}_\alpha(\psi, \phi)\|_{\mathbf{K}}}{\lambda_i}$

Algorithm to construct the α-level set of M



1. Construct the ellipsoid $\mathcal{E}_\alpha^1$
   For all directions $\mathbf{d}(\psi, \phi)$:
2. Take the point $\mathbf{c}_\alpha(\psi, \phi)$
3. Construct the ellipsoid $\mathcal{E}_\alpha^2(\psi, \phi)$
4. Choose the point of $\mathcal{E}_\alpha^2(\psi, \phi)$ in direction $\sqrt{\mathbf{K}}^{-1}\mathbf{d}(\psi, \phi)$