ABSTRACT

Title of Dissertation: Hamiltonian Dynamics of a Spatial Elastica and the Stability of Solitary Waves

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A novel approach to constrained Lagrangian systems is introduced in order to derive an unconstrained Hamiltonian system of partial differential equations whose solutions model the large amplitude, time evolution of an inextensible and unshearable elastic rod. The inextensibility and unshearability conditions are not imposed as constraints, but instead arise as integrals of motion for the Hamiltonian system. The new formulation also has applications outside rod dynamics.

For a uniform rod of infinite length it is known that solitary waves arise. The solitary waves have the unusual property that a given profile can move at any speed. In this thesis the Hamiltonian formulation of the dynamics is exploited to analyze the stability properties of the planar solitary waves. The analysis begins with a model of the planar dynamics. In the usual manner in Hamiltonian systems, the solitary wave profiles are characterized as critical points of an appropriate variational principle. For the case of rods this variational principle involves a previously unrecognized conservation law. Furthermore, for subsonic wave speeds the solitary wave profiles can actually be characterized as nonisolated minimizers of a time invariant functional F amongst planar perturbations. The implications of this fact upon nonlinear dynamic stability properties are described.

The model is then generalized to consider three-dimensional motion. A variational analysis of planar solitary wave profiles on an isotropic rod shows that there is a range of subsonic wave speeds for which the invariant F has index one amongst admissible spatial perturbations; outside of that range F is infinitely indefinite. The analysis is then generalized to two related anisotropic rod problems.

HAMILTONIAN DYNAMICS OF A SPATIAL ELASTICA AND THE STABILITY OF SOLITARY WAVES

By
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CHAPTER 1

INTRODUCTION

The subject of this dissertation is the dynamics of an inextensible and unshearable elastic rod. The title is motivated by the fact that the evolutionary system considered is a natural generalization to spatial dynamics of Euler's classic elastica model which describes the equilibria of such a planar rod. My primary goal in this dissertation has been to examine the nonlinear stability of planar solitary waves arising in the finite-amplitude motion of a uniform elastica of infinite length. The pursuit of that goal has led to two significant additional developments. First, an existing framework for the stability analysis of travelling waves motivated the reformulation of the dynamics as an unconstrained Hamiltonian system of partial differential equations governing the time evolution of the elastica. The approach taken here to develop such unconstrained Hamiltonian systems for constrained Lagrangian dynamics is quite general, but I have concentrated on its utility in the context of rod dynamics. Second, in order to establish a stability result in the Hamiltonian context it is natural to consider the invariants of motion. Toward this end I have established a previously unrecognized conservation law associated with the translational symmetry of an infinite uniform rod, and a consequent invariant of motion which I will call the linear impulse.

Our study of rod dynamics begins with a formulation of Lagrangian equations of motion corresponding to Newton's laws for an undamped, linearly elastic, geometrically exact rod. The starting point for the derivation of the equations of motion is Hamilton's principle of stationary action subject to constraints that describe the inextensibility and unshearability. This formulation is based upon the model for rod dynamics first proposed by Kirchhoff [35] in 1859. In the first part of the thesis it is assumed that the motion is planar and without twist. An associated Hamiltonian system of partial differential equations is derived in which the Hamiltonian is the total energy. However the technique used to obtain the Hamiltonian system has certain novel aspects. For example, the evolutionary system has the significant feature that the variables are unconstrained. Instead, the inextensibility and unshearability arise from certain integrals of motion of the system. A second unusual feature is that the Hamiltonian is only defined after minimization with respect to certain auxiliary variables.

The new Hamiltonian formulation of the equations of rod dynamics proved convenient for analyzing the orbital stability of solitary waves, using a now standard method (cf. Benjamin's seminal analysis of the stability of the solitary wave in the Korteweg-de Vries equation [10]). The solitary wave profiles are first characterized as critical points of the variational principle in which the total energy is made stationary subject to a prescribed value of the linear impulse. Provided that appropriate existence and continuation results for solutions to the Cauchy initial boundary value problem of the full time dynamics are assumed, we may, via a Lyapunov type argument, conclude nonlinear, orbital stability of those planar solitary waves that are actually minimizers of the variational principle.

In the case of the infinite rod, solitary waves of all speeds can be found in explicit form. Moreover for a certain range of wave speeds called subsonic, the solitary wave profiles realize unconstrained nonisolated minima of an appropriate time invariant functional. By contrast, supersonic solitary waves are not minima of any obvious variational principle (constrained or not), but no conclusion has been reached as to whether or not they are dynamically stable. However, an estimate of the forces involved indicates that the inextensibility and unshearability assumptions are inappropriate for waves with near sonic and supersonic speeds, so the significance of their stability in the context of our model would be questionable.

The rod model is then extended to describe three-dimensional motions. Euler parameters were chosen to represent the rotational kinematics for their analytical and computational advantages. Invariants of motion that arise in spatial dynamics are discussed, together with their associated symmetries. Planar solitary waves are again shown to be critical points of a variational principle involving an invariant of motion F. To determine the essential properties of the second variation of F we examine the special case of an isotropic rod. In contrast with the planar model, a planar solitary wave profile with subsonic wave speed on an isotropic rod is not a minimizer of F amongst admissible spatial variations. Instead, F has index one at the solitary wave for a certain range of subsonic wave speeds; otherwise F is infinitely indefinite.

Finally in the appendices two alternative Hamiltonian formulations of the dynamics of an elastica are presented, as well as some detailed variational calculations.

This dissertation does not address the problem of well-posedness of the initial value problem. A full nonlinear stability result for the solitary waves will depend on obtaining appropriate existence and continuation results for the initial value problem. In particular, it remains to justify rigorously the time-invariance of the formal integrals of motion. Nevertheless, error bounds in an L^2 sense are obtained which prove that a solitary wave profile is a true local minimum of an appropriate variational principle involving the integrals of motion.

As remarked previously, the technique used to cast a constrained Lagrangian dynamical system in the form of an unconstrained Hamiltonian system has applications outside of rod dynamics. For example, an unconstrained Hamiltonian system describing incompressible fluid flow is derived in [44].

The study of travelling waves in an evolutionary system with $1+1$ dimensions and translational symmetry is classic, tracing back at least as far as the work of Boussinesq [9] and Korteweg and de Vries [40]. In many conservative systems a travelling wave can be characterized as a critical point of the energy subject to a prescribed value of the invariant of motion associated with translational symmetry, such as linear impulse or momentum. Perhaps the first modern treatment of a stability analysis along these lines was the 1972 analysis of the Korteweg-de Vries equation by Benjamin [10], but as Benjamin pointed out, the basic idea is due to Boussinesq [9] among others. One then uses this variational characterization in constructing a Lyapunov function to prove stability. It is convenient to do this in a Hamiltonian formulation of the dynamics, where the time evolution is first order. Moreover the relations between symmetries and conserved quantities via Noether's Theorem are transparent in a Hamiltonian system, as described for example by Olver ([52], Chs. 4 and 7).

In this dissertation these ideas are applied in the context of rod dynamics. The difficulty in rod dynamics lies in the derivation of a well-defined evolutionary system that satisfies the constraints of inextensibility and unshearability. As a consequence of these constraints, the tensile and shear forces are basic unknowns that must be determined, but for which there are no time dynamics. Caflisch and Maddocks [12] studied the planar dynamics of a finite rod, from which Xu *et al.* [62] later developed a numerical scheme to study the dynamics of solitary waves. In my research, I first derived a nonlocal Hamiltonian formulation which appears in Appendix B, based upon the work in [12]. But there is a problem in extending this model to an infinite rod, as I was unable to determine the constraint forces at infinity. This difficulty caused a change in emphasis. Rather than enforcing the constraint, the time-differentiated constraint is enforced, which leads to the impetus-striction formulation. The multipliers (or strictions, which determine the forces) then couple with the definition of the conjugate momenta (or impetuses). A pre-Hamiltonian is formulated as a function of both the state variables and the strictions, and the strictions are determined as a function of the state variables through minimization. This approach seems to be quite general. It can be viewed as an infinite-dimensional variant of vakonomic mechanics [5]. A key distinction is that vakonomics does not use the minimization step that allows the extension to partial differential equations. Another difference is the impetus-striction approach concerns only holonomic constraints, whereas vakonomics focuses mainly on nonholonomic constraints.

A detailed review of the current literature as it relates to the results in this dissertation is presented in Chapter 7. To close this introduction, I will outline the overall structure of the dissertation.

We begin in Chapter 2 with a development of the standard theory of rod kinematics and dynamics. First the general director model for rod kinematics is described. Next we review the balance laws for linear and angular momentum, and the constitutive relations as they apply to both unconstrained and constrained motions. Finally the model is specialized to the case of an inextensible, unshearable rod, first for planar motion with no twist and then for generic spatial motion.

In Chapter 3 we study the conservation laws associated with rod dynamics. Along with the well-known conservation laws pertaining to energy and linear and angular momentum, a newly recognized conservation law which arises in the case of a uniform, hyperelastic rod is presented, as well as a generalization of a second conservation law for an isotropic rod.

Chapter 4 contains a finite dimensional description of the "impetus-striction" method for formulating a constrained Lagrangian system as an unconstrained Hamiltonian system. The method is first outlined, with emphasis place on contrasts with vakonomic mechanics. The distinctive features of the impetus-striction method are then illustrated against a familiar background, using the example of a spherical pendulum.

In Chapter 5 the impetus-striction method is exploited to obtain an unconstrained Hamiltonian formulation describing the planar motion of an inextensible, unshearable rod. The invariants of motion associated with this Hamiltonian system are discussed, and the solitary wave profiles are shown to be extremals of an appropriate linear combination, F say, of the Hamiltonian and a second invariant of motion. Moreover it is demonstrated that for subsonic wave speeds the solitary wave profile actually minimizes F. Finally the validity of some of the assumptions in the rod model is discussed.

In Chapter 6 the rod model is extended to consider the spatial motion of an elastica, represented in terms of Euler parameters. First a generalization of the Hamiltonian system from Chapter 5 is developed to describe three-dimensional motion of the rod, and the invariants of motion are delineated. The solitary wave profiles are again shown to be extremals of an appropriate invariant of motion F. The variational analysis then focuses upon the critical case of an isotropic rod, with the ends of the rod free to rotate. For this case it is demonstrated that, for a certain range of subsonic wave speeds, there exists a spatial perturbation of the planar solitary wave profile that satisfies the requisite linearized constraints related to the invariants of motion and also makes the second variation of the invariant F negative. Outside of that range, the second variation of F is infinitely indefinite. The analysis for the special case of an isotropic rod can then be applied to characterize the planar solitary wave in two related cases, where the rod is anisotropic and where the ends of the rod are clamped.

A detailed summary of our results in presented in Chapter 7, along with a discussion of the relation of our results with the current literature.

In Appendix A the impetus-striction formulation is applied once more to provide a second Hamiltonian formulation for the spatial motion of the elastica, this time in terms of Euler angles. This formulation is presented both because of the familiarity of Euler angles in classical mechanics, and for comparison with the Euler parameter representation described in Chapter 6.

An alternative Hamiltonian formulation for the planar motion of a finite elastica is presented in Appendix B. Unlike the other treatments, this formulation does not exploit the impetus-striction method. Rather it is obtained directly from a nonlocal Lagrangian formulation given in [12]. This Hamiltonian formulation was in fact the first model we considered, but we were unable to extend it to the case of an infinite rod.

Appendix C contains some detailed calculations relating to the variational analysis of Chapter 6.

CHAPTER 2

CLASSICAL ROD DYNAMICS

2.1 Introduction

This chapter summarizes the rod model that will be employed throughout this dissertation. The derivation of the equations of motion will only be outlined here, as the rod model that will be adopted is a comparatively standard one (cf. e.g. [2] or [4]). The kinematics of a rod are first described in section 2.2. Next the force and moment balance laws are presented in section 2.3, and the constitutive relations are discussed in section 2.4. The remainder of the chapter is devoted to the modelling of an inextensible, unshearable elastic rod, i.e. an elastica. A set of coordinates is established in the special case of a planar elastica with no twist in section 2.5. The coordinates are then generalized in section 2.6 to describe the spatial kinematics of an elastica using both Euler angles and Euler parameters.

2.2 Kinematics of Rods

We begin by reviewing the director model for rod kinematics. The dependent variables are a vector function $\mathbf{r}(s,t) \in \mathbf{R}^3$ and a right-handed orthonormal frame of directors $\mathbf{d}_k(s,t) \in \mathbf{R}^3$, $k = 1, 2, 3$. (Hereafter subscripts will be taken to run from 1 to 3, and repeated indices should be summed unless stated to the contrary.) The independent variables are two scalars, namely undeformed arc-length s, and time t. At each time t, the curve $\mathbf{r}(\cdot, t)$ can be interpreted as the configuration of a material line in a long, slender elastic body. The triad $\{\mathbf{d}_k\}$ can be interpreted as providing information concerning the orientation of the material cross-section of the rod. (See Figure 1 at the end of this chapter.) The kinematics of the rod are encapsulated in the relations

$$\mathbf{r}' = \mathbf{v}, \tag{2.2.1}$$

$$\mathbf{d}_k' = \mathbf{u} \times \mathbf{d}_k, \tag{2.2.2}$$

$$\dot{\mathbf{d}}_k = \boldsymbol{\omega} \times \mathbf{d}_k. \tag{2.2.3}$$

Here $\mathbf{d}_k'$ denotes the partial derivative with respect to arc-length s, while $\dot{\mathbf{d}}_k$ denotes the partial derivative with respect to t, etc. The components $v_k \equiv \mathbf{v} \cdot \mathbf{d}_k$ of the vector $\mathbf{v}(s,t)$

with respect to the triad $\{\mathbf{d}_k\}$ are the strains associated with stretching and shear, the components $u_k \equiv \mathbf{u}\cdot\mathbf{d}_k$ of the (Darboux) vector $\mathbf{u}$ are the strains associated with bending and twist, and the components $\omega_k \equiv \boldsymbol{\omega}\cdot\mathbf{d}_k$ of the vector $\boldsymbol{\omega}(s,t)$ can be regarded as the body components of the angular velocity of the material cross-section at s when viewed as a rigid body. Because v_k, u_k and ω_k are components with respect to a variable frame we obtain relations such as

$$\boldsymbol{\omega}' = \omega_k'\mathbf{d}_k + \mathbf{u}\times\boldsymbol{\omega}, \tag{2.2.4}$$

and

$$\mathbf{r}'' = \mathbf{v}' = v_k'\mathbf{d}_k + \mathbf{u}\times\mathbf{v}. \tag{2.2.5}$$

(By convention expressions such as v_k' involve the derivative of the ith component of $\mathbf{v}$ with respect to the triad $\{\mathbf{d}_j\}$, which is of course not the same as the ith component of $\mathbf{v}'$, etc., with respect to a fixed basis.) We shall exploit the compatibility conditions

$$\dot{\mathbf{u}} = \omega_k'\mathbf{d}_k, \tag{2.2.6}$$

which are implied by use of equations (2.2.2), (2.2.3) and (2.2.4) in the identities

$$\frac{\partial^2}{\partial s\partial t}\mathbf{d}_k = \frac{\partial^2}{\partial t\partial s}\mathbf{d}_k. \tag{2.2.7}$$

2.3 Balance Laws

The densities of linear and angular momentum are

$$\mathbf{p}(s,t) \equiv \rho(s)\dot{\mathbf{r}}, \tag{2.3.1}$$

$$\boldsymbol{\pi}(s,t) \equiv I_{ij}(s)\omega_i\mathbf{d}_j. \tag{2.3.2}$$

Here $\rho(s)$ is the mass per unit arc-length associated with the material cross-section at the point s, which is determined entirely by the reference configuration. The quantities $I_{ij}(s)$ are the components of the positive-definite inertia tensor of the material cross-section at s expressed with respect to the triad $\{\mathbf{d}_k\}$. It will be assumed that these inertia coefficients are independent of the strains u_k and v_k. Thus the quantities $I_{ij}(s)$ are also completely determined by the reference configuration and will therefore be regarded as known. In particular the symmetry conditions

$$I_{ij} = I_{ji} \tag{2.3.3}$$

are valid. It is consistent to restrict to the case in which the cross-section is modelled as a lamina with principal axes of inertia $\mathbf{d}_1$ and $\mathbf{d}_2$, so that

$$I_{ij} = 0, \quad i \neq j, \qquad I_{33} = I_{11} + I_{22}, \tag{2.3.4}$$

and we henceforth make this assumption.

In a rod model the stresses acting across each material cross-section are reduced to a net force $\boldsymbol{\lambda}(s,t)$ and moment $\mathbf{m}(s,t)$ (of the material on the side s^+ acting on the material on the side s^- at the point $\mathbf{r}(s,t)$). Then balance of linear and angular momentum yield the equations

$$\dot{\mathbf{p}} = \boldsymbol{\lambda}', \tag{2.3.5}$$

and

$$\dot{\boldsymbol{\pi}} = \mathbf{m}' + \mathbf{r}' \times \boldsymbol{\lambda}. \tag{2.3.6}$$

2.4 Constitutive Relations

To complete the general formulation of rod dynamics, we must discuss the constitutive relations for both constrained and unconstrained motions. The nine equations contained in (2.2.3) and the three equations each in (2.3.1), (2.3.5) and (2.3.6) together govern the time evolution of the motion of a rod. However these eighteen equations are not sufficient to determine the twenty-four dynamic variables, namely the components of the configuration $\mathbf{r}$, the directors $\{\mathbf{d}_k\}$, the linear momentum $\mathbf{p}$, the angular momentum $\boldsymbol{\pi}$, the force $\boldsymbol{\lambda}$ and $\mathbf{m}$. In the theory of elasticity it is postulated that the force $\boldsymbol{\lambda}$ and moment $\mathbf{m}$ can be determined as functions of the current configuration of the rod. These constitutive relations are unlike the balance laws in that they are specific to the material of the rod. The manner in which $\boldsymbol{\lambda}$ and $\mathbf{m}$ are determined depends upon whether the rod is materially unconstrained or constrained. Roughly speaking, the motion of a rod is said to be materially unconstrained if there are no restrictions upon the variables $\{v_k\}$ and $\{u_k\}$; otherwise, the motion is called constrained. This distinction must however be qualified, because most models of rod dynamics specify some conditions upon the strains, independent of the material properties, which are required for physically meaningful solutions. For example, one generally requires that

$$v_3 = \mathbf{v} \cdot \mathbf{d}_3 > 0 \tag{2.4.1}$$

be satisfied for all motions, so that a section of the rod of positive length cannot be compressed to zero length.

In this dissertation only simple materials are considered, for which $\boldsymbol{\lambda}(s,t)$ and $\mathbf{m}(s,t)$ are functions of s, $\mathbf{r}(s,t)$, $\partial_s\mathbf{r}(s,t)$, $\mathbf{d}_k(s,t)$, and $\partial_s\mathbf{d}_k(s,t)$ only. It is known by the principle of frame indifference (cf. e.g. [2]) that constitutive relations for simple materials can be written as

$$\mathsf{n}(s,t) = \hat{\mathsf{n}}(\mathsf{u}(s,t), \mathsf{v}(s,t), s), \tag{2.4.2}$$

$$\mathsf{m}(s,t) = \hat{\mathsf{m}}(\mathsf{u}(s,t), \mathsf{v}(s,t), s), \tag{2.4.3}$$

where $\mathsf{n} \equiv (n_1, n_2, n_3)$ with $n_k \equiv \boldsymbol{\lambda} \cdot \mathbf{d}_k$, and m, u and v are defined similarly. A case of special interest is the hyperelastic rod. A rod is said to be hyperelastic if there exists a scalar valued strain energy density function $W(v_i, u_i, s)$, dependent upon the six strains and arc-length, with the property that the components of stress satisfy the relations

$$n_i = W_{v_i}, \tag{2.4.4}$$

$$m_i = W_{u_i}. \tag{2.4.5}$$

The focus in this dissertation is upon rod dynamics subject to material constraints, specifically the constraints of inextensibility and unshearability which are described by

$$\mathbf{r}_s = \mathbf{d}_3, \tag{2.4.6}$$

or equivalently

$$v_1 = 0 = v_2 \qquad \text{and} \qquad v_3 = 1. \tag{2.4.7}$$

Note that equation (2.4.7) automatically implies the inequality (2.4.1). For brevity equation (2.4.6) will be referred to henceforth as the inextensibility condition. It is stipulated that the constraint (2.4.6) holds for all possible forces and moments. In the model under consideration a constitutive equation is assumed only for the moments, in the form of equation (2.4.3). The force must instead be determined by the requirement that the constraint (2.4.6) be maintained. More specifically the rod is assumed to be uniform and hyperelastic, with a strain energy density given by

$$W(\mathsf{u}) = \tfrac{1}{2}\mathsf{u} \cdot \mathsf{K}\mathsf{u}, \tag{2.4.8}$$

where K is a positive definite symmetric matrix. It follows by equation (2.4.5) that the moment is given by

$$\mathsf{m}(s,t) = \mathsf{K}\mathsf{u}(s,t). \tag{2.4.9}$$

Note that the constitutive relation (2.4.9) implies that the unstressed configuration of the rod is straight. Also, while a linear bending law (2.4.9) is assumed in this dissertation, much of the analysis carries over to the case where equation (2.4.9) is replaced a more general nonlinear constitutive relation. This point is discussed in greater detail in Chapter 7.

2.5 Dynamics of a Planar Elastica

To summarize, we are modelling the dynamics of an inextensible and unshearable uniform elastic rod whose unstressed configuration is straight. In this section it is assumed that the deformation is planar with no twist. The generalization to full three-dimensional dynamics is treated in the next section.

The Cartesian axes will be denoted by $\mathbf{e}_1 \equiv (1,0,0)$, $\mathbf{e}_2 \equiv (0,1,0)$ and $\mathbf{e}_3 \equiv (0,0,1)$. The rod is restricted to lie in the $(\mathbf{e}_1, \mathbf{e}_2)$ plane, so its cartesian coordinates are $\mathbf{r}(s,t) = (x(s,t), y(s,t), 0)$. The directors are parametrized by a single angle $\phi(s,t)$ from $\mathbf{e}_1$ to $\mathbf{d}_3$, so that

$$\begin{aligned} \mathbf{d}_3(s,t) &= \cos\phi(s,t)\mathbf{e}_1 + \sin\phi(s,t)\mathbf{e}_2, && (2.5.1)\\ \mathbf{d}_2(s,t) &= -\sin\phi(s,t)\mathbf{e}_1 + \cos\phi(s,t)\mathbf{e}_2, && (2.5.2)\\ \mathbf{d}_1(s,t) &= -\mathbf{e}_3. && (2.5.3) \end{aligned}$$

It follows from equations (2.5.1) – (2.5.3) that the derivatives in time and arc-length of the directors are given by

$$\partial_t \mathbf{d}_k(s,t) = \phi_t(s,t)\mathbf{e}_3 \times \mathbf{d}_k(s,t) \tag{2.5.4}$$

and

$$\partial_s \mathbf{d}_k(s,t) = \phi_s(s,t)\mathbf{e}_3 \times \mathbf{d}_k(s,t). \tag{2.5.5}$$

In the unstressed reference configuration, $\mathbf{r}(s,0) = s\mathbf{e}_1$ and $\phi(s,0) = 0$. It is convenient to represent the deformation of the rod by its displacement $\mathbf{h}(s,t) \equiv \mathbf{r}(s,t) - s\mathbf{e}_1$ from the reference configuration. In terms of $\mathbf{h}$, the constraint (2.4.6) can be rewritten as

$$\mathbf{h}_s(s,t) = \mathbf{d}_3(s,t) - \mathbf{e}_1. \tag{2.5.6}$$

For the planar deformation with no twist, the angular momentum density $\pi = I_1(s)\phi_t$ and the moment m are scalars. The balance laws of linear and angular momentum yield the equations

$$(\rho \mathbf{r}_t)_t = \boldsymbol{\lambda}_s, \tag{2.5.7}$$

and

$$(I_1\phi_t)_t = m_s + \mathbf{e}_3 \cdot (\mathbf{r}_s \times \boldsymbol{\lambda}). \tag{2.5.8}$$

In keeping with equation (2.4.9), it will be assumed that the moment $m(s,t)$ is related to the deformation at s through the linear constitutive law

$$m(s,t) \equiv K_1 \phi_s(s,t), \tag{2.5.9}$$

where the constant K_1 is the bending stiffness about the $\mathbf{d}_1$ axis.

The kinetic energy of the rod is

$$T(\mathbf{h}_t, \phi_t) = \int_{-L}^{L} \tfrac{1}{2} \left\{ \rho |\mathbf{h}_t(s)|^2 + I_1 \phi_t(s)^2 \right\} ds, \tag{2.5.10}$$

where $|\cdot|$ denotes the Euclidean norm. The potential energy, corresponding to the bending law (2.5.9) is

$$V(\phi) = \int_{-L}^{L} \tfrac{1}{2} K_1 \phi_s(s)^2 \, ds. \tag{2.5.11}$$

For boundary conditions the rod is required to be stationary at its ends, with no applied moment. That is, we require

$$\mathbf{h}_t(\pm L, t) = 0, \tag{2.5.12}$$

and

$$\phi_s(\pm L, t) = 0. \tag{2.5.13}$$

It is also specified that the two ends of the rod have been brought together by an amount $4a > 0$ along the x axis, so

$$\mathbf{e}_2 \cdot \mathbf{h}(\pm L, t) = 0, \tag{2.5.14}$$

$$\mathbf{e}_3 \cdot \mathbf{h}(\pm L, t) = 0, \tag{2.5.15}$$

and

$$\mathbf{e}_1 \cdot [\mathbf{h}(L, t) - \mathbf{h}(-L, t)] = -4a. \tag{2.5.16}$$

By a symmetric displacement of the ends, we can in fact set

$$\mathbf{e}_1 \cdot \mathbf{h}(-L, t) = 2a \tag{2.5.17}$$

and

$$\mathbf{e}_1 \cdot \mathbf{h}(L, t) = -2a. \tag{2.5.18}$$

In the case of an infinite rod, we consider only motions for which the total energy is finite. The conditions (2.5.12) - (2.5.18), with $L \to \infty$, are consistent with this restriction. In addition, it is specified that the rod asymptote to the x axis, so

$$\mathbf{d}_3(\pm\infty, t) = \mathbf{e}_1. \tag{2.5.19}$$

Consequently,

$$\mathbf{h}_s(\pm\infty, t) = \mathbf{0} \tag{2.5.20}$$

and

$$\phi_t(\pm\infty, t) = 0. \tag{2.5.21}$$

Equation (2.5.19) implies that $\phi(-\infty, t)$ and $\phi(\infty, t)$ are integer multiples of 2π. Without loss of generality, one can set $\phi(-\infty, t) = 0$.

The insertion of the constraint (2.5.6) and the definition (2.5.1) of $\mathbf{d}_3$ into the boundary condition (2.5.16) produces the following integral constraints on ϕ:

$$\int_{-L}^{L} [1 - \cos\phi(s)] \, ds = 4a \tag{2.5.22}$$

and

$$\int_{-L}^{L} \sin\phi(s)\, ds = 0. \tag{2.5.23}$$

As a consequence of the boundary condition (2.5.16), the director $\mathbf{d}_3$ cannot be constant as a function of arc-length. This fact is intuitively clear, since an inextensible rod must bend in response to displacement, and is easily proved by contradiction. If $\mathbf{d}_3$ is constant, then $\mathbf{d}_3 = \mathbf{e}_1$ by the boundary condition (2.5.19), and so $\mathbf{h}_s \equiv 0$ by the constraint (2.5.6). But then

$$\mathbf{e}_1 \cdot [\mathbf{h}(L,t) - \mathbf{h}(-L,t)] = \mathbf{e}_1 \cdot \int_{-L}^{L} \mathbf{h}_s(s)\, ds = 0 \tag{2.5.24}$$

which contradicts the boundary condition (2.5.16). It follows that $\mathbf{d}_2$ is also not constant. We shall later exploit these elementary observations.

The initial data are required to be consistent with the boundary conditions (2.5.12) – (2.5.18) for an finite rod, or (2.5.12) - (2.5.21) for an infinite rod. It is further stipulated that the initial data satisfy the constraint (2.5.6). Finally, for an infinite rod the initial data must have finite total energy. Hence the expressions (2.5.10) and (2.5.11) for the kinetic and potential energies imply that at time $t = 0$, $\mathbf{h}_t \in L^2(\mathbf{R})$, $\phi_t \in L^2(\mathbf{R})$, and $\phi_s \in L^2(\mathbf{R})$.

Using the constraint (2.4.6) one could alternatively eliminate the variable $\mathbf{r}$ from the model, leading to a nonlocal formulation of the kinematics. By integrating equation (2.4.6) and applying the boundary conditions (2.5.14) and (2.5.17) we obtain

$$x(s,t) = -L + 2a + \int_{-L}^{s} \cos\phi(u,t)\, du \tag{2.5.25}$$

$$y(s,t) = \int_{-L}^{s} \sin\phi(u,t)\, du, \tag{2.5.26}$$

This is the formulation employed in Appendix B.

2.6 Dynamics of a Spatial Elastica

To fully describe the kinematics of an elastica we must allow for general three-dimensional displacement and twist. In the development of a model for spatial motion it will again be assumed at first that the arc-length lies in the finite interval $[-L, L]$, with the limit $L \to \infty$ being taken for the analysis of solitary wave motion.

Because we now consider three-dimensional motion, a parametrization of the directors $\{\mathbf{d}_k\}$ is equivalent to selecting a representation for the group of orthogonal transformations $SO(3)$. In mechanics there are several common choices for such a representation. (Cf. e.g. [24].) Euler angles are often used because this representation uses the the minimum number of parameters, three. The essential facts about a representation by Euler angles is given in section 2.6.1. There are two significant drawbacks in using Euler angles. First, due to the topology of $SO(3)$ a set of three parameters cannot provide a global representation. In the case of Euler angles there is a direction in space at which the representation becomes singular. This singularity makes the study of dynamical systems a delicate matter, since the particular set of Euler angles must then be selected to avoid the singularity, or different charts must be used as the dynamics evolve. A second drawback in the use of Euler angles is the appearance of trigonometric functions, which can be computationally expensive.

Euler parameters provide a four-dimensional, global, two-to-one parametrization of $SO(3)$. In addition, only polynomial and rational functions arise in this representation. The difficulty with the Euler parameter representation is that the squares of the parameters must sum to one, which imposes a constraint upon the dynamics. An elementary discussion of Euler parameters can be found in [24]. Further discussions of Euler parameters as they apply to the motion of rigid spacecraft can also be found in [29] and [61]. The kinematics of Euler parameters are summarized in section 2.6.2.

2.6.1 Euler Angles

There are 12 distinct Euler angle sequences. Here we will consider only the sequence generated by a rotation about the 3-axis through an angle ϕ, followed by a rotation about the resulting 2-axis through an angle θ, followed by a rotation about the resulting 3-axis through an angle ψ. The Euler angle sequence will be denoted by $\boldsymbol{\phi} = (\phi, \theta, \psi)^T$, where the superscript T denotes the transpose. This sequence is especially convenient for modelling planar motion in the xy-plane because when $\theta = \pi/2$ and $\psi = 0$ the angle ϕ measures the angle in the plane from $\mathbf{e}_1$ to $\mathbf{d}_3$ as in the previous section. Moreover, in this representation the singularity is avoided so long as the angle θ is near $\pi/2$.

Let A denote the orthogonal matrix which transforms a vector from observer frame coordinates (relative to $\mathcal{O} \equiv \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$) into the director frame coordinates (relative to $\mathcal{D} \equiv \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$), so $\mathsf{A}\mathbf{d}_k = \mathbf{e}_k$. For our choice of coordinates,

$$\mathsf{A} = \mathbf{e}_j \otimes \mathbf{d}_j = \begin{pmatrix} c\psi c\theta c\phi - s\psi s\phi & c\psi c\theta s\phi + s\psi c\phi & -c\psi s\theta \\ -s\psi c\theta c\phi - c\psi s\phi & -s\psi c\theta s\phi + c\psi c\phi & s\psi s\theta \\ s\theta c\phi & s\theta s\phi & c\theta \end{pmatrix}, \tag{2.6.1.1}$$

where $\otimes$ denotes the outer product, c denotes the cosine and s denotes the sine. The three rows of A specify the coordinates in the reference frame of $\mathbf{d}_1$, $\mathbf{d}_2$, and $\mathbf{d}_3$, respectively. Thus

$$\mathbf{d}_1 = \begin{pmatrix} c\psi c\theta c\phi - s\psi s\phi \\ c\psi c\theta s\phi + s\psi c\phi \\ -c\psi s\theta \end{pmatrix}, \tag{2.6.1.2}$$

$$\mathbf{d}_2 = \begin{pmatrix} -s\psi c\theta c\phi - c\psi s\phi \\ -s\psi c\theta s\phi + c\psi c\phi \\ s\psi s\theta \end{pmatrix}, \tag{2.6.1.3}$$

and

$$\mathbf{d}_3 = \begin{pmatrix} s\theta c\phi \\ s\theta s\phi \\ c\theta \end{pmatrix}. \tag{2.6.1.4}$$

Note that $\mathbf{d}_3$ is independent of the angle ψ.

The strains u and the angular velocities $\boldsymbol{\omega}$ appearing in equations (2.2.2) and (2.2.3) are given in terms of the Euler angles by

$$\mathsf{u} = \mathsf{B}[\boldsymbol{\phi}]\,\boldsymbol{\phi}_s \tag{2.6.1.5}$$

and

$$\boldsymbol{\omega} = \mathsf{B}[\boldsymbol{\phi}]\,\boldsymbol{\phi}_t, \tag{2.6.1.6}$$

where

$$\mathsf{B}[\boldsymbol{\phi}] = \mathbf{e}_j \otimes \mathbf{b}^j(\boldsymbol{\phi}) = \begin{pmatrix} -c\psi s\theta & s\psi & 0 \\ s\psi s\theta & c\psi & 0 \\ c\theta & 0 & 1 \end{pmatrix}. \tag{2.6.1.7}$$

Note that in the planar case, $\theta = \pi/2$ and $\psi = 0$ imply

$$\mathsf{B}[\boldsymbol{\phi}] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.6.1.8}$$

The rows vectors $\{\mathbf{b}^j(\boldsymbol{\phi})\}$ of the matrix B are computed by exploiting equation (2.2.2) to obtain expressions for u_1, u_2 and u_3. For example,

$$\mathbf{d}_3' = (u_1\mathbf{d}_1 + u_2\mathbf{d}_2 + u_3\mathbf{d}_3) \times \mathbf{d}_3 = -u_1\mathbf{d}_2 + u_2\mathbf{d}_1, \tag{2.6.1.9}$$

from which we conclude that

$$u_1 = -\mathbf{d}_2 \cdot \mathbf{d}_3' = -\mathbf{d}_2 \cdot \mathsf{D}_3[\boldsymbol{\phi}]^T \boldsymbol{\phi}_s, \tag{2.6.1.10}$$

where

$$\mathsf{D}_j[\boldsymbol{\phi}] \equiv \left(\frac{\partial \mathbf{d}_j}{\partial \boldsymbol{\phi}}\right)^T. \tag{2.6.1.11}$$

Thus the first row of $\mathsf{B}[\boldsymbol{\phi}]$ is given by

$$\mathbf{b}^1(\boldsymbol{\phi}) = -\mathsf{D}_3[\boldsymbol{\phi}]\,\mathbf{d}_2 = \mathsf{D}_2[\boldsymbol{\phi}]\,\mathbf{d}_3 = \begin{pmatrix} -c\psi s\theta \\ s\psi \\ 0 \end{pmatrix}, \tag{2.6.1.12}$$

where the second equality follows from differentiation of the identity $\mathbf{d}_2(\boldsymbol{\phi}) \cdot \mathbf{d}_3(\boldsymbol{\phi}) = \delta_{jk}$ with respect to $\boldsymbol{\phi}$, in which δ_{jk} is the Kronecker delta. Similarly

$$\mathbf{b}^2(\boldsymbol{\phi}) = -\mathsf{D}_1[\boldsymbol{\phi}]\,\mathbf{d}_3 = \mathsf{D}_3[\boldsymbol{\phi}]\,\mathbf{d}_1 = \begin{pmatrix} s\psi s\theta \\ c\psi \\ 0 \end{pmatrix} \tag{2.6.1.13}$$

and

$$\mathbf{b}^3(\boldsymbol{\phi}) = -\mathsf{D}_2[\boldsymbol{\phi}]\,\mathbf{d}_1 = \mathsf{D}_1[\boldsymbol{\phi}]\,\mathbf{d}_2 = \begin{pmatrix} c\theta \\ 0 \\ 1 \end{pmatrix}. \tag{2.6.1.14}$$

We also observe the identity

$$\mathsf{D}_k[\boldsymbol{\phi}]\mathbf{d}_k(\boldsymbol{\phi}) = \mathbf{0} \qquad \text{(no sum)}, \tag{2.6.1.15}$$

which follows from the fact that $\mathbf{d}_k$ is a unit vector, so

$$\frac{1}{2}\frac{\partial}{\partial \boldsymbol{\phi}}|\mathbf{d}_k(\boldsymbol{\phi})|^2 = \mathsf{D}_k[\boldsymbol{\phi}]\mathbf{d}_k(\boldsymbol{\phi}) = 0. \tag{2.6.1.16}$$

Equations (2.6.1.12) - (2.6.1.15) allow us to express the matrices $\mathsf{D}_j[\boldsymbol{\phi}]$ in terms of the vectors $\{\mathbf{b}^j(\boldsymbol{\phi})\}$ and $\{\mathbf{d}_j(\boldsymbol{\phi})\}$ as

$$\mathsf{D}_1[\boldsymbol{\phi}] = \left(\frac{\partial \mathbf{d}_1}{\partial \boldsymbol{\phi}}\right)^T = \mathbf{b}^3(\boldsymbol{\phi}) \otimes \mathbf{d}_2(\boldsymbol{\phi}) - \mathbf{b}^2(\boldsymbol{\phi}) \otimes \mathbf{d}_3(\boldsymbol{\phi}), \tag{2.6.1.17}$$

$$\mathsf{D}_2[\boldsymbol{\phi}] = \left(\frac{\partial \mathbf{d}_2}{\partial \boldsymbol{\phi}}\right)^T = \mathbf{b}^1(\boldsymbol{\phi}) \otimes \mathbf{d}_3(\boldsymbol{\phi}) - \mathbf{b}^3(\boldsymbol{\phi}) \otimes \mathbf{d}_1(\boldsymbol{\phi}) \tag{2.6.1.18}$$

and

$$\mathsf{D}_3[\boldsymbol{\phi}] = \left(\frac{\partial \mathbf{d}_3}{\partial \boldsymbol{\phi}}\right)^T = \mathbf{b}^2(\boldsymbol{\phi}) \otimes \mathbf{d}_1(\boldsymbol{\phi}) - \mathbf{b}^1(\boldsymbol{\phi}) \otimes \mathbf{d}_2(\boldsymbol{\phi}). \tag{2.6.1.19}$$

Assuming $\sin\theta \neq 0$, the vectors $\{\mathbf{b}^j(\boldsymbol{\phi})\}$ define a basis for $\mathbf{R}^3$. However this basis is not orthonormal in general, so it is convenient to also define its dual basis $\{\mathbf{b}_j(\boldsymbol{\phi})\}$ which satisfies the identity

$$\mathbf{b}^j \cdot \mathbf{b}_k = \delta_{jk}. \tag{2.6.1.20}$$

The vectors $\{\mathbf{b}_j(\boldsymbol{\phi})\}$ are therefore the columns of the inverse of $\mathsf{B}[\boldsymbol{\phi}]$, so that

$$\mathsf{B}[\boldsymbol{\phi}]^{-1} = \mathbf{b}_j(\boldsymbol{\phi}) \otimes \mathbf{e}_j = \begin{pmatrix} \mathbf{b}_1(\boldsymbol{\phi}) & \mathbf{b}_2(\boldsymbol{\phi}) & \mathbf{b}_3(\boldsymbol{\phi}) \end{pmatrix} \tag{2.6.1.21}$$

and

$$\mathbf{b}_1(\boldsymbol{\phi}) = \begin{pmatrix} -c\psi/s\theta \\ s\psi \\ c\psi c\theta/s\theta \end{pmatrix}, \quad \mathbf{b}_2(\boldsymbol{\phi}) = \begin{pmatrix} s\psi/s\theta \\ c\psi \\ -s\psi c\theta/s\theta \end{pmatrix}, \quad \mathbf{b}_3(\boldsymbol{\phi}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.6.1.22}$$

The identity (2.6.1.20) together with the forms of the matrices $\mathsf{D}_j[\boldsymbol{\phi}]$ imply

$$\mathsf{D}_k[\boldsymbol{\phi}]^T \mathbf{b}_k(\boldsymbol{\phi}) = \mathbf{0} \qquad \text{(no sum)} \ . \tag{2.6.1.23}$$

We infer from equations (2.6.1.17) - (2.6.1.19) and (2.6.1.23) that the vector $\mathbf{b}_k(\boldsymbol{\phi})$ has a geometric interpretation as the instantaneous rotation about the axis $\mathbf{d}_k(\boldsymbol{\phi})$. For example, $\mathsf{D}_1[\boldsymbol{\phi}]^T\mathbf{b}_1(\boldsymbol{\phi}) = \mathbf{0}$ while $\mathsf{D}_2[\boldsymbol{\phi}]^T\mathbf{b}_1(\boldsymbol{\phi}) = \mathbf{d}_3(\boldsymbol{\phi})$ and $\mathsf{D}_3\boldsymbol{\phi}]^T\mathbf{b}_1(\boldsymbol{\phi}) = -\mathbf{d}_2(\boldsymbol{\phi})$.

Using equation (2.6.1.6) we can express the angular momentum with respect the reference frame as

$$\boldsymbol{\omega}^o = \mathsf{B}^o[\boldsymbol{\phi}]\,\boldsymbol{\phi}_t \tag{2.6.1.24}$$

where

$$\mathsf{B}^o[\boldsymbol{\phi}] \equiv \mathsf{A}^T[\boldsymbol{\phi}]\mathsf{B}[\boldsymbol{\phi}] = \begin{pmatrix} 0 & -s\phi & s\theta c\phi \\ 0 & c\phi & s\theta s\phi \\ 1 & 0 & c\theta \end{pmatrix} . \tag{2.6.1.25}$$

The kinetic energy of the rod for general three-dimensional motion is

$$T(\mathbf{h},\boldsymbol{\phi}) = \tfrac{1}{2}\int_{-L}^{L} \{\rho(s)|\mathbf{h}_t(s)|^2 + \boldsymbol{\omega}\cdot\mathsf{I}\,\boldsymbol{\omega}\}\; ds. \tag{2.6.1.26}$$

Using the expression (2.6.1.6) for the angular velocity, the kinetic energy can be rewritten as

$$\begin{aligned} T(\mathbf{h},\boldsymbol{\phi}) &= \tfrac{1}{2}\int_{-L}^{L} \left\{\rho(s)|\mathbf{h}_t(s)|^2 + \boldsymbol{\phi}_t\cdot\mathsf{I}_B[\boldsymbol{\phi}]\,\boldsymbol{\phi}_t\right\}\, ds \\ &= \tfrac{1}{2}\int_{-L}^{L} \left\{\rho(s)|\mathbf{h}_t(s)|^2 + I_j\left[\mathbf{b}^j(\boldsymbol{\phi})\cdot\boldsymbol{\phi}_t\right]^2\right\}\, ds, \end{aligned} \tag{2.6.1.27}$$

where $\mathsf{I}_B[\boldsymbol{\phi}] \equiv \mathsf{B}[\boldsymbol{\phi}]^T\,\mathsf{I}\,\mathsf{B}[\boldsymbol{\phi}]$ is given by

$$\mathsf{I}_B[\boldsymbol{\phi}] = \begin{pmatrix} \begin{array}{c}(I_1c^2\psi + I_2s^2\psi)s^2\theta \\ +I_3c^2\theta\end{array} & (I_2 - I_1)s\psi c\psi s\theta & I_3c\theta \\ \cdot & I_1s^2\psi + I_2c^2\psi & 0 \\ \cdot & \cdot & I_3 \end{pmatrix}, \tag{2.6.1.28}$$

and the entry $\cdot$ denotes the presence of the symmetric element. Similarly the potential energy in terms of Euler angles is

$$\begin{aligned} V(\boldsymbol{\phi}) &= \tfrac{1}{2}\int_{-L}^{L} \boldsymbol{\phi}_s\cdot\mathsf{K}_B[\boldsymbol{\phi}]\boldsymbol{\phi}_s,\, ds \\ &= \tfrac{1}{2}\int_{-L}^{L} K_j\left[\mathbf{b}^j(\boldsymbol{\phi})\cdot\boldsymbol{\phi}_s\right]^2\, ds, \end{aligned} \tag{2.6.1.29}$$

where $\mathsf{K}_B[\boldsymbol{\phi}] \equiv \mathsf{B}[\boldsymbol{\phi}]^T \mathsf{K} \mathsf{B}[\boldsymbol{\phi}]$ is given by

$$\mathsf{K}_B[\boldsymbol{\phi}] = \begin{pmatrix} \begin{array}{c}(K_1 c^2\psi + K_2 s^2\psi)s^2\theta \\ +K_3 c^2\theta\end{array} & (K_2 - K_1)s\psi c\psi s\theta & K_3 c\theta \\ \cdot & K_1 s^2\psi + K_2 c^2\psi & 0 \\ \cdot & \cdot & K_3 \end{pmatrix}. \qquad (2.6.1.30)$$

2.6.2 Euler Parameters

A set of Euler parameters is a quadruple of real numbers $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)^T$ that satisfies the identity

$$\boldsymbol{\beta} \cdot \boldsymbol{\beta} = {\beta_1}^2 + {\beta_2}^2 + {\beta_3}^2 + \beta_4^2 = 1. \qquad (2.6.2.1)$$

The set of all quadruples satisfying (2.6.2.1) forms the unit sphere $\mathbf{S}^3$ in $\mathbf{R}^4$. A set of Euler parameters $\boldsymbol{\beta}$ is often called a unit quaternion, in connection with the algebra of quaternions [56]. In fact one reason that Euler parameters are so useful for the representation of $SO(3)$ is because the composition of two rotations may be expressed an elegant manner which is closely related to the multiplication of quaternions. Euler parameters are also closely associated with the Cayley-Klein parameters used to represent $SU(2)$ [36]. However we will not explore these aspects in this dissertation.

To obtain a representation of the directors in terms of Euler parameters, first recall that by Euler's Theorem each element of $\mathsf{A} \in SO(3)$ is equivalent to a rotation through an angle Φ about an axis determined by a unit vector $\mathbf{k} = k_j \mathbf{e}_j$. Specifically A is given in terms of $\mathbf{k}$ and Φ by

$$\begin{aligned} \mathsf{A} &= (\cos\Phi)\mathbf{1} + (1 - \cos\Phi)\mathbf{k} \otimes \mathbf{k} - \sin\Phi[\mathbf{k}\times] \\ &= \begin{pmatrix} c\Phi + k_1^2(1 - c\Phi) & k_1 k_2(1 - c\Phi) + k_3 s\Phi & k_1 k_3(1 - c\Phi) - k_2 s\Phi \\ k_1 k_2(1 - c\Phi) - k_3 s\Phi & c\Phi + k_2^2(1 - c\Phi) & k_2 k_3(1 - c\Phi) + k_1 s\Phi \\ k_1 k_3(1 - c\Phi) + k_2 s\Phi & k_2 k_3(1 - c\Phi) - k_1 s\Phi & c\Phi + k_3^2(1 - c\Phi) \end{pmatrix}, \end{aligned} \qquad (2.6.2.2)$$

where $[\mathbf{k}\times]$ is the skew matrix

$$(\mathbf{k}\times) \equiv \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} \qquad (2.6.2.3)$$

corresponding to the cross product with the vector $\mathbf{k}$. The Euler parameters $(\beta_1, \beta_2, \beta_3, \beta_4)$ are defined in terms of $\mathbf{k}$ and Φ by

$$\beta_j = k_j \sin(\Phi/2), \qquad \text{for } j = 1, 2, 3 \qquad (2.6.2.4)$$

and

$$\beta_4 = \cos(\Phi/2). \tag{2.6.2.5}$$

Euler parameters have the unusual feature that if $\boldsymbol{\beta}$ corresponds to the pair $(\mathbf{k}, \Phi)$, then $-\boldsymbol{\beta}$ corresponds to the pair $(\mathbf{k}, \Phi + 2\pi)$ which represents a coterminal rotation. Thus there is a two-to-one correspondence between Euler parameters and rotations. Exploiting the identities $\cos\Phi = 2\cos^2(\Phi/2) - 1$ and $\sin\Phi = 2\sin(\Phi/2)\cos(\Phi/2)$ together with equations (2.6.2.1), (2.6.2.5), (2.6.2.4) and (2.6.2.2) one can represent A in terms of $\boldsymbol{\beta}$ as

$$\mathsf{A} = \frac{1}{|\boldsymbol{\beta}|^2}\begin{pmatrix} \beta_1^2 - \beta_2^2 - \beta_3^2 + \beta_4^2 & 2(\beta_1\beta_2 + \beta_3\beta_4) & 2(\beta_1\beta_3 - \beta_2\beta_4) \\ 2(\beta_1\beta_2 - \beta_3\beta_4) & -\beta_1^2 + \beta_2^2 - \beta_3^2 + \beta_4^2 & 2(\beta_1\beta_4 + \beta_2\beta_3) \\ 2(\beta_1\beta_3 + \beta_2\beta_4) & 2(-\beta_1\beta_4 + \beta_2\beta_3) & -\beta_1^2 - \beta_2^2 + \beta_3^2 + \beta_4^2 \end{pmatrix}. \tag{2.6.2.6}$$

The factor of $1/|\boldsymbol{\beta}|^2$ appearing in equation (2.6.2.6) is not generally used in this representation. However it is advantageous in the formulation of a Hamiltonian system to introduce this term which makes A explicitly orthogonal.

The matrix A is again used to denote the transformation from the reference frame $\mathcal{O} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the director frame $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$. Thus

$$\mathbf{d}_1 = \frac{1}{|\boldsymbol{\beta}|^2}\begin{pmatrix} \beta_1^2 - \beta_2^2 - \beta_3^2 + \beta_4^2 \\ 2(\beta_1\beta_2 + \beta_3\beta_4) \\ 2(\beta_1\beta_3 - \beta_2\beta_4) \end{pmatrix}, \tag{2.6.2.7}$$

$$\mathbf{d}_2 = \frac{1}{|\boldsymbol{\beta}|^2}\begin{pmatrix} 2(\beta_1\beta_2 - \beta_3\beta_4) \\ -\beta_1^2 + \beta_2^2 - \beta_3^2 + \beta_4^2 \\ 2(\beta_1\beta_4 + \beta_2\beta_3) \end{pmatrix}, \tag{2.6.2.8}$$

and

$$\mathbf{d}_3 = \frac{1}{|\boldsymbol{\beta}|^2}\begin{pmatrix} 2(\beta_1\beta_3 + \beta_2\beta_4) \\ 2(-\beta_1\beta_4 + \beta_2\beta_3) \\ -\beta_1^2 - \beta_2^2 + \beta_3^2 + \beta_4^2 \end{pmatrix}. \tag{2.6.2.9}$$

If we assume $\beta_4 \neq 0$, then the Euler parameters may be recovered from the rotation matrix (2.6.2.6) and equation (2.6.2.1) by the formulæ

$$\beta_4 = \pm\frac{1}{2}\sqrt{1 + A_{1,1} + A_{2,2} + A_{3,3}}, \tag{2.6.2.10}$$

$$\beta_1 = \frac{1}{4\beta_4}(A_{2,3} - A_{3,2}), \tag{2.6.2.11}$$

$$\beta_2 = \frac{1}{4\beta_4}(A_{3,1} - A_{1,3}), \tag{2.6.2.12}$$

and

$$\beta_3 = \frac{1}{4\beta_4}(A_{1,2} - A_{2,1}). \tag{2.6.2.13}$$

(A more robust conversion algorithm that handles the case where $\beta_4 = 0$ is presented in [57].) The two-to-one correspondence of Euler parameters to each rotation is reflected by the sign ambiguity in equation (2.6.2.10) for β_4. In the particular case of planar motion where the directors are given by equations (2.5.1) - (2.5.3), the Euler parameters derived from equations (2.6.2.10) - (2.6.2.13) are

$$\boldsymbol{\beta} = \tfrac{1}{\sqrt{2}}\left(-\sin\left(\tfrac{1}{2}\phi\right), \cos\left(\tfrac{1}{2}\phi\right), \sin\left(\tfrac{1}{2}\phi\right), \cos\left(\tfrac{1}{2}\phi\right)\right). \tag{2.6.2.14}$$

It is worth mentioning that, had we instead considered planar motion in the $(\mathbf{e}_2, \mathbf{e}_3)-$ plane for which $\mathbf{d}_1 = \mathbf{e}_1$, $\mathbf{d}_2 = \cos\phi\,\mathbf{e}_2 - \sin\phi\,\mathbf{e}_3$ and $\mathbf{d}_3 = \sin\phi\,\mathbf{e}_2 + \cos\phi\,\mathbf{e}_3$, then the associated Euler parameters would simply be $\boldsymbol{\beta} = \left(\sin\left(\tfrac{1}{2}\phi\right), 0, 0, \cos\left(\tfrac{1}{2}\phi\right)\right)$. However in this dissertation we chose to analyze the stability of solitary waves in terms of both Euler parameters and Euler angles. Our choice of coordinates is therefore determined primarily by the need to avoid the singularities in Euler angles, as discussed in the previous subsection. By contrast, the particular choice of the coordinate system does not significantly affect the analysis in terms of Euler parameters.

The strains u and the angular velocities $\boldsymbol{\omega}$ appearing in equations (2.2.2) and (2.2.3) are given in terms of the Euler parameters by

$$\mathsf{u} = \frac{2}{|\boldsymbol{\beta}|^2}\,\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_s \tag{2.6.2.15}$$

and

$$\boldsymbol{\omega} = \frac{2}{|\boldsymbol{\beta}|^2}\,\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_t, \tag{2.6.2.16}$$

where

$$\mathsf{B}(\boldsymbol{\beta}) = \mathbf{e}_j \otimes \mathsf{B}_j\boldsymbol{\beta} = \begin{pmatrix} \beta_4 & \beta_3 & -\beta_2 & -\beta_1 \\ -\beta_3 & \beta_4 & \beta_1 & -\beta_2 \\ \beta_2 & -\beta_1 & \beta_4 & -\beta_3 \end{pmatrix}. \tag{2.6.2.17}$$

The matrix $\mathsf{B}[\boldsymbol{\beta}]$ can be computed by using the same method used in the previous section to obtain $\mathsf{B}[\boldsymbol{\phi}]$, and the vectors $\{\mathsf{B}_j\boldsymbol{\beta}\}$ that form the rows of of $\mathsf{B}[\boldsymbol{\beta}]$ are analogous to the vectors $\{\mathbf{b}^j(\bar{\phi})\}$ defined there. However the vectors $\{\mathsf{B}_j\boldsymbol{\beta}\}$ are orthonormal, so there is no need to define a distinct dual basis. Moreover $\mathsf{B}_j\boldsymbol{\beta}$ is orthogonal to $\boldsymbol{\beta}$ and is linear in the Euler parameters, which simplifies both our analysis and our computations. I am not aware of any prior work in the kinematics of rods or of rigid bodies where the vectors $\mathsf{B}_j\boldsymbol{\beta}$ have been utilized explicitly.

As with the Euler angles we define the matrices

$$\mathsf{D}_j[\boldsymbol{\beta}] \equiv \left(\frac{\partial \mathbf{d}_j}{\partial \boldsymbol{\beta}}\right)^T, \tag{2.6.2.18}$$

and note that these matrices can be expressed in terms of $\{\mathsf{B}_j\boldsymbol{\beta}\}$ and $\{\mathbf{d}_j(\boldsymbol{\beta})\}$ as

$$\mathsf{D}_1[\boldsymbol{\beta}] \equiv \left(\frac{\partial \mathbf{d}_1}{\partial \boldsymbol{\beta}}\right)^T = \frac{2}{|\boldsymbol{\beta}|^2}\,\mathsf{B}_3\boldsymbol{\beta}\otimes\mathbf{d}_2 - \frac{2}{|\boldsymbol{\beta}|^2}\,\mathsf{B}_2\boldsymbol{\beta}\otimes\mathbf{d}_3, \tag{2.6.2.19}$$

$$\mathsf{D}_2[\boldsymbol{\beta}] \equiv \left(\frac{\partial \mathbf{d}_2}{\partial \boldsymbol{\beta}}\right)^T = \frac{2}{|\boldsymbol{\beta}|^2}\,\mathsf{B}_1\boldsymbol{\beta}\otimes\mathbf{d}_3 - \frac{2}{|\boldsymbol{\beta}|^2}\,\mathsf{B}_3\boldsymbol{\beta}\otimes\mathbf{d}_1 \tag{2.6.2.20}$$

and

$$\mathsf{D}_3[\boldsymbol{\beta}] = \left(\frac{\partial \mathbf{d}_3}{\partial \boldsymbol{\beta}}\right)^T = \frac{2}{|\boldsymbol{\beta}|^2}\,\mathsf{B}_2\boldsymbol{\beta}\otimes\mathbf{d}_1 - \frac{2}{|\boldsymbol{\beta}|^2}\,\mathsf{B}_1\boldsymbol{\beta}\otimes\mathbf{d}_2. \tag{2.6.2.21}$$

As a consequence of equations (2.6.2.19) - (2.6.2.21) we observe the identities

$$\begin{aligned} \mathsf{D}_k[\boldsymbol{\beta}]\mathbf{d}_k(\boldsymbol{\beta}) &= \mathbf{0} \qquad \text{(no sum)}\ , && (2.6.2.22)\\ \mathsf{D}_k[\boldsymbol{\beta}]^T\mathsf{B}_k\boldsymbol{\beta} &= \mathbf{0} \qquad \text{(no sum)}\ , && (2.6.2.23)\\ \mathsf{D}_k[\boldsymbol{\beta}]^T\boldsymbol{\beta} &= \mathbf{0}. && (2.6.2.24) \end{aligned}$$

Equation (2.6.2.22) follows from the fact that $\mathbf{d}_k$ is a unit vector, which implies

$$\frac{1}{2}\frac{\partial}{\partial \boldsymbol{\beta}}|\mathbf{d}_k(\boldsymbol{\beta})|^2 = \mathsf{D}_k[\boldsymbol{\beta}]\mathbf{d}_k(\boldsymbol{\beta}) = 0. \tag{2.6.2.25}$$

Equation (2.6.2.23) is a manifestation of the geometric interpretation of $\mathsf{B}_k\boldsymbol{\beta}$ as the instantaneous rotation about the $\mathbf{d}_k$ axis, which does not change $\mathbf{d}_k$ itself. Equation (2.6.2.24) is a consequence of our definition of $\mathbf{d}_k$ as a function homogeneous in $\boldsymbol{\beta}$ with degree zero.

The angular momentum with respect the reference frame is

$$\boldsymbol{\omega}^o = \frac{2}{|\boldsymbol{\beta}|^2}\,\mathsf{B}^o[\boldsymbol{\beta}]\boldsymbol{\beta}_t, \tag{2.6.2.26}$$

where

$$\begin{aligned} \mathsf{B}^o[\boldsymbol{\beta}] \equiv \mathsf{A}[\boldsymbol{\beta}]^T\mathsf{B}[\boldsymbol{\beta}] &= \mathbf{d}_j(\boldsymbol{\beta})\otimes\mathsf{B}_j\boldsymbol{\beta}\\ &= \mathbf{e}_j\otimes\mathsf{B}^o_j\boldsymbol{\beta}\\ &= \begin{pmatrix} \beta_4 & -\beta_3 & \beta_2 & -\beta_1\\ \beta_3 & \beta_4 & -\beta_1 & -\beta_2\\ -\beta_2 & \beta_1 & \beta_4 & -\beta_3 \end{pmatrix}. \end{aligned} \tag{2.6.2.27}$$

We now note a few properties of the matrices B and B^o which will be used later. Observe first that the 4-by-4 matrix given by

$$\mathsf{R}[\boldsymbol{\beta}] \equiv \frac{1}{|\boldsymbol{\beta}|}\begin{pmatrix} \mathsf{B}[\boldsymbol{\beta}]\\ \boldsymbol{\beta}^T \end{pmatrix} \tag{2.6.2.28}$$

is orthogonal. It then follows from the identity $\mathsf{R}[\boldsymbol{\beta}]\mathsf{R}[\boldsymbol{\beta}]^T = \mathbf{1}$ that

$$\mathsf{B}[\boldsymbol{\beta}]\mathsf{B}^T[\boldsymbol{\beta}] = |\boldsymbol{\beta}|^2\mathbf{1}, \tag{2.6.2.29}$$

and that the 3-by-4 matrix $\mathsf{B}[\boldsymbol{\beta}]$ has rank 3, with its null space spanned by $\boldsymbol{\beta}$:

$$\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta} = \mathbf{0}. \tag{2.6.2.30}$$

Similarly the identity $\mathsf{R}[\boldsymbol{\beta}]^T\mathsf{R}[\boldsymbol{\beta}] = \mathbf{1}$ implies

$$\mathsf{B}^T[\boldsymbol{\beta}]\mathsf{B}[\boldsymbol{\beta}] = |\boldsymbol{\beta}|^2\boldsymbol{\Pi}[\boldsymbol{\beta}], \tag{2.6.2.31}$$

where

$$\boldsymbol{\Pi}[\boldsymbol{\beta}] \equiv \mathbf{1} - \frac{\boldsymbol{\beta}\otimes\boldsymbol{\beta}}{|\boldsymbol{\beta}|^2} \tag{2.6.2.32}$$

is the projection onto the subspace orthogonal to $\boldsymbol{\beta}$. Notice that equations (2.6.2.17), (2.6.2.27) and (2.6.2.29) imply

$$\mathsf{B}[\boldsymbol{\beta}]\ (\mathsf{B}^o[\boldsymbol{\beta}])^T = \mathsf{A}[\boldsymbol{\beta}]. \tag{2.6.2.33}$$

Equation (2.6.2.33) shows that the matrices $\mathsf{B}[\boldsymbol{\beta}]$ and $\mathsf{B}^o[\boldsymbol{\beta}]$ provide a factorization of the rotation matrix $\mathsf{A}[\boldsymbol{\beta}]$, analogous to the factorization of a rotation matrix in terms of three elementary rotations using Euler angles. Apparently Euler first introduced Euler parameters precisely for the purpose of producing such a factorization. (Cf. [25], [56].)

The matrices which determine the vectors $\mathsf{B}_j\boldsymbol{\beta}$ are

$$\mathsf{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.6.2.34}$$

$$\mathsf{B}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{2.6.2.35}$$

and

$$\mathsf{B}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{2.6.2.36}$$

The skew matrices B_k satisfy the commutation relations

$$\mathsf{B}_k^2 = -\mathbf{1} \qquad \text{for } k = 1, 2, 3, \tag{2.6.2.37}$$

$$\mathsf{B}_1\mathsf{B}_2 = -\mathsf{B}_3, \tag{2.6.2.38}$$

$$\mathsf{B}_2\mathsf{B}_3 = -\mathsf{B}_1, \tag{2.6.2.39}$$

$$\mathsf{B}_3\mathsf{B}_1 = -\mathsf{B}_2, \tag{2.6.2.40}$$

which can be summarized as

$$\mathsf{B}_j\mathsf{B}_k = -\delta_{jk}\mathbf{1} - \epsilon_{ijk}\mathsf{B}_i, \tag{2.6.2.41}$$

where δ_{jk} is the Kronecker delta, and ϵ_{ijk} is the permutation tensor. Thus group of matrices $\{\pm\mathbf{1}, \pm\mathsf{B}_k\}$ is isomorphic to the quaternion group. (Cf. [30].) The skewness of the matrices B_j together with the property (2.6.2.37) imply that each exponential $\mathsf{Q}_j \equiv \exp(\frac{1}{2}\epsilon\mathsf{B}_j)$ is an orthogonal transformation in $\mathbf{R}^4$ given by

$$\mathsf{Q}_j = \cos(\tfrac{1}{2}\epsilon)\mathbf{1} + \sin(\tfrac{1}{2}\epsilon)\mathsf{B}_j. \tag{2.6.2.42}$$

In fact each transformation Q_j on the Euler parameters corresponds to a rotation $\tilde{\mathsf{Q}}_j$ in $\mathbf{R}^3$ through an angle ϵ about the axis $\mathbf{d}_j$. In particular,

$$\begin{aligned}
\mathbf{d}_1\,[\mathsf{Q}_3\boldsymbol{\beta}] &= \cos(\epsilon)\,\mathbf{d}_1[\boldsymbol{\beta}] + \sin(\epsilon)\,\mathbf{d}_2[\boldsymbol{\beta}],\\
\mathbf{d}_2\,[\mathsf{Q}_3\boldsymbol{\beta}] &= -\sin(\epsilon)\,\mathbf{d}_1[\boldsymbol{\beta}] + \cos(\epsilon)\,\mathbf{d}_2[\boldsymbol{\beta}],\\
\mathbf{d}_3\,[\mathsf{Q}_3\boldsymbol{\beta}] &= \mathbf{d}_3[\boldsymbol{\beta}].
\end{aligned} \tag{2.6.2.43}$$

The transformation Q_3 describes the rotational symmetry about the $\mathbf{d}_3$ axis in the dynamics of an isotropic rod. In addition, because the vector $\mathsf{B}_1\boldsymbol{\beta}$ is the generator of rotations about the $\mathbf{d}_1$ axis it plays an important role in our study of planar motion. For example, the derivative of equation (2.6.2.14) with respect to s yields

$$\boldsymbol{\beta}_s = \tfrac{1}{\sqrt{2}}\left(-\cos\left(\tfrac{1}{2}\phi\right), -\sin\left(\tfrac{1}{2}\phi\right), \cos\left(\tfrac{1}{2}\phi\right), -\sin\left(\tfrac{1}{2}\phi\right)\right)\tfrac{1}{2}\phi_s \tag{2.6.2.44}$$

or

$$\boldsymbol{\beta}_s = -\tfrac{1}{2}\phi_s\mathsf{B}_1\boldsymbol{\beta}. \tag{2.6.2.45}$$

We use expressions like (2.6.2.45) repeatedly in our analysis, rather than appealing to the specific form (2.6.2.14) of the Euler parameters.

Similarly the vectors $\mathsf{B}^o_j\boldsymbol{\beta}$ are given by the matrices

$$\mathsf{B}^o_1 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.6.2.46}$$

$$\mathsf{B}^o_2 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{2.6.2.47}$$

and

$$\mathsf{B}_3^o = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.6.2.48}$$

The skew matrices B_k^o satisfy the commutation relations

$$\begin{aligned} (\mathsf{B}_k^o)^2 &= -\mathbf{1} \qquad \text{for } k = 1, 2, 3, && (2.6.2.49) \\ \mathsf{B}_1^o \mathsf{B}_2^o &= \mathsf{B}_3^o, && (2.6.2.50) \\ \mathsf{B}_2^o \mathsf{B}_3^o &= \mathsf{B}_1^o, && (2.6.2.51) \\ \mathsf{B}_3^o \mathsf{B}_1^o &= \mathsf{B}_2^o, && (2.6.2.52) \end{aligned}$$

which can be summarized as

$$\mathsf{B}_j^o \mathsf{B}_k^o = -\delta_{jk}\mathbf{1} + \epsilon_{ijk}\mathsf{B}_i^o. \tag{2.6.2.53}$$

The group of matrices $\{\pm\mathbf{1}, \pm\mathsf{B}_k^o\}$ is also isomorphic to the quaternion group. The matrices B_k are also closely related to the Pauli spin matrices. (Cf. e.g. [24].) In addition, while we are not aware of their prior application in rod dynamics, Seywald *et al.* [55] have exploited the matrices B_k^o and their commutation relations (2.6.2.53) to analyze the optimal control problem of reorienting a rigid spacecraft using the least amount of fuel. Each orthogonal transformation $\mathsf{Q}_j^o \equiv \exp(\frac{1}{2}\epsilon B_j^o)$ corresponds to a rotation $\tilde{\mathsf{Q}}_j^o$ in $\mathbf{R}^3$ about the axis $\mathbf{e}_j$ through an angle ϵ. These transformations also have the property that

$$\left(\mathsf{Q}_j^o\right)^T \mathsf{B}_k \, \mathsf{Q}_j^o = \mathsf{B}_k. \tag{2.6.2.54}$$

This invariance of the B_k matrices is related to the fact that the strains u_j and angular velocities ω_j relative to the $\{\mathbf{d}_k\}$ basis are unchanged by a rigid rotation of the entire rod.

Using the matrices B_k and B_k^o one can rewrite equation (2.6.2.33) in terms of the entries of matrix A as

$$\left(|\boldsymbol{\beta}|^2\mathbf{d}_j\right)\cdot\mathbf{e}_k = \left(|\boldsymbol{\beta}|^2\mathbf{d}_j\right)_k = \boldsymbol{\beta}\cdot\mathsf{D}_j^k\boldsymbol{\beta}, \tag{2.6.2.55}$$

where the 4-by-4 matrix D_j^k is given by

$$\mathsf{D}_j^k \equiv -\mathsf{B}_j \, \mathsf{B}_k^o. \tag{2.6.2.56}$$

One can show that each matrix D_j^k is symmetric, which together with the skew-symmetry of the matrices B_k and B_k^o implies

$$\mathsf{D}_j^k \equiv -\mathsf{B}_k^o \, \mathsf{B}_j. \tag{2.6.2.57}$$

Explicitly the matrices D_j^k are

$$\mathsf{D}_1^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (2.6.2.58)$$

$$\mathsf{D}_1^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad (2.6.2.59)$$

$$\mathsf{D}_1^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad (2.6.2.60)$$

$$\mathsf{D}_2^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad (2.6.2.61)$$

$$\mathsf{D}_2^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (2.6.2.62)$$

$$\mathsf{D}_2^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad (2.6.2.63)$$

$$\mathsf{D}_3^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad (2.6.2.64)$$

$$\mathsf{D}_3^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad (2.6.2.65)$$

$$\mathsf{D}_3^3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (2.6.2.66)$$

$$(2.6.2.67)$$

The matrices $\{\pm\mathbf{1}, \pm\mathsf{B}_k, \pm\mathsf{B}_k^o, \pm\mathsf{D}_j^k\}$ form a group under multiplication, in which the algebraic properties of B_j and B_k and the definition (2.6.2.56) of D_j^k imply

$$\begin{aligned}
\mathsf{D}_j^k\,\mathsf{B}_j &= \mathsf{B}_k^o, && (2.6.2.68)\\
\mathsf{D}_j^k\,\mathsf{B}_l &= -\epsilon_{jli}\,\mathsf{D}_i^k \qquad \text{for } l \neq j, && (2.6.2.69)\\
\mathsf{D}_j^k\,\mathsf{B}_k^o &= \mathsf{B}_j, && (2.6.2.70)\\
\mathsf{D}_j^k\,\mathsf{B}_l^o &= -\epsilon_{ikl}\,\mathsf{D}_j^i \qquad \text{for } l \neq k, && (2.6.2.71)\\
\mathsf{D}_i^j\,\mathsf{D}_i^j &= \mathbf{1}, && (2.6.2.72)\\
\mathsf{D}_i^j\,\mathsf{D}_i^l &= -\epsilon_{jln}\,\mathsf{B}_n^o \qquad \text{for } l \neq j, && (2.6.2.73)\\
\mathsf{D}_i^j\,\mathsf{D}_k^j &= \epsilon_{ikm}\,\mathsf{B}_m \qquad \text{for } k \neq i, && (2.6.2.74)\\
\mathsf{D}_i^j\,\mathsf{D}_k^l &= -\epsilon_{ikm}\,\epsilon_{jln}\,\mathsf{B}_m\,\mathsf{B}_n^o \qquad \text{for } k \neq i, l \neq j. && (2.6.2.75)
\end{aligned}$$

These relations are utilized in the variational analysis performed in Chapter 6. I am not aware of any prior work in kinematics where this group has been recorded.

Substituting equation (2.6.2.16) for the angular velocity into the expression (2.6.1.26) for the kinetic energy, we obtain

$$\begin{aligned}
T(\mathbf{h},\boldsymbol{\beta}) &= \int_{-L}^{L} \left\{ \tfrac{1}{2}\rho(s)|\mathbf{h}_t(s)|^2 + \tfrac{2}{|\boldsymbol{\beta}|^4}\,\boldsymbol{\beta}_t \cdot \mathsf{B}[\boldsymbol{\beta}]^T\,\mathsf{I}\,\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_t \right\} ds \\
&= \int_{-L}^{L} \left\{ \tfrac{1}{2}\rho(s)|\mathbf{h}_t(s)|^2 + \tfrac{2}{|\boldsymbol{\beta}|^4}\, I_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_t)^2 \right\} ds. \qquad (2.6.2.76)
\end{aligned}$$

Similarly the potential energy becomes

$$\begin{aligned}
V[\boldsymbol{\beta}] &= \int_{-L}^{L} \frac{2}{|\boldsymbol{\beta}|^4}\,\boldsymbol{\beta}_s \cdot \mathsf{B}[\boldsymbol{\beta}]^T\,\mathsf{K}\,\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_s.\, ds \\
&= \int_{-L}^{L} \frac{2}{|\boldsymbol{\beta}|^4}\, K_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)^2\, ds. \qquad (2.6.2.77)
\end{aligned}$$

Figure 1: A representative curve describing the configuration of a planar rod. The curve is parameterized by its arc-length s. The location of the material point at s is given by both its position $\mathbf{r}$ with respect to the origin and its displacement $\mathbf{h}$ from the reference configuration. The unit tangent vector is $\mathbf{d}_3$.

CHAPTER 3

CONSERVATION LAWS

3.1 Introduction

This chapter describes the conservation laws that are implied by the equations of motion of a rod in the presence of a continuous symmetry group. For our purposes a conservation law means an expression of the form

$$D_t(s,t) = F_s(s,t), \tag{3.1.1}$$

where D, the density, and F, the (negative of) the flux, are each functions of the (scalar) independent variables s and t, and the (vector) dependent variable $\mathbf{y}$ and its derivatives:

$$\begin{aligned} D(s,t) &= \tilde{D}(\mathbf{y}(s,t), \mathbf{y}'(s,t), \dot{\mathbf{y}}(s,t), s, t) \quad (3.1.2)\\ F(s,t) &= \tilde{F}(\mathbf{y}(s,t), \mathbf{y}'(s,t), \dot{\mathbf{y}}(s,t), s, t) \quad (3.1.3) \end{aligned}$$

Conservation laws arise naturally in rod dynamics. It may be observed immediately that the balance of linear momentum (2.3.5) is in the form of a conservation law with density $\mathbf{p}$ and flux $\boldsymbol{\lambda}$. Moreover,we can use (2.3.1) and (2.3.5) to rewrite equation (2.3.6) in the conservation form

$$(\boldsymbol{\pi} + \mathbf{r} \times \mathbf{p})_t = (\mathbf{m} + \mathbf{r} \times \boldsymbol{\lambda})_s \, . \tag{3.1.4}$$

Consequently, for appropriate boundary conditions, such as the periodic ones pertinent for a closed loop, the total linear momentum $\int \mathbf{p}\, ds$ and total angular momentum $\int (\boldsymbol{\pi} + \mathbf{r} \times \mathbf{p})\, ds$ about a fixed origin are integrals of the motion, as could have been surmised at the outset. Equations (2.3.5) and (3.1.4) are conservation laws independent of the constitutive relations. By contrast, the additional conservation laws that we discuss in this chapter are associated with specific symmetries in the constitutive properties.

The primary reason for our interest in conservation laws is that for boundary conditions such that the flux vanishes at the boundaries, each conservation law (3.1.1) can be integrated over the space variable s to yield an associated integral of motion

$$\frac{d}{dt} \int D\, ds = 0. \tag{3.1.5}$$

In fact it is the intimate connection between symmetries and integrals of motion, associated with Noether's Theorem, that provides the basis for the stability analysis we utilize here.

In the remainder of this chapter we consider three special cases of rods with associated conservation laws. In Section 3.2 the conservation law and associated integral for uniform, hyperelastic rods are described, where the symmetry is translation in arc-length. In this case the integral will be called the impulse (3.2.14). The impulse is not simply related to total linear momentum, which, for rod dynamics, is the integral generated by translational symmetry in space. In Section 3.3 the analogous quantities are derived in the appropriate (sub-)case of transverse isotropy, where the symmetry is rotation about the centerline of the rod. In the case of an isotropic rod the integral is the total component of the angular momentum along the local axis of transverse isotropy, and will therefore be called the isotropic angular momentum (3.3.25). The particular manifestations of the conservation laws in the special cases of string models, of inextensible, unshearable rods, and of planar motion are outlined in Section 3.4.

3.2 Uniform Hyperelastic Rods

It will now be demonstrated that for a uniform, hyperelastic rod, the equations (2.3.5) and (2.3.6) governing the dynamics imply another conservation law and associated first integral. The rod is said to be uniform if the coefficients ρ and I_{ij} are actually constants,

$$\rho' \equiv I'_{ij} \equiv 0, \tag{3.2.6}$$

and, in addition, the constitutive law (2.4.4) and (2.4.5) has no explicit dependence on the arc-length s. Because the constitutive laws specify the components of $\boldsymbol{\lambda}$ and $\mathbf{m}$ with respect to the variable basis $\{\mathbf{d}_i\}$, the kinematic relations

$$\boldsymbol{\lambda}' = n'_i \mathbf{d}_i + \mathbf{u} \times \boldsymbol{\lambda}, \tag{3.2.7}$$

and

$$\mathbf{m}' = m'_i \mathbf{d}_i + \mathbf{u} \times \mathbf{m} \tag{3.2.8}$$

are implied.

The following conservation law is then a consequence of the governing equations (2.3.5) and (2.3.6) that express balance of linear and angular momentum:

$$(\boldsymbol{\pi} \cdot \mathbf{u} + \mathbf{p} \cdot \mathbf{v})_t = \Big(\mathbf{m} \cdot \mathbf{u} + \boldsymbol{\lambda} \cdot \mathbf{v} - W + \tfrac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\pi} + \tfrac{1}{2}\dot{\mathbf{r}} \cdot \mathbf{p}\Big)_s \ . \tag{3.2.9}$$

To verify (3.2.9), the derivative on the left-hand side can be expanded, using (2.2.1), as

$$\dot{\boldsymbol{\pi}} \cdot \mathbf{u} + \boldsymbol{\pi} \cdot \dot{\mathbf{u}} + \dot{\mathbf{p}} \cdot \mathbf{v} + \mathbf{p} \cdot \dot{\mathbf{r}}', \tag{3.2.10}$$

which, upon elimination of $\dot{\mathbf{u}}$, $\mathbf{p}$, $\boldsymbol{\pi}$, $\dot{\mathbf{p}}$ and $\dot{\boldsymbol{\pi}}$ by use of (2.2.6), (2.3.1), (2.3.2), (2.3.5) and (2.3.6), can be seen to be the same as

$$(\mathbf{m}' + \mathbf{r}' \times \boldsymbol{\lambda}) \cdot \mathbf{u} + \omega_i I_{ij} \omega_j' + \boldsymbol{\lambda}' \cdot \mathbf{v} + \rho \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}'. \tag{3.2.11}$$

Then by the kinematic relations (2.2.1), (2.2.5), (3.2.7), (3.2.8) and symmetry condition (2.3.3), we obtain

$$m_i' u_i + n_i' v_i + \left(\tfrac{1}{2}\omega_i I_{ij} \omega_j\right)_s + \left(\tfrac{1}{2}\rho \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right)_s, \tag{3.2.12}$$

or, with constitutive laws (2.4.4) and (2.4.5),

$$\left(u_i W_{u_i} + v_i W_{v_i} - W + \tfrac{1}{2}\omega_i I_{ij} \omega_j \;+ \tfrac{1}{2}\rho \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right)_s, \tag{3.2.13}$$

which can be seen to coincide with the right-hand side of (3.2.9) because of the constitutive relations (2.3.1), (2.3.2), (2.4.4) and (2.4.5).

It should be remarked that it has nowhere been assumed that the constitutive relations (2.4.4) and (2.4.5) are such that the stresses $(\boldsymbol{\lambda}, \mathbf{m})$ vanish for vanishing strains $(\mathbf{v}, \mathbf{u})$. Consequently the analysis presented here is valid for the dynamics of uniform hyperelastic rods whose unstressed state is helical.

The quantity

$$\int (\boldsymbol{\pi} \cdot \mathbf{u} + \mathbf{p} \cdot \mathbf{v}) \; ds \tag{3.2.14}$$

is called the impulse, where it is understood that the range of integration is the entire length of the rod. The impulse is clearly a finite quantity if the rod is of finite length (and the evolution is assumed to be smooth), but the impulse is also finite for an infinite rod if either the strains $(\mathbf{v}, \mathbf{u})$ or the momenta $(\mathbf{p}, \boldsymbol{\pi})$ decay sufficiently rapidly.

The impulse is actually a time-invariant quantity, i.e., a conserved quantity or integral of the dynamics, for systems with appropriate boundary conditions. For example in a finite closed loop of a rod, all the dependent variables must be periodic functions of s, and the impulse is therefore a conserved quantity. For an infinite rod the impulse can also be a (finite) conserved quantity. For example, the impulse will be a constant of the motion if the momenta satisfy $\lim_{s\to\pm\infty}(\mathbf{p}, \boldsymbol{\pi}) = 0$ and the strains satisfy $\lim_{s\to\pm\infty}(v_i, u_i) = (\hat{v}_i, \hat{u}_i)$ for some constants $(\hat{v}_i, \hat{u}_i)$.

Notice that the flux appearing in the conservation law (3.2.9) is the Legendre transform of the density of total energy of the rod. As such it is reminiscent of the Eshelby Energy-Momentum tensor which arises in a conservation law for the elasto-statics of a homogeneous elastic material in several space dimensions. This problem is discussed among other places, by Olver [52], Example 4.32, p. 281, with a brief discussion of related issues and literature on p. 288.

It should also be remarked that linear combinations of the conservation laws (2.3.5), (3.1.4), and (3.2.9) can be taken to yield new conservation laws. In some circumstances such a construction may have nontrivial implications. For example the boundary conditions may be such that a linear combination of the fluxes may vanish and so provide a conserved quantity, while the individual conservation laws do not imply any integral of motion. Moreover, in the case of an infinite rod some linear combination of the densities may provide a finite conserved quantity while none of the individual conservation laws have associated convergent integrals. This last phenomenon is related to the choice of frame from which the rod dynamics is viewed.

3.3 Isotropic Rods

We now drop the hypotheses that the rod is uniform and hyperelastic and instead assume that the rod is transversely isotropic. By transversely isotropic it is meant that the constitutive relations for the stresses and momenta are symmetric with respect to rotations about one of the directors, $\mathbf{d}_3$ say. As full isotropy, i.e., independence of the constitutive relation to arbitrary rotations of the basis $\{\mathbf{d}_i\}$, is not a pertinent concept in the context of rods, we shall henceforth allow the adjective transversely to stand by implication. Antman ([2], Theorem 9.16) demonstrates that the hypothesis of isotropic elastic response is equivalent to the specialized constitutive relations

$$\boldsymbol{\lambda} = N_1\mathbf{v} + N_2\mathbf{u} + N_3\mathbf{d}_3, \tag{3.3.15}$$

and

$$\mathbf{m} = M_1\mathbf{v} + M_2\mathbf{u} + M_3\mathbf{d}_3, \tag{3.3.16}$$

where the N_i, and M_i are each scalar functions of the six scalar arguments $|\mathbf{u}|$, $|\mathbf{v}|$, $\mathbf{u}\cdot\mathbf{v}$, u_3, v_3 and s. Isotropy of the dynamic properties of the rod follows when the expression (2.3.2) for the angular momentum is specialized to

$$\boldsymbol{\pi} \equiv I_1\boldsymbol{\omega} + (I_3 - I_1)\omega_3\mathbf{d}_3. \tag{3.3.17}$$

Here $\mathbf{d}_3$ is a principal axis of inertia with associated principal inertia $I_3(s)$, and the other two principal inertias both equal $I_1(s)$ (cf. (2.3.4) with $I_1 = I_2$). The general expression (2.3.1) for the linear momentum is already isotropic.

With the constitutive relations (3.3.15), (3.3.16) and (3.3.17), and the additional restriction

$$M_1(|\mathbf{u}|, \mathbf{u}\cdot\mathbf{v}, |\mathbf{v}|, u_3, v_3, s) \equiv N_2(|\mathbf{u}|, \mathbf{u}\cdot\mathbf{v}, |\mathbf{v}|, u_3, v_3, s), \tag{3.3.18}$$

the conservation law

$$(\boldsymbol{\pi}\cdot\mathbf{d}_3)_t = (\mathbf{m}\cdot\mathbf{d}_3)_s, \tag{3.3.19}$$

is a consequence of the angular momentum balance (2.3.6). In particular the derivation of (3.3.19) does not exploit the linear momentum balance (2.3.5) so the conservation law remains valid even for systems with external forces. The delimitation (3.3.18) is actually implied in the case of an isotropic hyperelastic rod where there is a strain energy function of the form $W(|\mathbf{u}|, \mathbf{u}\cdot\mathbf{v}, |\mathbf{v}|, u_3, v_3, s)$. Another plausible special case is $M_1 \equiv N_2 \equiv 0$.

The validity of (3.3.19) can be seen after three preliminary calculations. First we note that because of hypothesis (3.3.16)

$$\mathbf{m}\cdot(\mathbf{u}\times\mathbf{d}_3) = M_1\mathbf{v}\cdot(\mathbf{u}\times\mathbf{d}_3). \tag{3.3.20}$$

Similarly by (3.3.17)

$$\boldsymbol{\pi}\cdot(\boldsymbol{\omega}\times\mathbf{d}_3) = 0. \tag{3.3.21}$$

Thirdly equations (2.2.1) and (3.3.15) are used to write the balance of angular momentum (2.3.6) in the form

$$\mathbf{m}' + \mathbf{v}\times(N_2\mathbf{u} + N_3\mathbf{d}_3) = \dot{\boldsymbol{\pi}}, \tag{3.3.22}$$

from which it may be concluded that

$$\mathbf{m}'\cdot\mathbf{d}_3 + N_2(\mathbf{v}\times\mathbf{u})\cdot\mathbf{d}_3 = \dot{\boldsymbol{\pi}}\cdot\mathbf{d}_3. \tag{3.3.23}$$

Now we use (2.2.2), (2.2.3), (3.3.18), (3.3.20), (3.3.21) and (3.3.23) to calculate that

$$(\boldsymbol{\pi}\cdot\mathbf{d}_3)_t - (\mathbf{m}\cdot\mathbf{d}_3)_s = \dot{\boldsymbol{\pi}}\cdot\mathbf{d}_3 + \boldsymbol{\pi}\cdot(\boldsymbol{\omega}\times\mathbf{d}_3) - \mathbf{m}'\cdot\mathbf{d}_3 - \mathbf{m}\cdot(\mathbf{u}\times\mathbf{d}_3) = 0, \tag{3.3.24}$$

as is required to validate (3.3.19).

The quantity

$$\int \boldsymbol{\pi}\cdot\mathbf{d}_3\, ds \tag{3.3.25}$$

will be called the (component of) isotropic angular momentum. The isotropic angular momentum is an integral of the motion for appropriate boundary conditions, for example the periodic conditions associated with a closed loop, as has already been observed by Simo *et al.* in [58]. The remarks made at the end of Section 3.4 concerning infinite rods and linear combinations of conservation laws apply with equal validity here.

3.4 Strings, Inextensible Rods and Planar Dynamics

In this section we consider the conservation laws that arise in certain degenerate cases of the rod model described in Section 3.2. The treatment of special cases is by no means exhaustive, but it does relate the conservation laws (3.2.9) and (3.3.19) to quantities found in previous works.

A model of a string is obtained if the strain $\mathbf{v}$ is declared to satisfy $v_1 = v_2 = 0$, so that (2.2.1) reduces to

$$\mathbf{r}' = v_3 \mathbf{d}_3, \tag{3.4.26}$$

and the constitutive laws for the force and moment are taken to be

$$n_1 = n_2 = 0, \qquad \boldsymbol{\lambda} = N_3 \mathbf{d}_3, \tag{3.4.27}$$

and

$$m_1 = m_2 = 0, \qquad \mathbf{m} = M_3 \mathbf{d}_3. \tag{3.4.28}$$

Here N_3 and M_3 are scalar functions of the arguments (v_3, u_1, u_2, u_3, s). Conditions (3.4.26) and (3.4.27) require that the force in the string is always tangential. Condition (3.4.28) states that there are no bending moments, although the string can support a twisting moment about the unit tangent vector $\mathbf{d}_3$.

For this string model the balance of angular momentum (2.3.6) simplifies to

$$\dot{\boldsymbol{\pi}} = \mathbf{m}', \tag{3.4.29}$$

which is trivially in conservation form. Thus for the dynamics of any such string there are always three conservation laws, namely (2.3.5), (3.4.29), and, because (3.1.4) is still valid,

$$(\mathbf{r} \times \mathbf{p})_t = (\mathbf{r} \times \boldsymbol{\lambda})_s. \tag{3.4.30}$$

It could be argued that (3.4.30) is not independent of (2.3.5), and (3.4.29), but because it is expressed in terms of different variables it has the potential to yield integrals of motion under different sets of boundary conditions.

The string is uniform and hyperelastic if there is a strain energy density function $W(v_3, u_3)$ such that $N_3 = W_{v_3}$ and $M_3 = W_{u_3}$. (W must be independent of u_1 and u_2 to be consistent with the requirements $n_1 = n_2 = 0$.) For such a string the conservation of impulse (3.2.9) takes the form

$$(\boldsymbol{\pi} \cdot \mathbf{u} + \mathbf{p} \cdot \mathbf{r}')_t = \left(m_3 u_3 + n_3 v_3 - W + \tfrac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\pi} + \tfrac{1}{2}\dot{\mathbf{r}} \cdot \mathbf{p}\right)_s . \tag{3.4.31}$$

Notice that the constitutive laws for a hyperelastic string automatically satisfy the constitutive isotropy assumptions (3.3.15), (3.3.16), and (3.3.18) (with $N_1 \equiv N_2 \equiv M_1 \equiv M_2 \equiv 0$), so if the dynamic isotropy condition (3.3.17) is also satisfied we obtain the law of conservation of isotropic angular momentum (3.3.19) as before. Notice that the conservation laws (3.4.31) and (3.3.19) are truly independent. In the absence of the dynamic isotropy condition (3.3.17), equation (3.3.19) does not hold. A final special feature arises when for a uniform, hyperelastic, isotropic string, it is further assumed, as in classic string models, that the strain energy density function is of the decoupled form

$$W(v_3, u_3) = W_1(v_3) + W_2(u_3). \tag{3.4.32}$$

Then the balance of linear momentum (2.3.5) decouples from the balance of angular momentum (2.3.6), and the additional conservation law

$$(\mathbf{p} \cdot \mathbf{r}')_t = \left(n_3 v_3 - W_1 + \tfrac{1}{2}\dot{\mathbf{r}} \cdot \mathbf{p}\right)_s \tag{3.4.33}$$

is implied. The associated impulse is

$$\int \mathbf{p} \cdot \mathbf{r}' \, ds \left(= \rho \int \dot{\mathbf{r}} \cdot \mathbf{r}' \, ds\right). \tag{3.4.34}$$

The conservation law (3.4.33) was found and exploited by Healey [26] in an analysis of the dynamics of closed loops of string. Healey also remarked that for uniform closed loops the impulse (3.4.34) is a conserved quantity that can be interpreted as a total circulation.

A different special case arises in the example of an inextensible, unshearable rod. In this model we prescribe

$$v_1 = v_2 = 0, \qquad v_3 = 1, \tag{3.4.35}$$

so that $\mathbf{r}' = \mathbf{d}_3$. The constitutive relations (2.4.4) are discarded and the force $\boldsymbol{\lambda}(s,t)$ assumes the role of an additional dependent variable. The equations of motion are unaffected, and so in the uniform hyperelastic case the conservation laws (2.3.5) and (3.2.9) imply that

$$(\boldsymbol{\pi} \cdot \mathbf{u} + \mathbf{p} \cdot (\mathbf{d}_3 - \mathbf{e}))_t = \left(\mathbf{m} \cdot \mathbf{u} + \boldsymbol{\lambda} \cdot (\mathbf{d}_3 - \mathbf{e}) - W + \tfrac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\pi} + \tfrac{1}{2}\dot{\mathbf{r}} \cdot \mathbf{p}\right)_s . \tag{3.4.36}$$

Here $\mathbf{e} \in \mathbf{R}^3$ is any constant vector. The case of an infinite rod with conditions at infinity requiring that the rod be at rest and $\mathbf{d}_3 \to \mathbf{e}$ provides an example where the conservation law (3.4.36) implies the integral of motion

$$\int (\boldsymbol{\pi} \cdot \mathbf{u} + \mathbf{p} \cdot (\mathbf{d}_3 - \mathbf{e})) \, ds \tag{3.4.37}$$

while, if $\boldsymbol{\lambda}(\infty) \neq \boldsymbol{\lambda}(-\infty)$, neither the total linear momentum $\int \mathbf{p}\, ds$ nor the impulse (3.2.14) are individually invariants of the motion.

A further special case arises when the planar dynamics of an inextensible, unshearable rod with a linear constitutive relation between bending moment and curvature are considered. This model is the dynamic version of the classic elastica of Euler. Then we can set $\mathbf{r}' = \mathbf{d}_3 = (\cos\phi, \sin\phi, 0)$, and calculate that $\mathbf{u} = (0, 0, \phi_s)$ and (after nondimensionalization) $\boldsymbol{\pi} = (0, 0, \phi_t)$. For the planar elastica, (3.4.37) reduces to

$$\int (\phi_t \phi_s + \mathbf{p} \cdot (\mathbf{d}_3 - \mathbf{e})) \, ds, \tag{3.4.38}$$

which is precisely the integral of motion discovered and exploited in [17]. Motivated by (3.4.38), but independent of the present work, Coleman *et al* ([14], eq. 3.32a) found the invariant

$$\int (\dot{\mathbf{r}} \cdot \mathbf{d}_3 + \dot{\mathbf{d}}_1 \cdot \mathbf{d}_1' + \dot{\mathbf{d}}_2 \cdot \mathbf{d}_2')\, ds \tag{3.4.39}$$

which is the equivalent of a nondimensional form of (3.4.37) when there is a linear bending law, the inertia is of the particular form (2.3.4), and $\mathbf{e} = 0$.

CHAPTER 4

THE IMPETUS-STRICTION FORMULATION

4.1 Introduction

In this chapter we introduce a novel formulation of constrained Lagrangian dynamics as an unconstrained Hamiltonian system. The terms impetus and striction in the title refer to two new dynamic variables central to the formulation, the impetus being a variable conjugate to the configuration variable, while the striction is a multiplier associated with the constraint. We define these variables precisely in the next section. The new Hamiltonian formulation employed here is a variant and extension of the method of 'vakonomic' mechanics as described in [5], p. 33, which treats constrained motion for finite dimensional systems governed by ordinary differential equations. The theory of vakonomics considers both holonomic and nonholonomic constraints, with a strong emphasis on nonholonomic constraints. In contrast we consider only holonomic constraints, but in the infinite-dimensional context of partial differential equations.

In section 4.2 we outline the method and note its significant features. In section 4.3 we illustrate the technique by obtaining a novel Hamiltonian formulation describing the motion of a spherical pendulum.

4.2 Formulation

The essential features of the impetus-striction formulation can be effectively illustrated in the following finite dimensional context. We seek to determine the motion $\mathbf{q}(\cdot) \in \mathbf{R}^n$ for which the action

$$\int_{t_0}^{t_1} L(\mathbf{q}(t), \dot{\mathbf{q}}(t))\, dt \tag{4.2.1}$$

is stationary for all t_0 and t_1, subject to the single constraint

$$f(\mathbf{q}) = 0. \tag{4.2.2}$$

It is assumed that the Lagrangian L is strictly convex in $\dot{\mathbf{q}}$, and that $f_{\mathbf{q}}$ is nonzero when $\mathbf{q}$ satisfies the constraint (4.2.2). Here, as usual, subscripts denote partial derivatives.

The standard reformulation of the constrained action principle is in terms of a Lagrange multiplier, with the requirement that

$$\int_{t_0}^{t_1} \{L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) - \lambda(t)\, f(\mathbf{q}(t))\}\, dt \tag{4.2.3}$$

be stationary for all t_0 and t_1. The associated system of Euler-Lagrange equations is

$$\frac{d}{dt} L_{\dot{\mathbf{q}}} = L_{\mathbf{q}} - \lambda f_{\mathbf{q}}. \tag{4.2.4}$$

The first step leading to an unconstrained Hamiltonian formulation of the dynamics is to replace the constraint (4.2.2) by its time derivative

$$f_{\mathbf{q}}(\mathbf{q}) \cdot \dot{\mathbf{q}} = 0. \tag{4.2.5}$$

Assuming the initial data $\mathbf{q}(0)$ satisfies the constraint (4.2.2), conditions (4.2.2) and (4.2.5) are equivalent. The variational problem (4.2.1) can now be reformulated as the requirement that

$$\int_{t_0}^{t_1} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), \Lambda(t))\, dt \tag{4.2.6}$$

be stationary for all t_0 and t_1, where the function $\mathcal{L}$ is defined by

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, \Lambda) \equiv L(\mathbf{q}, \dot{\mathbf{q}}) + \Lambda\, f_{\mathbf{q}}(\mathbf{q}) \cdot \dot{\mathbf{q}}. \tag{4.2.7}$$

We call the Lagrange multiplier Λ associated with the time-differentiated constraint the striction[1], to distinguish it from the multiplier λ in (4.2.3). The Euler-Lagrange equation associated with the action principle (4.2.6) can be written as

$$\frac{d}{dt} L_{\dot{\mathbf{q}}} = L_{\mathbf{q}} - \Lambda_t f_{\mathbf{q}}. \tag{4.2.8}$$

Because we assumed $f_{\mathbf{q}}$ to be nonzero, a comparison of equations (4.2.4) and (4.2.8) shows that $\Lambda_t = \lambda$. That is, the striction Λ is a time anti-derivative of the Lagrange multiplier λ. (The constraint term was added to L is equation (4.2.7) rather than subtracted as is equation (4.2.3) to reflect the integration by parts in time which relates the two expressions.)

In the development above, it is important to note that the constraint (4.2.5) is holonomic. In the case of a nonholonomic constraint, there is controversy as to whether the action principle (4.2.6) yields the physically appropriate time dynamics. Such issues are of no concern here, because only the holonomic case will be considered.

In order to obtain a Hamiltonian system associated with the Lagrangian (4.2.7), we define the conjugate variable

$$\mathbf{p} \equiv \mathcal{L}_{\dot{\mathbf{q}}} = L_{\dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) + \Lambda\, f_{\mathbf{q}}(\mathbf{q}), \tag{4.2.9}$$

[1]The word striction is an archaic term which is synonymous with constriction. Our choice of the name striction for the multiplier Λ is then motivated by one of the more obscure meanings of constriction, namely the force maintaining a constraint.

and also the momentum

$$\boldsymbol{\zeta} \equiv L_{\dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}). \tag{4.2.10}$$

Thus $\mathbf{p}$ and $\boldsymbol{\zeta}$ are related by $\boldsymbol{\zeta} = \mathbf{p} - \Lambda f_{\mathbf{q}}(\mathbf{q})$. The conjugate variable $\mathbf{p}$ is thus similar to, but distinct from, the classical quantity of momentum. To emphasize this distinction we refer to $\mathbf{p}$ as the impetus[2].

In vakonomic mechanics, one solves the system of equations (4.2.5) and (4.2.9) simultaneously for $\dot{\mathbf{q}}$ and Λ. We take a modified, but equivalent, approach. By the assumed strict convexity of L with respect to $\dot{\mathbf{q}}$, equation (4.2.10) can be solved for $\dot{\mathbf{q}}$:

$$\dot{\mathbf{q}} = \mathbf{v}(\mathbf{q}, \boldsymbol{\zeta}) = \mathbf{v}(\mathbf{q}, \mathbf{p} - \Lambda f_{\mathbf{q}}). \tag{4.2.11}$$

Because $\dot{\mathbf{q}}$ depends upon the unknown striction Λ, a well-defined evolutionary system cannot yet be defined. Instead, from the Legendre transform

$$L^*(\mathbf{q}, \boldsymbol{\zeta}) \equiv \boldsymbol{\zeta} \cdot \mathbf{v}(\mathbf{q}, \boldsymbol{\zeta}) - L(\mathbf{q}, \mathbf{v}(\mathbf{q}, \boldsymbol{\zeta})), \tag{4.2.12}$$

we define a pre-Hamiltonian

$$M(\mathbf{q}, \mathbf{p}, \Lambda) \equiv L^*(\mathbf{q}, \mathbf{p} - \Lambda f_{\mathbf{q}}) = \mathbf{p} \cdot \mathbf{v} - \mathcal{L}(\mathbf{q}, \mathbf{v}, \Lambda). \tag{4.2.13}$$

The crucial step in our formulation of the unconstrained Hamiltonian system is to define the striction to be that Λ which minimizes the pre-Hamiltonian M for specified values of $\mathbf{q}$ and $\mathbf{p}$. To see that M has a unique minimizer, recall that the Legendre transform L^* is strictly convex in $\boldsymbol{\zeta}$ when L is strictly convex in $\dot{\mathbf{q}}$. Thus M is strictly convex in Λ for fixed $\mathbf{q}$ and $\mathbf{p}$, so a unique minimizer $\tilde{\Lambda}(\mathbf{q}, \mathbf{p})$ exists. Moreover, at the minimizer the first derivative M_Λ vanishes, which is readily shown to imply that

$$f_{\mathbf{q}}(\mathbf{q}) \cdot \mathbf{v}(\mathbf{q}, \tilde{\boldsymbol{\zeta}}(\mathbf{q}, \mathbf{p})) = 0, \tag{4.2.14}$$

where

$$\tilde{\boldsymbol{\zeta}}(\mathbf{q}, \mathbf{p}) \equiv \mathbf{p} - \tilde{\Lambda}(\mathbf{q}, \mathbf{p})\, f_{\mathbf{q}}(\mathbf{q}). \tag{4.2.15}$$

Therefore by our definition of the striction $\tilde{\Lambda}(\mathbf{q}, \mathbf{p})$, the constraint (4.2.5) is an identity. We can now define the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) \equiv \min_{\Lambda} M(\mathbf{q}, \mathbf{p}, \Lambda) = M(\mathbf{q}, \mathbf{p}, \tilde{\Lambda}(\mathbf{q}, \mathbf{p})). \tag{4.2.16}$$

[2]In non-technical dictionaries, the words impetus, momentum and impulse are generally described as synonymous. We are not aware of a prior technical usage for the term impetus, and as the quantity in our definition is related to both impulse and momentum, impetus would appear to be an appropriate choice of name. In the context of rod dynamics, momentum, impulse and impetus all play significant, related, but distinct roles.

The associated canonical Hamiltonian system gives the evolution equations:

$$
\begin{aligned}
\dot{\mathbf{q}} &= H_{\mathbf{p}}(\mathbf{q},\mathbf{p}) = \mathbf{v}(\mathbf{q},\tilde{\boldsymbol{\zeta}}(\mathbf{q},\mathbf{p})), && (4.2.17)\\
\dot{\mathbf{p}} &= -H_{\mathbf{q}}(\mathbf{q},\mathbf{p}) = L_{\mathbf{q}}(\mathbf{q},\mathbf{p}) + \tilde{\Lambda}(\mathbf{q},\mathbf{p}) f_{\mathbf{qq}}(\mathbf{q})\mathbf{v}(\mathbf{q},\tilde{\boldsymbol{\zeta}}(\mathbf{q},\mathbf{p})). && (4.2.18)
\end{aligned}
$$

We remark that, because $\tilde{\Lambda}(\mathbf{q},\mathbf{p})$ minimizes the pre-Hamiltonian M, the partial derivatives of H can be computed as

$$
\begin{aligned}
H_{\mathbf{q}}(\mathbf{q},\mathbf{p}) &= M_{\mathbf{q}}(\mathbf{q},\mathbf{p},\Lambda)|_{\Lambda=\tilde{\Lambda}(\mathbf{q},\mathbf{p})}, && (4.2.19)\\
H_{\mathbf{p}}(\mathbf{q},\mathbf{p}) &= M_{\mathbf{p}}(\mathbf{q},\mathbf{p},\Lambda)|_{\Lambda=\tilde{\Lambda}(\mathbf{q},\mathbf{p})}. && (4.2.20)
\end{aligned}
$$

Further, the time anti-derivative of the identity (4.2.14) implies that $f(\mathbf{q})$ is an integral of motion. Therefore, if the initial data satisfy (4.2.2), then (4.2.2) holds for all $t \geq 0$.

The minimization procedure developed here is superficially reminiscent of the Pontryagin Maximum Principle in optimal control theory, but in that context the pre-Hamiltonian is extremized with respect to the control variable, rather than with respect to a multiplier as is the case here. (Cf. e.g. [22].)

A noteworthy feature of this approach is a certain indeterminacy in the initial value of the impetus $\mathbf{p}$. For given values of $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$, the momentum $\boldsymbol{\zeta} = \mathbf{p} - \Lambda f_{\mathbf{q}}$ is completely determined by equation (4.2.10). However, one must specify the initial value of the striction, $\Lambda(0) \equiv \tilde{\Lambda}(\mathbf{q}(0),\mathbf{p}(0))$, in order to determine the value of $\mathbf{p}(0)$ in equation (4.2.9). In this sense, the initial values of $\mathbf{p}$ and $\tilde{\Lambda}$ are not individually observable. However, once $\Lambda(0)$ has been specified, the evolution of $\mathbf{q}$ and $\mathbf{p}$ is completely determined. Moreover, the values of $\mathbf{q}$ and $\dot{\mathbf{q}}$ are unchanged for different choices of $\Lambda(0)$, so the physical variables are unambiguously determined by the dynamics. Also notice that if $(\mathbf{q},\mathbf{p})$ is a solution of the Hamiltonian system, with striction $\tilde{\Lambda}(\mathbf{q},\mathbf{p})$, then for any constant Λ^*, $(\mathbf{q},\mathbf{p}+\Lambda^* f_{\mathbf{q}})$ is also a solution, with striction $\tilde{\Lambda}(\mathbf{q},\mathbf{p}) + \Lambda^*$. This symmetry corresponds, via Noether's Theorem, to the invariant of motion (4.2.2).

For our purposes, the advantage of formulating the Hamiltonian in terms of minimization is that it carries over to the analysis of partial differential equations in an effective manner. We demonstrate in Chapter 5 how the impetus-striction formulation can be applied to obtain a Hamiltonian system governing the motion of an inextensible, unshearable rod.

4.3 An Elementary Application: the Spherical Pendulum

To illustrate the features of the impetus-striction formulation, it is instructive to examine the case of a heavy spherical pendulum with unit mass and unit length in the framework above. This example was also treated using vakonomics by Arnold ([5], p. 37). In this example $n = 3$, the constraint is

$$|\mathbf{q}|^2 - 1 = 0, \tag{4.3.1}$$

and the associated Lagrangian is

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \tfrac{1}{2}|\dot{\mathbf{q}}|^2 - g\mathbf{k} \cdot \mathbf{q}, \tag{4.3.2}$$

where g is the acceleration due to gravity and $\mathbf{k}$ is the upward unit vector.

The time derivative of the constraint (4.3.1) is

$$\mathbf{q} \cdot \dot{\mathbf{q}} = 0, \tag{4.3.3}$$

so following the impetus-striction formulation we define

$$\mathcal{L} \equiv \tfrac{1}{2}|\dot{\mathbf{q}}|^2 - g\mathbf{k} \cdot \mathbf{q} + \Lambda \mathbf{q} \cdot \dot{\mathbf{q}}. \tag{4.3.4}$$

The impetus $\mathbf{p}$ associated with $\mathbf{q}$ is

$$\mathbf{p} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}} + \Lambda \mathbf{q}, \tag{4.3.5}$$

which is easily inverted to obtain an expression for the velocity

$$\dot{\mathbf{q}} = \mathbf{p} - \Lambda \mathbf{q}. \tag{4.3.6}$$

From here we define the pre-Hamiltonian

$$M(\mathbf{q}, \mathbf{p}, \Lambda) \equiv L^*(\mathbf{q}, \mathbf{p} - \Lambda f_{\mathbf{q}}) = \mathbf{p} \cdot \mathbf{v} - \mathcal{L}(\mathbf{q}, \mathbf{v}, \Lambda). \tag{4.3.7}$$

Because the Lagrangian L has the form of quadratic kinetic energy minus potential energy, the Legendre transform L^* has the form of quadratic kinetic energy plus potential energy, namely

$$M(\mathbf{q}, \mathbf{p}, \Lambda) = \tfrac{1}{2}|\mathbf{p} - \Lambda \mathbf{q}|^2 + g\mathbf{k} \cdot \mathbf{q}. \tag{4.3.8}$$

We define the striction $\tilde{\Lambda}(\mathbf{q}, \mathbf{p})$ to be that Λ which minimizes the pre-Hamiltonian M for specified values of $\mathbf{q}$ and $\mathbf{p}$. We can therefore determine $\tilde{\Lambda}(\mathbf{q}, \mathbf{p})$ from the condition that the first derivative of M with respect to Λ vanish, namely

$$-[\mathbf{p} - \tilde{\Lambda}(\mathbf{q}, \mathbf{p})\mathbf{q}] \cdot \mathbf{q} = 0, \tag{4.3.9}$$

from which we find

$$\tilde{\Lambda}(\mathbf{q}, \mathbf{p}) = \frac{\mathbf{q} \cdot \mathbf{p}}{|\mathbf{q}|^2}. \tag{4.3.10}$$

Inserting equation (4.3.10) into the expression (4.3.6), we find

$$\dot{\mathbf{q}} = \left(\mathbf{1} - \frac{\mathbf{q} \otimes \mathbf{q}}{|\mathbf{q}|^2}\right) \mathbf{p} = \frac{\mathbf{q} \times (\mathbf{p} \times \mathbf{q})}{|\mathbf{q}|^2}, \tag{4.3.11}$$

where $\mathbf{1}$ is the identity matrix, and $\otimes$ denotes the dyadic product. Notice that the right-hand side of equation (4.3.11) yields the component of $\mathbf{p}$ orthogonal to $\mathbf{q}$. We can now define the Hamiltonian as

$$\begin{aligned} H(\mathbf{q}, \mathbf{p}) \equiv M(\mathbf{q}, \mathbf{p}, \tilde{\Lambda}(\mathbf{q}, \mathbf{p})) &= \tfrac{1}{2} \left| \left(\mathbf{1} - \tfrac{\mathbf{q} \otimes \mathbf{q}}{|\mathbf{q}|^2}\right) \mathbf{p} \right|^2 + g \mathbf{k} \cdot \mathbf{q} \\ &= \tfrac{1}{2} \left(\tfrac{|\mathbf{q} \times \mathbf{p}|}{|\mathbf{q}|}\right)^2 + g \mathbf{k} \cdot \mathbf{q} \end{aligned} \tag{4.3.12}$$

The associated Hamiltonian system is

$$\dot{\mathbf{q}} = H_{\mathbf{p}} = \left(\mathbf{1} - \frac{\mathbf{q} \otimes \mathbf{q}}{|\mathbf{q}|^2}\right) \mathbf{p}, \tag{4.3.13}$$

$$\dot{\mathbf{p}} = -H_{\mathbf{q}} = \frac{\mathbf{q} \cdot \mathbf{p}}{|\mathbf{q}|^2} \left(\mathbf{1} - \frac{\mathbf{q} \otimes \mathbf{q}}{|\mathbf{q}|^2}\right) \mathbf{p} - g \mathbf{k}, \tag{4.3.14}$$

which can be rewritten as

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \mathbf{J} \begin{pmatrix} H_{\mathbf{q}} \\ H_{\mathbf{p}} \end{pmatrix} \tag{4.3.15}$$

where $\mathbf{J}$ is the skew matrix

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{4.3.16}$$

The Hamiltonian H is time-independent, and so is an invariant of motion. The Hamiltonian is also invariant under the transformation $\mathbf{p} \to \mathbf{p} + \epsilon \mathbf{q}$ for arbitrary ϵ, in accordance with our earlier observation regarding the incomplete observability of $\mathbf{p}$ and $\tilde{\Lambda}$. By Noether's Theorem there is an associated, additional invariant of motion. Following the framework described, for example, in [52] (Theorem 4.29), we recognize that the infinitesimal generator $(\mathbf{0}, \mathbf{q})$ of this transformation is related to the invariant $N(\mathbf{q}) \equiv \frac{1}{2}(|\mathbf{q}|^2 - 1)$ through the relation

$$\begin{pmatrix} 0 \\ \mathbf{q} \end{pmatrix} = \mathbf{J} \begin{pmatrix} N_{\mathbf{q}} \\ N_{\mathbf{p}} \end{pmatrix}. \tag{4.3.17}$$

In addition, the Hamiltonian is invariant under rotation about the $\mathbf{k}$ axis, a symmetry whose infinitesimal generator is $(\mathbf{k} \times \mathbf{q}, \mathbf{k} \times \mathbf{p})$. From Noether's Theorem we observe that the associated invariant of motion is the component Ω_k of angular momentum about the $\mathbf{k}$ axis, namely

$$\begin{aligned} \Omega_k &\equiv \mathbf{k} \cdot (\mathbf{q} \times \mathbf{p}) \\ &= \mathbf{k} \cdot (\mathbf{q} \times \dot{\mathbf{q}}), \end{aligned} \tag{4.3.18}$$

where the second equality is a consequence of the definition of $\mathbf{p}$ (4.3.5).

The focus in this dissertation is on analysis of stability properties of special solutions which reflect a symmetry in a Hamiltonian system. We illustrate that analysis here by investigating the steadily rotating motions $(\bar{\mathbf{q}}(t), \bar{\mathbf{p}}(t))$ of a spherical pendulum. The same approach will then be used in Chapters 5 and 6 for the stability analysis of planar solitary waves in an inextensible, unshearable rod. The procedure is as follows. First, we step outside of the realm of time dynamics and determine which pairs $(\mathbf{q}_0, \mathbf{p}_0)$ are critical points of the invariant

$$F \equiv H - \omega \mathbf{k} \cdot (\mathbf{q} \times \mathbf{p}) - \tfrac{1}{2}\mu(|\mathbf{q}|^2 - 1) \tag{4.3.19}$$

for some scalars ω and μ. This is equivalent to finding the critical points of H, subject to prescribed values of the invariants $\Omega_k = \mathbf{k} \cdot (\mathbf{q} \times \mathbf{p})$ and $|\mathbf{q}|^2 - 1$. It is shown that for initial data given by a critical point $(\mathbf{q}_0, \mathbf{p}_0)$ of F, the solution $(\mathbf{q}(t), \mathbf{p}(t))$ of the Hamiltonian dynamics describes a steadily rotating motion. Next we determine whether the point $(\mathbf{q}_0, \mathbf{p}_0)$ is in fact a minimizer of F. In general such a minimizer is nonisolated, due to the rotational symmetry of F. A minimizer $(\mathbf{q}_0, \mathbf{p}_0)$ is significant because the invariant F may then be used as a Lyapunov function to prove orbital stability for the associated motion $(\mathbf{q}(t), \mathbf{p}(t))$. However the question of instability in the case where $(\mathbf{q}_0, \mathbf{p}_0)$ is not a minimizer of F is more delicate, and will not be discussed here.

The critical points of F (4.3.19) are given by the pair of equations

$$\begin{aligned} F_{\mathbf{q}} &= -\frac{\mathbf{q}_0 \cdot \mathbf{p}_0}{|\mathbf{q}_0|^2}\left(\mathbf{1} - \frac{\mathbf{q}_0 \otimes \mathbf{q}_0}{|\mathbf{q}_0|^2}\right)\mathbf{p}_0 + g\mathbf{k} - \omega \mathbf{p}_0 \times \mathbf{k} - \mu \mathbf{q}_0 = \mathbf{0}, && (4.3.20) \\ F_{\mathbf{p}} &= \left(\mathbf{1} - \frac{\mathbf{q}_0 \otimes \mathbf{q}_0}{|\mathbf{q}_0|^2}\right)\mathbf{p}_0 - \omega \mathbf{k} \times \mathbf{q}_0 = \mathbf{0}. && (4.3.21) \end{aligned}$$

We restrict our attention to critical points for which $|\mathbf{q}_0| = 1$. Equation (4.3.21) then implies that

$$\mathbf{p}_0 = \Lambda_0 \mathbf{q}_0 + \omega \mathbf{k} \times \mathbf{q}_0 \tag{4.3.22}$$

where $\Lambda_0 \equiv (\mathbf{q}_0 \cdot \mathbf{p}_0)$ is indeterminate due to the gauge symmetry. When equation (4.3.22) is inserted into equation (4.3.20) and the terms are rearranged, we find

$$\left(\mu + \omega^2\right)\mathbf{q}_0 = \left[g + \omega^2(\mathbf{q}_0 \cdot \mathbf{k})\right]\mathbf{k}. \tag{4.3.23}$$

There are two types of solutions for equation (4.3.23), depending on whether $\mathbf{q}_0$ is parallel to $\mathbf{k}$ or not.

If $\mathbf{q}_0$ is parallel to $\mathbf{k}$, then in view of the constraint $|\mathbf{q}_0| = 1$ the only two solutions are

$$\mathbf{q}_0 = \pm\mathbf{k}. \tag{4.3.24}$$

Equations (4.3.23) and (4.3.22) then imply

$$\mu = \pm g \tag{4.3.25}$$

$$\mathbf{p}_0 = \pm\Lambda_0 \mathbf{k}. \tag{4.3.26}$$

The multiplier ω is arbitrary. For our analysis we take

$$\omega = 0. \tag{4.3.27}$$

The angular momentum vanishes at these critical points. If the point $(\mathbf{q}_0, \mathbf{p}_0)$ is used as initial data for the Hamiltonian system (4.3.13) - (4.3.14), the solution is

$$\bar{\mathbf{q}}(t) = \pm\mathbf{k}, \tag{4.3.28}$$

$$\bar{\mathbf{p}}(t) = (gt \pm \Lambda_0)\mathbf{k}. \tag{4.3.29}$$

That is, the impetus $\bar{\mathbf{p}}(t)$ grows linearly in time as the solution evolves. This behavior will be seen again in the solitary waves of an inextensible rod in Chapter 5. The solutions (4.3.28) correspond to the two equilibria of the spherical pendulum, with $\dot{\mathbf{q}}(t) = \mathbf{0}$. (By contrast, the Hamiltonian system (4.3.13) - (4.3.14) has no equilibrium solutions for $g \neq 0$.)

If $\mathbf{q}_0$ is not parallel to $\mathbf{k}$ then we express $\mathbf{q}_0$ as

$$\mathbf{q}_0 = -\mathbf{k}\cos\theta_0 + \mathbf{i}\sin\theta_0 \tag{4.3.30}$$

where θ_0 is the angle from $-\mathbf{k}$ to $\mathbf{q}_0$ with $0 \leq \theta_0 \leq \pi$, and $\mathbf{i}$ is a unit vector orthogonal to $\mathbf{k}$. For (4.3.30) to be a solution of (4.3.23), the multipliers ω and μ must be

$$\omega = \pm\sqrt{\frac{g}{\cos\theta_0}} \tag{4.3.31}$$

and

$$\mu = -\omega^2 = -\frac{g}{\cos\theta_0}. \tag{4.3.32}$$

From equation (4.3.31) we infer that $\cos\theta_0 > 0$, so $0 \leq \theta_0 < \pi/2$. That is, $\mathbf{q}_0$ must lie below the horizontal plane $\theta_0 = \pi/2$. Equation (4.3.22) implies that the corresponding impetus is

$$\mathbf{p}_0 = \Lambda_0(-\mathbf{k}\cos\theta_0 + \mathbf{i}\sin\theta_0) + \omega\mathbf{k}\times\mathbf{i}\sin\theta_0. \tag{4.3.33}$$

For these critical points the $\mathbf{k}$ component of angular momentum is $\omega\sin^2\theta_0$. If the point $(\mathbf{q}_0, \mathbf{p}_0)$ is used as initial data for the Hamiltonian system (4.3.13) - (4.3.14), the solution is

$$\bar{\mathbf{q}}(t) = -\mathbf{k}\cos\theta_0 + \sin\theta_0\left[\mathbf{i}\,\cos(\omega t) + \mathbf{k}\times\mathbf{i}\,\sin(\omega t)\right] \tag{4.3.34}$$

and

$$\bar{\mathbf{p}}(t) = (\omega^2 t + \Lambda_0)\bar{\mathbf{q}}(t) + \omega\mathbf{k}\times\bar{\mathbf{q}}(t). \tag{4.3.35}$$

Once again the impetus $\bar{\mathbf{p}}(t)$ is observed to grow linearly in time as the solution evolves. Each solution $\bar{\mathbf{q}}(t)$ of the form (4.3.34) describes a steadily rotating motion, with

$$\dot{\bar{\mathbf{q}}}(t) = \omega\mathbf{k}\times\bar{\mathbf{q}}(t). \tag{4.3.36}$$

Next we determine which of the critical points $(\mathbf{q}_0, \mathbf{p}_0)$ are minimizers of F, for the associated values of ω and μ. For further analysis it is convenient to decompose F as

$$F = F_1 + F_2, \tag{4.3.37}$$

where

$$\begin{aligned} F_1(\mathbf{q}, \mathbf{p}) &\equiv \tfrac{1}{2}\tfrac{1}{|\mathbf{q}|^2}\left|\mathbf{q} \times (\mathbf{p} - \omega\mathbf{k} \times \mathbf{q})\right|^2 \\ &= \tfrac{1}{2}\left|\tfrac{\mathbf{q}\times(\mathbf{p}\times\mathbf{q})}{|\mathbf{q}|^2} - \omega\mathbf{k} \times \mathbf{q})\right|^2 \end{aligned} \tag{4.3.38}$$

and

$$\begin{aligned} F_2(\mathbf{q}, \mathbf{p}) &\equiv g\mathbf{k} \cdot \mathbf{q} - \tfrac{1}{2}\mu(|\mathbf{q}|^2 - 1) - \tfrac{1}{2}\omega^2|\mathbf{k} \times \mathbf{q}|^2 n \\ &= -\tfrac{1}{2}\mu|\mathbf{q} - g\mathbf{k}/\mu|^2 - \tfrac{1}{2}\omega^2|\mathbf{k} \times (\mathbf{q} - g\mathbf{k}/\mu)|^2 + \tfrac{1}{2}(\mu + g^2/\mu). \end{aligned} \tag{4.3.39}$$

This decomposition is chosen so that the terms involving $\mathbf{p}$ appear only in F_1. In fact, the function F_1 may be thought of as the kinetic energy, measured relative to a frame rotating with angular velocity ω about $\mathbf{k}$. It is apparent from (4.3.38) that F_1 is nonnegative. Moreover from equation (4.3.22) we see that $F_1(\mathbf{q}_0, \mathbf{p}_0) = 0$. Therefore $(\mathbf{q}_0, \mathbf{p}_0)$ minimizes F_1. Because $(\mathbf{q}_0, \mathbf{p}_0)$ extremizes F by construction, it follows that $(\mathbf{q}_0, \mathbf{p}_0)$ also extremizes F_2. This fact can be ascertained directly from equation (4.3.23) and the derivative of F_2,

$$\begin{aligned} DF_2 &= g\mathbf{k} - \mu\mathbf{q} - \omega^2\left[\mathbf{q} - (\mathbf{q} \cdot \mathbf{k})\mathbf{q}\right] \\ &= \left[g + (\mathbf{q} \cdot \mathbf{k})\right]\mathbf{k} - (\mu + \omega^2)\mathbf{q}. \end{aligned} \tag{4.3.40}$$

We therefore focus most our analysis upon the function F_2.

First consider critical points given by (4.3.24) and (4.3.26). For the critical point $\mathbf{q}_0 = -\mathbf{k}$ we have $\mu = -g$ and $\omega = 0$, so $\mathbf{q}_0$ is an unconstrained minimizer of F_2 (4.3.39), and so $(\mathbf{q}_0, \mathbf{p}_0)$ is an unconstrained minimizer of F. We also obtain the inequalities

$$|\mathbf{q} - \mathbf{q}_0|^2 \leq 2\left[F(\mathbf{q}, \mathbf{p}) - F(\mathbf{q}_0, \mathbf{p}_0)\right]/g \tag{4.3.41}$$

and

$$|\mathbf{q} \times \mathbf{p}|^2/|\mathbf{q}|^2 \leq 2\left[F(\mathbf{q}, \mathbf{p}) - F(\mathbf{q}_0, \mathbf{p}_0)\right]. \tag{4.3.42}$$

Because the impetus $\mathbf{p}$ is partially unobservable, it would be difficult make a statement about the stability of $(\mathbf{q}_0, \mathbf{p}_0)$ in terms of the Hamiltonian variables. However the bounds (4.3.41) and (4.3.42) can be used to state a stability result in terms of Lagrangian variables $(\mathbf{q}, \dot{\mathbf{q}})$. Assuming the initial data $(\mathbf{q}(0), \mathbf{p}(0))$ are chosen such that $|F(\mathbf{q}(0), \mathbf{p}(0)) - F(\mathbf{q}_0, \mathbf{p}_0)| < \frac{1}{2}\epsilon^2$, the invariance of F implies $|F(\mathbf{q}(t), \mathbf{p}(t)) - F(\mathbf{q}_0, \mathbf{p}_0)| < \frac{1}{2}\epsilon^2$. Equation (4.3.41) then implies

$$|\mathbf{q}(t) - \mathbf{q}_0|^2 \leq \epsilon^2/g, \tag{4.3.43}$$

and equation (4.3.11) for $\dot{\mathbf{q}}$ can be introduced into equation (4.3.42) to obtain

$$|\dot{\mathbf{q}}(t)|^2 \leq \epsilon^2. \tag{4.3.44}$$

Thus the critical point with $\mathbf{q}_0 = -\mathbf{k}$ is stable, as is well known.

By contrast at the critical point $(\mathbf{q}_0, \mathbf{p}_0) = (\mathbf{k}, \Lambda_0\mathbf{k})$, $\mu = g$ so $\mathbf{q}_0$ is a maximum of F_2. We also note that F_1 is flat in any direction $(\delta\mathbf{q}, \Lambda_0\delta\mathbf{q})$ from $(\mathbf{k}, \Lambda_0\mathbf{k})$. Therefore $(\mathbf{k}, \Lambda_0\mathbf{k})$ is not a minimizer of F. While it is well known that the equilibrium $\mathbf{q}_0 = \mathbf{k}$ is unstable, we do not pursue the instability proof here.

Finally consider a critical point $(\mathbf{q}_0, \mathbf{p}_0)$ given by (4.3.30) and (4.3.33) for some θ_0 such that $0 \leq \theta_0 < \pi/2$. Using equation (4.3.30) and expressions (4.3.31) and (4.3.32) for the multipliers ω and μ, we can rewrite F_2 as

$$\begin{aligned} F_2(\mathbf{q}, \mathbf{p}) &= \tfrac{1}{2}\omega^2|\mathbf{q} + \mathbf{k}\cos\theta_0|^2 - \tfrac{1}{2}\omega^2|\mathbf{k} \times (\mathbf{q} + \mathbf{k}\cos\theta_0)|^2 + \tfrac{1}{2}(\mu + g^2/\mu) \\ &= \tfrac{1}{2}\omega^2|\mathbf{k} \cdot (\mathbf{q} - \mathbf{q}_0)|^2 + \tfrac{1}{2}(\mu + g^2/\mu). \end{aligned} \tag{4.3.45}$$

Thus $(\mathbf{q}_0, \mathbf{p}_0)$ is a minimizer of F_2 and so also F, which is nonisolated due to the rotational symmetry of F. From equations (4.3.31), (4.3.38) and (4.3.45) one can also obtain the bounds

$$|\mathbf{k} \cdot (\mathbf{q} - \mathbf{q}_0)|^2 \leq 2[F(\mathbf{q}, \mathbf{p}) - F(\mathbf{q}_0, \mathbf{p}_0)]/\omega^2 \tag{4.3.46}$$

and

$$\left|\frac{\mathbf{q} \times (\mathbf{p} \times \mathbf{q})}{|\mathbf{q}|^2} - \omega\mathbf{k} \times \mathbf{q}\right|^2 \leq 2[F(\mathbf{q}, \mathbf{p}) - F(\mathbf{q}_0, \mathbf{p}_0)]. \tag{4.3.47}$$

Equations (4.3.46) and (4.3.47) together with the expression (4.3.11) for $\dot{\mathbf{q}}$ imply that if $|F(\mathbf{q}(0), \mathbf{p}(0)) - F(\mathbf{q}_0, \mathbf{p}_0)| < \frac{1}{2}\epsilon^2$, then

$$|\mathbf{k} \cdot (\mathbf{q}(t) - \bar{\mathbf{q}}(t))|^2 \leq \epsilon^2/\omega^2 \tag{4.3.48}$$

and

$$|\dot{\mathbf{q}}(t) - \omega\mathbf{k} \times \mathbf{q}(t)| \leq \epsilon^2. \tag{4.3.49}$$

Using the inequalities (4.3.46) and (4.3.47) and the constant of motion $|\mathbf{q}| = 1$, one can show that at each time t there exists a translation τ such that

$$|\mathbf{q}(t) - \bar{\mathbf{q}}(t + \tau)|^2 \leq C_1\epsilon \tag{4.3.50}$$

and

$$|\dot{\mathbf{q}}(t) - \dot{\mathbf{q}}(t + \tau)|^2 \leq C_2\epsilon \tag{4.3.51}$$

for appropriate constants C_1 and C_2. Therefore the solution $(\bar{\mathbf{q}}, \dot{\mathbf{q}})$ given by (4.3.34) and (4.3.36) is orbitally stable.

CHAPTER 5

THE PLANAR ELASTICA

5.1 Introduction

In this chapter we employ the impetus-striction method described in Chapter 4 in order to derive an unconstrained Hamiltonian system of partial differential equations whose solutions model the planar, large amplitude, time evolution of an inextensible and unshearable elastic rod. The structure of the chapter is as follows. In section 5.2 we obtain the Hamiltonian formulation based on the rod model in section 2.5. In section 5.3 we present the proof that, for a given state $\mathbf{z}$ in an appropriate function space, the minimizer $\tilde{\Lambda}(\mathbf{z})$ of the pre-Hamiltonian M exists and is unique. In section 5.4 we discuss the integrals of the Hamiltonian system, and their relationship to the various symmetries. In section 5.5, the solitary wave profile is characterized as an extremal of an appropriate variational principle. In section 5.6 we prove rigorously that for wave speeds less than a critical value, the solitary wave profile is actually a minimum of the associated variational principle, and discuss the implications of our analysis upon dynamic stability properties of the solitary wave.

5.2 A Hamiltonian Formulation of the Dynamics

The fundamental balance laws (2.5.7) and (2.5.8) can be shown to be equivalent to the form of Hamilton's Principle (expressed in Lagrangian variables) which requires that the action

$$\int_{t_0}^{t_1} L(\phi, \mathbf{h}_t, \phi_t)\, ds\, dt \tag{5.2.1}$$

be stationary for all t_0 and t_1, amongst all motions satisfying the boundary conditions and the constraint (2.5.6), where L is the Lagrangian

$$L(\phi, \mathbf{h}_t, \phi_t) = T(\mathbf{h}_t, \phi_t) - V(\phi). \tag{5.2.2}$$

By the Lagrange multiplier rule, Hamilton's Principle can be rewritten as the requirement that the modified action

$$\int_{t_0}^{t_1} \left\{ L(\phi, \mathbf{h}_t, \phi_t) - \int_{-L}^{L} \boldsymbol{\lambda} \cdot (\mathbf{r}_s - \mathbf{d}_3)\ ds \right\} dt \tag{5.2.3}$$

be stationary for all t_0 and t_1, among all motions satisfying the boundary conditions. These action principles serve as a convenient starting point for our derivation of a Hamiltonian evolutionary system.

Another way to implement the constraint (2.5.6) would be to express the displacement $\mathbf{h}$ explicitly in terms of ϕ, by integrating (2.5.6) subject to the initial data and boundary conditions. Then the dynamic variables would be ϕ and ϕ_t only. This approach, taken in [12], leads to a nonlocal Lagrangian formulation of the dynamics. An associated nonlocal Hamiltonian formulation is derived for the finite rod in Appendix B. However with such a formulation we were unable to obtain a well-defined system of equations for the infinite rod. The difficulty lies in determining the multiplier (that is, the force) $\boldsymbol{\lambda}$ as a function of the state. As described in the Introduction, we will take a different approach to obtain a well-defined local Hamiltonian system, through a novel minimization procedure.

A key feature of our approach is the reformulation of the constraint (2.5.6), by differentiation in time to obtain

$$\mathbf{h}_{st} - (\mathbf{d}_3)_t = 0. \tag{5.2.4}$$

Since we require the initial conditions to satisfy the constraint (2.5.6), the conditions (2.5.6) and (5.2.4) are equivalent. Equation (5.2.4) can be rewritten, using equation (2.5.4), as

$$\mathbf{h}_{st} - \phi_t \mathbf{d}_2 = 0. \tag{5.2.5}$$

Now, with the constraint in the form (5.2.5), Hamilton's Principle can be expressed as the requirement that

$$\int_{t_0}^{t_1} \mathcal{L}(\phi, \mathbf{h}_t, \phi_t, \mathbf{\Lambda})\, dt \tag{5.2.6}$$

be stationary for all t_0 and t_1, among all motions satisfying the boundary conditions, where $\mathcal{L}$ is

$$\mathcal{L}(\phi, \mathbf{h}_t, \phi_t, \mathbf{\Lambda}) \equiv T(\mathbf{h}_t, \phi_t) - V(\phi) + \int_{-L}^{L} \mathbf{\Lambda} \cdot (\mathbf{h}_{st} - \phi_t \mathbf{d}_2)\, ds. \tag{5.2.7}$$

We refer to the vector field of Lagrange multipliers $\mathbf{\Lambda}$ associated with the time differentiated constraints as the striction. Notice that in (5.2.7) we have added the constraint term, whereas in (5.2.3), the constraint term is subtracted. This is merely a sign convention that simplifies the interpretation of $\mathbf{\Lambda}$. The physical significance of the striction will become apparent after we obtain the equations of motion from the variational problem (5.2.6).

The first variation of the kinetic energy term in (5.2.6) is

$$\begin{aligned} &\delta \int_{t_0}^{t_1} \int_{-L}^{L} \tfrac{1}{2} \left\{ \rho |\mathbf{h}_t|^2 + I_1 \phi_t^2 \right\} ds\, dt \\ &= \int_{t_0}^{t_1} \int_{-L}^{L} \left\{ \rho \mathbf{h}_t \cdot \delta \mathbf{h}_t + I_1 \phi_t \delta \phi_t \right\} ds\, dt \\ &= \int_{t_0}^{t_1} \int_{-L}^{L} \left\{ -(\rho \mathbf{h}_t)_t \cdot \delta \mathbf{h} - I_1 \phi_{tt} \delta \phi \right\} ds\, dt, \end{aligned} \tag{5.2.8}$$

where the second equality is obtained through an integration by parts. Similarly, the first variation of the potential energy term in (5.2.6) is

$$\delta \int_{t_0}^{t_1} \int_{-L}^{L} \tfrac{1}{2} K_1 \phi_s^2 \, ds \, dt = \textstyle\int_{t_0}^{t_1} \int_{-L}^{L} -K_1 \phi_{ss} \delta\phi \, ds \, dt. \tag{5.2.9}$$

From the expressions (2.5.2) and (2.5.1) for $\mathbf{d}_2$ and $\mathbf{d}_3$, their first variations can be expressed as

$$\delta \mathbf{d}_2 = -\delta\phi \mathbf{d}_3 \tag{5.2.10}$$

and

$$\delta \mathbf{d}_3 = \delta\phi \mathbf{d}_2. \tag{5.2.11}$$

One obtains the first variation of the constraint term in (5.2.6) using equations (5.2.10) and(5.2.11) together with integrations by parts to yield

$$\begin{aligned} &\delta \int_{t_0}^{t_1} \int_{-L}^{L} \boldsymbol{\Lambda} \cdot (\mathbf{h}_{st} - \phi_t \mathbf{d}_2) \, ds \, dt \\ &\quad = \int_{t_0}^{t_1} \int_{-L}^{L} \left\{ \boldsymbol{\Lambda}_{st} \cdot \delta \mathbf{h} + [(\mathbf{d}_2 \cdot \boldsymbol{\Lambda})_t + (\mathbf{d}_3 \cdot \boldsymbol{\Lambda})\phi_t] \, \delta\phi \right\} \, ds \, dt. \end{aligned} \tag{5.2.12}$$

Combining equations (5.2.8), (5.2.9), and (5.2.12), the total first variation of the Lagrangian is

$$\int_{t_0}^{t_1} \int_{-L}^{L} \{ -(\rho \mathbf{h}_t - \boldsymbol{\Lambda}_s)_t \cdot \delta \mathbf{h} + [-(I_1 \phi_t - \mathbf{d}_2 \cdot \boldsymbol{\Lambda}))_t - K_1 \phi_{ss} + (\mathbf{d}_3 \cdot \boldsymbol{\Lambda})\phi_t] \delta\phi \} \, ds \, dt. \tag{5.2.13}$$

Stationarity of the Lagrangian implies the Euler-Lagrange equations of motion:

$$(\rho \mathbf{h}_t - \boldsymbol{\Lambda}_s)_t = \mathbf{0}, \tag{5.2.14}$$

$$(I_1 \phi_t - \mathbf{d}_2 \cdot \boldsymbol{\Lambda})_t = K_1 \phi_{ss} + (\mathbf{d}_3 \cdot \boldsymbol{\Lambda}) \phi_t. \tag{5.2.15}$$

We are now in a position to interpret the physical meaning of the striction $\boldsymbol{\Lambda}$. When the identity (2.5.4), the constraint (2.4.6) and the constitutive relation (2.5.9) are substituted into the Euler-Lagrange equations, equation (5.2.14) can be rewritten as

$$(\rho \mathbf{h}_t)_t = \boldsymbol{\Lambda}_{ts} \tag{5.2.16}$$

and equation (5.2.15) as

$$(I_1 \phi_t)_t = m_s + \mathbf{e}_3 \cdot (\mathbf{r}_s \times \boldsymbol{\Lambda}_t). \tag{5.2.17}$$

Comparison of equations (5.2.16) and (5.2.17) with the equations of motion (2.5.7) and (2.5.8) reveals that

$$(\boldsymbol{\Lambda}_t - \boldsymbol{\lambda})_s = 0 \tag{5.2.18}$$

and

$$\mathbf{r}_s \times (\boldsymbol{\Lambda}_t - \boldsymbol{\lambda}) = 0. \tag{5.2.19}$$

Equation (5.2.18) implies that $(\mathbf{\Lambda}_t - \boldsymbol{\lambda})$ is independent of s, and equation (5.2.19) implies that $(\mathbf{\Lambda}_t - \boldsymbol{\lambda})$ is parallel to $\mathbf{d}_3$ for all s. But, as we remarked at the end of Section 2, the tangent vector $\mathbf{r}_s = \mathbf{d}_3$ cannot be constant as a function of s. Therefore $\mathbf{\Lambda}_t = \boldsymbol{\lambda}$. That is, the striction $\mathbf{\Lambda}$ is a time anti-derivative of the force $\boldsymbol{\lambda}$.

In order to obtain a Hamiltonian system, we follow the formulation described in Chapter 2 and define the impetuses

$$\begin{aligned} \mathbf{g} &\equiv \frac{\delta \mathcal{L}}{\delta \mathbf{h}_t} = \rho \mathbf{h}_t - \mathbf{\Lambda}_s, && (5.2.20) \\ \chi &\equiv \frac{\delta \mathcal{L}}{\delta \phi_t} = I_1 \phi_t - \mathbf{d}_2 \cdot \mathbf{\Lambda}. && (5.2.21) \end{aligned}$$

We will refer to the composite variable $\mathbf{z} \equiv (\mathbf{h}, \phi, \mathbf{g}, \chi)$ as the phase of the Hamiltonian system. Note that, because we consider only planar motion, the $\mathbf{e}_3$ components of $\mathbf{g}$ and $\mathbf{\Lambda}$ can be set to zero. Equations (5.2.20) and (5.2.21) are easily inverted to express $\mathbf{h}_t$ and ϕ_t in terms of the phase $\mathbf{z}$ and the striction $\mathbf{\Lambda}$:

$$\begin{aligned} \mathbf{h}_t &= \rho^{-1}(\mathbf{g} + \mathbf{\Lambda}_s) \equiv \mathbf{v}(\mathbf{z}, \mathbf{\Lambda}), && (5.2.22) \\ \phi_t &= I_1^{-1}(\chi + \mathbf{d}_2 \cdot \mathbf{\Lambda}) \equiv \omega(\mathbf{z}, \mathbf{\Lambda}). && (5.2.23) \end{aligned}$$

The Legendre transform of the Lagrangian (5.2.2) with respect to $(\mathbf{h}_t, \phi_t)$ produces a function of both $\mathbf{z}$ and $\mathbf{\Lambda}$ which we call the pre-Hamiltonian. In our case the Lagrangian (5.2.2) is the difference of the kinetic and potential energies (2.5.10) and (2.5.11), so the Legendre transform is the sum of the kinetic and potential energies, and the pre-Hamiltonian can be written as

$$M(\mathbf{z}, \mathbf{\Lambda}) = \int_{-L}^{L} \tfrac{1}{2} \left\{ \rho^{-1} |\mathbf{g} + \mathbf{\Lambda}_s|^2 + I_1^{-1}(\chi + \mathbf{d}_2 \cdot \mathbf{\Lambda})^2 + K_1 \phi_s^2 \right\} ds. \qquad (5.2.24)$$

The problem remains to determine the striction $\mathbf{\Lambda}$ as a function of the phase $\mathbf{z}$, such that the constraint (5.2.5) is satisfied. The important observation is that by choosing the striction such that the pre-Hamiltonian M is minimized, the constraint (5.2.5) is satisfied identically. For if $\mathbf{\Lambda}$ minimizes M for given $\mathbf{z}$, then the first variation of the pre-Hamiltonian (5.2.24) with respect to $\mathbf{\Lambda}$ must vanish identically. Using the expressions (5.2.22) for $\mathbf{v}$ and (5.2.23) for ω, together with an integration by parts, we find that the first variation of M with respect to $\mathbf{\Lambda}$ is

$$\begin{aligned} \langle \frac{\delta M}{\delta \mathbf{\Lambda}}, \delta \mathbf{\Lambda} \rangle &= \int_{-L}^{L} \left\{ \rho^{-1}(\mathbf{g} + \mathbf{\Lambda}_s) \cdot \delta \mathbf{\Lambda}_s + I_1^{-1}(\chi + \mathbf{d}_2 \cdot \mathbf{\Lambda})(\mathbf{d}_2 \cdot \delta \mathbf{\Lambda}) \right\} ds \\ &= \int_{-L}^{L} \{ \mathbf{v} \cdot \delta \mathbf{\Lambda}_s + \omega \mathbf{d}_2 \cdot \delta \mathbf{\Lambda} \} \, ds \\ &= -\int_{-L}^{L} \{ \mathbf{v}_s - \omega \mathbf{d}_2 \} \cdot \delta \mathbf{\Lambda} \, ds + \mathbf{v} \cdot \delta \mathbf{\Lambda} |_{-L}^{L}. && (5.2.25) \end{aligned}$$

The first variation (5.2.25) vanishes identically if and only if

$$\mathbf{v}_s - \omega \mathbf{e}_3 \times \mathbf{d}_3 = \rho^{-1}(\mathbf{g} + \mathbf{\Lambda}_s)_s - I_1^{-1}(\chi + \mathbf{d}_2 \cdot \mathbf{\Lambda})\mathbf{d}_2 = \mathbf{0}, \tag{5.2.26}$$

and

$$\mathbf{v} = \rho^{-1}(\mathbf{g} + \mathbf{\Lambda}_s) = \mathbf{0} \quad \text{at } s = \pm L. \tag{5.2.27}$$

Referring again to equations (5.2.22) and (5.2.23), we see that equation (5.2.26) is precisely the constraint (5.2.5), expressed in terms of the phase $\mathbf{z}$. Thus the constraint (5.2.5) is satisfied identically. Because the initial data satisfy the inextensibility condition (2.5.6), we further conclude that (2.5.6) is satisfied for all times $t \geq 0$.

To justify the above calculations it is necessary to show that, for any given state $\mathbf{z}$ in an appropriate set of functions $\mathbf{Z}$, there exists a unique minimizer $\tilde{\mathbf{\Lambda}}(\mathbf{z})$. The proof of existence and uniqueness of this minimizer for the case of the infinite rod appears in section 5.3. The proof for the finite rod is much easier.

Given the striction as a function of the state $\mathbf{z}$, we can define the Hamiltonian

$$H(\mathbf{z}) \equiv \min_{\mathbf{\Lambda}} M(\mathbf{z}, \mathbf{\Lambda}) = M(\mathbf{z}, \tilde{\mathbf{\Lambda}}(\mathbf{z})), \tag{5.2.28}$$

which we can rewrite, using the expression (5.2.24) for M, as

$$H(\mathbf{z}) = \int_{-L}^{L} \tfrac{1}{2} \left\{ \rho^{-1}|\mathbf{g} + \tilde{\mathbf{\Lambda}}_s(\mathbf{z})|^2 + I_1^{-1}[\chi + \mathbf{d}_2 \cdot \tilde{\mathbf{\Lambda}}(\mathbf{z})]^2 + K_1 \phi_s^2 \right\} ds, \tag{5.2.29}$$

where $\tilde{\mathbf{\Lambda}}(\mathbf{z})$ is the unique solution of (5.2.26). We can now define the Hamiltonian evolution equations for the rod dynamics

$$\begin{aligned}
\mathbf{h}_t &= \delta_{\mathbf{g}} H &&= \rho^{-1}[\mathbf{g} + \tilde{\mathbf{\Lambda}}_s(\mathbf{z})], && (5.2.30)\\
\phi_t &= \delta_{\chi} H &&= I_1^{-1}\left[\chi + \mathbf{d}_2 \cdot \tilde{\mathbf{\Lambda}}(\mathbf{z})\right], && (5.2.31)\\
\mathbf{g}_t &= -\delta_{\mathbf{h}} H &&= \mathbf{0}, && (5.2.32)\\
\chi_t &= -\delta_{\phi} H &&= I_1^{-1}\left[\chi + \mathbf{d}_2 \cdot \tilde{\mathbf{\Lambda}}(\mathbf{z})\right][\mathbf{d}_3 \cdot \tilde{\mathbf{\Lambda}}(\mathbf{z})] + K_1 \phi_{ss}. && (5.2.33)
\end{aligned}$$

As was remarked in the introduction, because $\tilde{\mathbf{\Lambda}}(\mathbf{z})$ minimizes M, the partial derivatives of $H(\mathbf{z})$ and $M(\mathbf{z}, \mathbf{\Lambda})$ with respect to the phase variable $\mathbf{z}$ coincide. This observation simplifies the calculation of the partial derivatives of H appearing in equations (5.2.30) - (5.2.33). It is a straightforward calculation to show that equations (5.2.30) and (5.2.31) are equivalent to equations (5.2.22) and (5.2.23) by which $\mathbf{h}_t$ and ϕ_t were expressed in terms of the state $\mathbf{z}$, while equations (5.2.32) and (5.2.33) are equivalent to the Euler-Lagrange equations (5.2.14) and (5.2.15).

The Hamiltonian system can be rewritten more compactly as

$$\mathbf{z}_t = \mathbf{J}\, \frac{\delta H}{\delta \mathbf{z}} \tag{5.2.34}$$

where $\mathbf{J}$ is the standard 8-by-8 skew-symmetric matrix

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{5.2.35}$$

where $\mathbf{0}$ is the 4-by-4 zero matrix, and $\mathbf{1}$ is the 4-by-4 identity matrix.

To solve the Hamiltonian system (5.2.34) starting from given initial rod displacement and velocity fields, we must specify the initial conditions for the impetuses $\mathbf{g}$ and χ. Because the striction $\mathbf{\Lambda}$ is a well-defined function of $\mathbf{z}$, it is clear from equations (5.2.22) and (5.2.23) that given a phase $\mathbf{z} = (\mathbf{h}, \phi, \mathbf{g}, \chi)$ as a function of s, the corresponding displacement and velocity fields $(\mathbf{h}, \phi, \mathbf{h}_t, \phi_t)$ are uniquely determined.

However, the converse is not true. Given fields $(\mathbf{h}, \phi, \mathbf{h}_t, \phi_t)$, if $\mathbf{z} = (\mathbf{h}, \phi, \mathbf{g}, \chi)$ satisfies equations (5.2.22) and (5.2.23), then for any absolutely continuous function $\mathbf{\Lambda}^*$ of s alone, the phase of the form

$$\mathbf{z}^* \equiv (\mathbf{h}, \phi, \mathbf{g} - \mathbf{\Lambda}^*_s, \chi - \mathbf{d}_2 \cdot \mathbf{\Lambda}^*) \tag{5.2.36}$$

also satisfies equations (5.2.22) and (5.2.23). The minimizer of M corresponding to the phase $\mathbf{z}^*$ is then

$$\tilde{\mathbf{\Lambda}}(\mathbf{z}^*) = \tilde{\mathbf{\Lambda}}(\mathbf{z}) + \mathbf{\Lambda}^*. \tag{5.2.37}$$

and $H(\mathbf{z}^*) = H(\mathbf{z})$. This feature of our model stands in direct analogy with the partial unobservability of the impetuses and strictions that we observed in Chapter 4.

Moreover, suppose $\mathbf{z}(s,t)$ is a solution of the Hamiltonian system (5.2.34) whose initial data $\mathbf{z}(s,0)$ correspond, via equations (5.2.22) and (5.2.23), to the initial data for $(\mathbf{h}, \phi, \mathbf{h}_t, \phi_t)$. Then $\mathbf{z}^*(s,t)$ is also a solution of (5.2.34) whose initial data correspond to the initial data for $(\mathbf{h}, \phi, \mathbf{h}_t, \phi_t)$. Since it is the state $(\mathbf{h}, \phi, \mathbf{h}_t, \phi_t)$ that we observe, the motions $\mathbf{z}(s,t)$ and $\mathbf{z}^*(s,t)$ are indistinguishable. We therefore say that the field $\mathbf{\Lambda}$ is unobservable. Equation (5.2.36) defines a gauge symmetry for this Hamiltonian system, and the arbitrary function $\mathbf{\Lambda}^*(s)$ is the gauge freedom.

Finally, we note that the dynamics of ϕ and χ in the Hamiltonian system (5.2.30) - (5.2.33) decouple from the dynamics of $\mathbf{h}$. Equation (5.2.32) implies that $\mathbf{g}$ is constant in time, so $\mathbf{g}$ is fixed by the choice of initial data. In addition, since the pre-Hamiltonian M (5.2.24) is independent of $\mathbf{h}$, the striction $\tilde{\mathbf{\Lambda}}(\mathbf{z})$ which minimizes M depends only on ϕ, χ and $\mathbf{g}$. Therefore, the pair of equations (5.2.31) and (5.2.33) form a Hamiltonian evolutionary system in ϕ and χ only, with parameter $\mathbf{g}$. Given $\phi(s,t)$ and the initial data $\mathbf{h}(s,0)$, the constant of motion

$$\mathbf{h}_s(s,t) - \mathbf{d}_3(s,t) + \mathbf{e}_1 = \mathbf{h}_s(s,0) - \mathbf{d}_3(s,0) + \mathbf{e}_1 = \mathbf{0} \tag{5.2.38}$$

can be integrated in s to determine the displacement field $\mathbf{h}(s,t)$. (If (5.2.38) takes a nonzero value, then the Hamiltonian system is still well-defined, but the solution $\mathbf{h}$ differs from the displacement of the inextensible rod by a function of s determined by the initial data.) Thus for this model of rod dynamics, the variables $\mathbf{h}$ and $\mathbf{g}$ serve primarily to motivate the derivation of the Hamiltonian dynamics. However, if we were to introduce a position-dependent body force into the model, it would be necessary to retain $\mathbf{h}$ and $\mathbf{g}$ as dynamic variables.

The decoupling just described is precisely analogous to the case of ignorable coordinates in finite dimensional systems. The Hamiltonian obtained by fixing $\mathbf{g}$ is then referred to as the Routhian. For such a reduced Hamiltonian system, different values of the variable conjugate to the ignorable coordinate generally lead to different time evolutions of the system. However for our model, the choice of $\mathbf{g}$ simply specifies the initial value of the striction, and as we have noted it does not affect the time evolution. Moreover, the specification of $\mathbf{g}$ for the reduced system eliminates the gauge freedom which arises in the full Hamiltonian system.

5.3 Existence and Uniqueness of the Striction

In this section we prove that, for a fixed state $\mathbf{z} = (\mathbf{h}, \phi, \mathbf{g}, \chi)$ in an appropriate set of functions, there exists a unique striction $\mathbf{\Lambda}$ that minimizes the pre-Hamiltonian

$$M(\mathbf{z}, \mathbf{\Lambda}) = \int_{-\infty}^{\infty} \tfrac{1}{2} \left\{ \rho^{-1} |\mathbf{g} + \mathbf{\Lambda}'|^2 + I_1^{-1} (\chi + \mathbf{d}_2 \cdot \mathbf{\Lambda})^2 + K_1 \phi'^2 \right\} ds. \tag{5.3.1}$$

The proof is somewhat technical, and may be omitted without breaking the chain of reasoning in this chapter. The essence of the proof is in the following theorem:

Theorem 5.1. *Let $\mathbf{g}$ and χ be functions in $L^2(\mathbf{R})$, and let ϕ be an absolutely continuous, nonconstant function on $\mathbf{R}$. Define the functional*

$$T(\phi, \mathbf{g}, \chi, \mathbf{\Lambda}) = \int_{-\infty}^{\infty} \tfrac{1}{2} \left\{ \rho^{-1} |\mathbf{g} + \mathbf{\Lambda}'|^2 + I_1^{-1} \left(\chi + \mathbf{n} \cdot \mathbf{\Lambda}\right)^2 \right\} ds, \tag{5.3.2}$$

where $\mathbf{n} = (-\sin(\phi), \cos(\phi))$. Then there exists a unique function $\tilde{\mathbf{\Lambda}}(\phi, \mathbf{g}, \chi)$ that minimizes T. The minimizer $\tilde{\mathbf{\Lambda}}(\phi, \mathbf{g}, \chi)$ is absolutely continuous as a function of s, with $\tilde{\mathbf{\Lambda}}'$ and $\mathbf{n} \cdot \tilde{\mathbf{\Lambda}}$ in $L^2(\mathbf{R})$.

Before proving Theorem 5.1, we remark that for ϕ fixed, the potential energy term is constant and so any minimizer $\mathbf{\Lambda}$ of T also minimizes M. Also we must be able to define the striction for functions $\mathbf{g}$ and χ not in $L^2(\mathbf{R})$, as occurs for the solitary wave. We can achieve this generalization using the gauge symmetry of M. For if

$$\mathbf{g} = -(\mathbf{\Lambda}^*)' + \mathbf{g}^*, \tag{5.3.3}$$

and

$$\chi = -\mathbf{n} \cdot \mathbf{\Lambda}^* + \chi^*, \tag{5.3.4}$$

where $\mathbf{g}^*$ and χ^* are in $L^2(\mathbf{R})$, and $\mathbf{\Lambda}^*$ is an absolutely continuous function, then Theorem 5.1 implies M has the unique minimizer $\tilde{\mathbf{\Lambda}}(\phi, \mathbf{g}, \chi) = \mathbf{\Lambda}^* + \tilde{\mathbf{\Lambda}}(\phi, \mathbf{g}^*, \chi^*)$. Finally if ϕ is constant, the proof of Theorem 5.1 can still be used to show the existence of a minimizer $\tilde{\mathbf{\Lambda}}$, but the minimizer is only unique up to an additive constant in the direction $\mathbf{t} = (\cos\phi, \sin\phi)$.

Proof of Theorem 5.1: In this proof we take $\mathbf{z}$ to be fixed, and suppress the dependence of T upon $\mathbf{z}$. Because $T(\mathbf{\Lambda})$ is bounded below by zero, and is finite for $\mathbf{\Lambda} = \mathbf{0}$, there exists a minimizing sequence $\{\mathbf{\Lambda}^j\}$ such that

$$\mathbf{u}^j(s) \equiv \mathbf{g}(s) + (\mathbf{\Lambda}^j)'(s) \tag{5.3.5}$$

converges weakly in $L^2(\mathbf{R})$, and

$$v^j(s) \equiv \chi(s) + \mathbf{d}_2(s) \cdot \mathbf{\Lambda}^j(s) \tag{5.3.6}$$

converges weakly in $L^2(\mathbf{R})$. We can use equation (5.3.5) to express $\mathbf{\Lambda}^j$ as

$$\mathbf{\Lambda}^j(s) = \mathbf{\Lambda}^j(0) - \int_0^s \mathbf{g}(\sigma)\, d\sigma + \mathbf{w}^j(s), \tag{5.3.7}$$

where

$$\mathbf{w}^j(s) \equiv \int_0^s \mathbf{u}^j(\sigma)\, d\sigma. \tag{5.3.8}$$

We will prove that there exists a subsequence of $\{\mathbf{\Lambda}^j\}$ that converges uniformly on bounded sets to a continuous function $\mathbf{\Lambda}$. In order to do this, we first prove that $\{\mathbf{w}^j\}$ has a convergent subsequence, and then go to the subsequence to show that $\{\mathbf{\Lambda}^j(0)\}$ converges.

We use the Arzelà-Ascoli Lemma together with a diagonalization argument to show that $\{\mathbf{w}^j\}$ has a convergent subsequence. The proof is as follows. The sequence $\{\mathbf{u}^j\}$ converges weakly in $L^2(\mathbf{R})$, which implies that the sequence $\{\|\mathbf{u}^j\|\}$ is bounded by some K. Therefore $\{\mathbf{w}^j\}$ is equicontinuous because

$$|\mathbf{w}^j(s_2) - \mathbf{w}^j(s_1)| = |\int_0^s \mathbf{u}^j(\sigma)\, d\sigma| \le \|\mathbf{u}^j\| \sqrt{|s_2 - s_1|} \le K\sqrt{|s_2 - s_1|}. \tag{5.3.9}$$

Similarly, on each interval $[-N, N]$, the functions $\{\mathbf{w}^j\}$ are uniformly bounded because

$$|\mathbf{w}^j(s)| \le K\sqrt{|s|} \le K\sqrt{N}. \tag{5.3.10}$$

It follows by the Arzelà-Ascoli Lemma that the sequence $\{\mathbf{w}^j\}$ has a subsequence, which we denote by $\{\mathbf{w}^j_N\}$, that converges uniformly on the interval $[-N, N]$ to a continuous function $\mathbf{w}_N$. We can repeat the above argument to show that, on the interval $[-2N, 2N]$, the sequence $\{\mathbf{w}^j_N\}$ has a subsequence $\{\mathbf{w}^j_{2N}\}$ that converges to a continuous function $\mathbf{w}_{2N}$. Moreover, by construction, $\mathbf{w}_{2N} = \mathbf{w}_N$ on $[-N, N]$. Proceeding inductively, we can apply a diagonalization argument to show that there is a subsequence of $\{\mathbf{w}^j\}$ that converges uniformly on bounded sets to a continuous function $\mathbf{w}$ on the whole real line. For simplicity of notation, we will denote the subsequence by $\{\mathbf{w}^j\}$.

To show that $\{\mathbf{\Lambda}^j(0)\}$ converges, it is essential that there are two points s_k, for $k=1$ and 2 such that the vectors $\{\mathbf{d}_3(s_k)\}$ and so also $\{\mathbf{d}_2(s_k)\}$ are linearly independent. This fact was demonstrated at the end of section 2. We can therefore choose smooth scalar functions $\{\eta^k\}$ such that η^k has compact support on a neighborhood of s_k, and the vectors

$$\mathbf{b}^k \equiv \int_{-\infty}^{\infty} \eta^k(s)\mathbf{d}_2(s)\, ds \tag{5.3.11}$$

are linearly independent. Because the vectors $\{\mathbf{b}^k\}$ are linearly independent, to prove that the sequence $\{\mathbf{\Lambda}^j(0)\}$ converges it is sufficient to prove that the sequences $\{\mathbf{\Lambda}^j(0)\cdot\mathbf{b}^k\}$ converge for $k=1$ and 2. From the definition of $\mathbf{b}^k$ (5.3.11),

$$\mathbf{\Lambda}^j(0)\cdot\mathbf{b}^k = \mathbf{\Lambda}^j(0)\cdot\int_{-\infty}^{\infty} \eta^k(s)\mathbf{d}_2(s)\, ds = \int_{-\infty}^{\infty} \eta^k(s)\mathbf{d}_2\cdot\mathbf{\Lambda}^j(0)\, ds. \tag{5.3.12}$$

We can use equations (5.3.6), (5.3.7) and (5.3.8) to rewrite the integrand in this last equation using

$$\begin{aligned}
\mathbf{d}_2\cdot\mathbf{\Lambda}^j(0) &= \mathbf{d}_2\cdot\mathbf{\Lambda}^j - \mathbf{d}_2\cdot\mathbf{w}^j + \mathbf{d}_2\cdot\int_0^s \mathbf{g}(\sigma)\, d\sigma \\
&= v^j - \mathbf{d}_2\cdot\mathbf{w}^j + \mathbf{d}_2\cdot\int_0^s \mathbf{g}(\sigma)\, d\sigma - \chi.
\end{aligned} \tag{5.3.13}$$

Equations (5.3.12) and (5.3.13) can then be combined to rewrite $\mathbf{\Lambda}^j(0)\cdot\mathbf{b}^k$ as

$$\begin{aligned}
\mathbf{\Lambda}^j(0)\cdot\mathbf{b}^k &= \int_{-\infty}^{\infty} \eta^k(s)v^j(s)\, ds - \int_{-\infty}^{\infty} \eta^k(s)\mathbf{d}_2\cdot\mathbf{w}^j(s)\, ds \\
&\quad + \int_{-\infty}^{\infty} \eta^k(s)\mathbf{d}_2\cdot\int_0^s \mathbf{g}(\sigma)\, d\sigma\, ds - \int_{-\infty}^{\infty} \eta^k(s)\chi(s)\, ds.
\end{aligned} \tag{5.3.14}$$

The first integral in equation (5.3.14) converges because $\{v^j\}$ converges weakly in $L^2(\mathbf{R})$. The second integral in (5.3.14) converges to $\int_{-\infty}^{\infty}\eta^k\mathbf{d}_2\cdot\mathbf{w}$, because η^k has compact support. The third and fourth integrals in (5.3.14) are constant. Therefore $\{\mathbf{\Lambda}^j(0)\cdot\mathbf{b}^k\}$ converges, and $\{\mathbf{\Lambda}^j(0)\}$ converges. The proof of the existence of a minimizer $\mathbf{\Lambda}$ for $T(\mathbf{\Lambda})$ is complete.

The uniqueness of the minimizer $\mathbf{\Lambda}$ is a consequence of the positivity of the second variation of T with respect to $\mathbf{\Lambda}$. The second variation is

$$\langle\delta\mathbf{\Lambda}, \frac{\delta^2 T}{\delta\mathbf{\Lambda}^2}\,\delta\mathbf{\Lambda}\rangle = \tfrac{1}{2}\int_{-\infty}^{\infty}\left\{\rho^{-1}|\delta\mathbf{\Lambda}'|^2 + I_1^{-1}|\mathbf{d}_2\cdot\delta\mathbf{\Lambda}|^2\right\} ds. \tag{5.3.15}$$

If

$$\langle\delta\mathbf{\Lambda}, \frac{\delta^2 T}{\delta\mathbf{\Lambda}^2}\,\delta\mathbf{\Lambda}\rangle = 0, \tag{5.3.16}$$

then $\delta\mathbf{\Lambda}' = \mathbf{0}$ and $\mathbf{d}_2 \cdot \delta\mathbf{\Lambda} = 0$. The first of these conditions implies that $\delta\mathbf{\Lambda}$ is constant, while the second implies that $\delta\mathbf{\Lambda}$ and $\mathbf{d}_2$ are orthogonal. These two conditions are inconsistent for $\delta\mathbf{\Lambda} \neq 0$, because $\mathbf{d}_2$ is not constant. Thus the second variation of M with respect to $\mathbf{\Lambda}$ is positive.

Now suppose $\mathbf{\Lambda}^1$ and $\mathbf{\Lambda}^2$ both minimize T. The kinetic energy T is quadratic and so convex in $\mathbf{\Lambda}$, and so all the points on the line between $\mathbf{\Lambda}^1$ and $\mathbf{\Lambda}^2$ are also minimizers. This implies

$$\langle T(\mathbf{\Lambda}^1 + \tau(\mathbf{\Lambda}^2 - \mathbf{\Lambda}^1)), \delta\mathbf{\Lambda}\rangle = \mathbf{0} \tag{5.3.17}$$

for all $\tau \in [0, 1]$. But then if we differentiate the first variation (5.3.17) with respect to τ and set $\delta\mathbf{\Lambda} = \mathbf{\Lambda}^2 - \mathbf{\Lambda}^1$, we obtain equation (5.3.16) which implies that $\mathbf{\Lambda}^2 - \mathbf{\Lambda}^1 = \mathbf{0}$. Therefore the minimizer of M is unique.

Finally we remark that standard arguments imply that the minimizer $\mathbf{\Lambda}$ satisfies the differential equation (5.2.26) in the sense of distributions on $\mathbf{R}$. □

5.4 Hamiltonian Form of the Conservation Laws

Henceforth we shall restrict our attention to the motion of an infinite rod, letting $L \to \infty$ in H (5.2.29). The Hamiltonian H appearing in the dynamical system that describes the motion of the rod is autonomous, and accordingly is an invariant of motion. The dynamical system for the infinite rod is also invariant under the additional symmetry of translation in the arc-length s, so by Noether's Theorem one expects an associated, additional invariant of motion. As is section 4.3 we follow the framework described in [52] (Theorem 4.29) and recognize that the infinitesimal generator of translational symmetry is ∂_s, so that we must seek a functional $I(\mathbf{z})$ satisfying

$$\mathbf{z}_s = \mathbf{J}\,\frac{\delta I}{\delta \mathbf{z}} \tag{5.4.1}$$

where $\mathbf{J}$ is the 8-by-8 matrix defined in (5.2.35), or equivalently

$$\begin{aligned}
\delta_{\mathbf{h}} I(\mathbf{z}) &= -\mathbf{g}_s, &(5.4.2)\\
\delta_{\phi} I(\mathbf{z}) &= -\chi_s, &(5.4.3)\\
\delta_{\mathbf{g}} I(\mathbf{z}) &= \mathbf{h}_s, &(5.4.4)\\
\delta_{\chi} I(\mathbf{z}) &= \phi_s. &(5.4.5)
\end{aligned}$$

The functional

$$I(\mathbf{z}) \equiv \int_{-\infty}^{\infty} \{\mathbf{g}(s)\cdot\mathbf{h}_s(s) + \chi(s)\cdot\phi_s(s)\}\, ds \tag{5.4.6}$$

satisfies these conditions, and it is straightforward to verify directly that I is indeed time-invariant under the dynamics (5.2.30) and (5.2.31). By analogy with the work of Benjamin [11], we call I the linear impulse.

The linear impulse can of course be rewritten in the original Lagrangian variables. From equations (5.2.20) and (5.2.21) for $\mathbf{g}$ and χ,

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \{(\rho \mathbf{h}_t - \boldsymbol{\Lambda}_s) \cdot \mathbf{h}_s + [I_1 \phi_t - \mathbf{d}_2 \cdot \boldsymbol{\Lambda}] \phi_s\} \, ds \\ &= \int_{-\infty}^{\infty} \{\rho \mathbf{h}_t \cdot \mathbf{h}_s + I_1 \phi_t \cdot \phi_s + \boldsymbol{\Lambda} \cdot (\mathbf{h}_{ss} - \phi_s \mathbf{d}_2)\} \, ds, \end{aligned} \tag{5.4.7}$$

where the second equality is obtained through an integration by parts and manipulation of the scalar triple product. The integral of motion (2.5.6) implies

$$\mathbf{h}_{ss} - \phi_s \mathbf{d}_2 = \mathbf{0}, \tag{5.4.8}$$

so the linear impulse can be rewritten as

$$I = \int_{-\infty}^{\infty} \{\rho \mathbf{h}_t \cdot \mathbf{h}_s + I_1 \phi_t \cdot \phi_s\} \, ds. \tag{5.4.9}$$

The expression (5.4.9) is the particular manifestation of the linear impulse for a planar inextensible, unshearable rod with linear bending law (2.5.9). It was shown in Chapter 3 that the linear impulse satisfies a conservation law for the motion of any uniform, hyperelastic rod, irrespective of the constraint (2.5.6).

For the reduced Hamiltonian system governing ϕ and χ only, the Hamiltonian (5.2.29) remains translation-invariant in the arc-length s, provided that the parameter $\mathbf{g}$ is chosen to be constant in s as well as in t. In this case, the linear impulse associated with the reduced system is

$$I_{\text{red}}(\phi, \chi) \equiv \int_{-\infty}^{\infty} \chi(s) \cdot \phi_s(s) \, ds. \tag{5.4.10}$$

As a consequence of the gauge freedom in the Hamiltonian system, it is always possible to choose $\mathbf{g}$ to be constant. Subject to the boundary conditions on $\mathbf{h}$, the linear impulses in the unreduced and reduced systems then differ by a constant,

$$I - I_{\text{red}} = \mathbf{g} \cdot \int_{-\infty}^{\infty} \mathbf{h}_s(s) \, ds = 4a\mathbf{g} \cdot \mathbf{e}_1. \tag{5.4.11}$$

The inextensibility condition (2.5.6) for our model also implies an infinite number of invariants of motion for the Hamiltonian system (5.2.34), one for each value of s. One would expect these invariants to be associated with symmetries of the Hamiltonian system, and this is in fact the case. As was noted in the last section, the Hamiltonian system has a gauge symmetry given by equation (5.2.36), which is parametrized by the gauge freedom $\boldsymbol{\Lambda}^*$, with the infinitesimal generator $(\mathbf{0}, 0, \boldsymbol{\Lambda}_s^*, \mathbf{d}_2 \cdot \boldsymbol{\Lambda}^*)$. Each symmetry generator $\boldsymbol{\gamma}(\boldsymbol{\Lambda}^*)$ is related to the invariant

$$C(\mathbf{z}, \boldsymbol{\Lambda}^*) \equiv \int_{-\infty}^{\infty} \boldsymbol{\Lambda}^*(s) \cdot [\mathbf{h}_s(s,t) - \mathbf{d}_3(s,t) + \mathbf{e}_1] \, ds \tag{5.4.12}$$

through the formula

$$(\mathbf{0}, 0, \mathbf{\Lambda}_s^*, \mathbf{d}_2 \cdot \mathbf{\Lambda}^*) = \mathbf{J}\,\frac{\delta C}{\delta \mathbf{z}}. \tag{5.4.13}$$

The inextensibility condition (2.5.6) is equivalent to the requirement that (5.4.12) be time-invariant for all $\mathbf{\Lambda}^*$.

Finally we note that

$$\int_{-\infty}^{\infty} [1 - \cos\phi(s)]\, ds = 4a \tag{5.4.14}$$

and

$$\int_{-\infty}^{\infty} \sin\phi(s)\, ds = 0 \tag{5.4.15}$$

are also invariants of motion. This fact is demonstrated by integrating the inextensibility condition (5.2.5) (which has the form of a conservation law, $(\mathbf{h}_t)_s = (\mathbf{d}_3 - \mathbf{e}_1)_t$) and noting that the integrals (5.4.14) and (5.4.15) correspond to the $\mathbf{e}_1$ and $\mathbf{e}_2$ components of $\int_{-\infty}^{\infty} \{\mathbf{d}_3 - \mathbf{e}_1\}\, ds$. The values of the integrals are then determined by the boundary conditions (2.5.14) and (2.5.16) on $\mathbf{h}$. Moreover, (5.4.14) and (5.4.15) are invariants of motion for the reduced Hamiltonian system, as can be shown using equations (2.5.4), (5.2.31) and (5.2.26):

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty} \{\mathbf{d}_3 - \mathbf{e}_1\}\, ds &= \int_{-\infty}^{\infty} \phi_t \mathbf{d}_2\, ds \\
&= \int_{-\infty}^{\infty} I_1^{-1}[\chi - \mathbf{d}_2 \cdot \mathbf{\Lambda}]\mathbf{d}_2\, ds \\
&= \int_{-\infty}^{\infty} \rho^{-1}(\mathbf{g} + \mathbf{\Lambda}_s)_s\, ds \\
&= \rho^{-1}(\mathbf{g} + \mathbf{\Lambda}_s)|_{-\infty}^{\infty} = \mathbf{0}.
\end{aligned} \tag{5.4.16}$$

5.5 A Variational Characterization of Solitary Wave Profiles

In this section we show that a solitary wave profile can be characterized as a critical point of the invariant

$$F(\mathbf{z}) \equiv H(\mathbf{z}) + cI(\mathbf{z}), \tag{5.5.1}$$

over the set of functions $\mathbf{z}$ specified in section 5.3. We remark that in the reduced system in which $\mathbf{g}$ is chosen to be a specified constant independent of both s and t, a solitary wave profile can be similarly characterized as an extremal of the variational principle in which F is replaced by

$$F_{\text{red}}(\phi, \chi) \equiv H(\phi, \chi) + cI_{\text{red}}(\phi, \chi). \tag{5.5.2}$$

In Section 7 we will prove rigorously that when the wave speed c satisfies the bound
$K_1 - c^2 I_1 > 0$, the solitary wave profile actually realizes a minimum of F_{red}. Implications of this fact upon the stability properties will also be discussed.

A travelling wave motion of the rod is given by a solution of (5.2.34) of the form

$$\mathbf{z}(s,t) = \bar{\mathbf{z}}(\xi), \tag{5.5.3}$$

where $\xi = s - ct$, and c is the wave speed. We refer to the function $\bar{\mathbf{z}}$ in equation (5.5.3) as the travelling wave profile. For a function f of ξ, $\partial_s f(\xi) = f'(\xi)$ and $\partial_t f(\xi) = -cf'(\xi)$, where $'$ denotes a derivative with respect to ξ. Our notion of a solitary wave is a travelling wave where the cartesian coordinates x, y rapidly approach the x axis as $|s| \to \pm\infty$. (Cf. Figure 1.)

We first show that a solitary wave $\mathbf{z}(s,t) = \bar{\mathbf{z}}(s-ct)$ is a motion of the Hamiltonian system (5.2.34) if and only if $\bar{\mathbf{z}}$ is an extremal of F. If $\mathbf{z}(s,t) = \bar{\mathbf{z}}(s-ct)$ is a solitary wave motion, then

$$\mathbf{z}_t + c\mathbf{z}_s = \mathbf{0}. \tag{5.5.4}$$

Substitution of the Hamiltonian system (5.2.34) and the characterization (5.4.1) of the linear impulse I into (5.5.4) produces

$$\frac{\delta H}{\delta \mathbf{z}}(\bar{\mathbf{z}}) + c\frac{\delta I}{\delta \mathbf{z}}(\bar{\mathbf{z}}) = \frac{\delta F}{\delta \mathbf{z}}(\bar{\mathbf{z}}) = \mathbf{0}. \tag{5.5.5}$$

That is, the profile $\bar{\mathbf{z}}$ of any solitary wave motion of the Hamiltonian system (5.2.34) is an extremal of F. By the same reasoning, if $\bar{\mathbf{z}}$ is an extremal of F, then the function $\mathbf{z}(s,t) = \bar{\mathbf{z}}(s-ct)$ is a solitary wave motion of the Hamiltonian system (5.2.34).

One can in fact derive explicit formulæ for the solitary wave profiles. A linear combination of the first variations of H given by (5.2.30) - (5.2.33) and of I given by (5.4.2) - (5.4.5) yields the conditions for an extremal of F,

$$\begin{aligned}
\mathbf{0} &= \delta_{\mathbf{h}} F = -c\bar{\mathbf{g}}', && (5.5.6)\\
0 &= \delta_{\phi} F = -I_1^{-1}\left[\bar{\chi} + \bar{\mathbf{d}}_2\cdot\bar{\mathbf{\Lambda}}\right][\bar{\mathbf{d}}_3\cdot\bar{\mathbf{\Lambda}}] - K_1\bar{\phi}'' - c\bar{\chi}', && (5.5.7)\\
\mathbf{0} &= \delta_{\mathbf{g}} F = \rho^{-1}(\bar{\mathbf{g}} + \bar{\mathbf{\Lambda}}') + c\bar{\mathbf{h}}', && (5.5.8)\\
0 &= \delta_{\chi} F = I_1^{-1}\left[\bar{\chi} + \bar{\mathbf{d}}_2\cdot\bar{\mathbf{\Lambda}}\right] + c\bar{\phi}', && (5.5.9)
\end{aligned}$$

in which $\bar{\mathbf{\Lambda}}$ denotes $\tilde{\mathbf{\Lambda}}(\bar{\mathbf{z}})$.

First note that an appropriate linear combination of equations (5.5.8), (5.5.9) and (5.2.26) implies that

$$(\bar{\mathbf{h}}' - \bar{\mathbf{d}}_3)' = \mathbf{0}. \tag{5.5.10}$$

Integration of equation (5.5.10) and application of the boundary conditions (2.5.19) and (2.5.20) then further implies that

$$\bar{\mathbf{h}}' = \bar{\mathbf{d}}_3 - \mathbf{e}_1. \tag{5.5.11}$$

Thus a solution of equations (5.5.6) - (5.5.9), subject to the specified boundary conditions, must satisfy the inextensibility condition (2.5.6). By integration of (5.5.11), subject to boundary condition (2.5.16), we further obtain

$$\int_{-\infty}^{\infty} \{1 - \cos\bar{\phi}(s)\}\ ds = 4a. \tag{5.5.12}$$

To solve equations (5.5.6) - (5.5.9) explicitly, we first solve (5.5.9) for $\bar{\chi}$:

$$\bar{\chi} = -cI_1\phi' - \bar{\mathbf{d}}_2 \cdot \bar{\mathbf{\Lambda}}. \tag{5.5.13}$$

When (5.5.13) together with its time derivative are inserted into equation (5.5.7) and the result is simplified, one obtains

$$(K_1 - c^2 I_1)\bar{\phi}'' = c\bar{\mathbf{d}}_2 \cdot \bar{\mathbf{\Lambda}}'. \tag{5.5.14}$$

In order to rewrite equation (5.5.14) explicitly in terms of $\bar{\phi}$, we use equation (5.5.8) for $\bar{\mathbf{\Lambda}}'$ and the inextensibility condition (5.5.11) to write

$$\bar{\mathbf{\Lambda}}' = -\bar{\mathbf{g}} - c\rho\bar{\mathbf{h}}' = -\bar{\mathbf{g}} - c\rho\left(\bar{\mathbf{d}}_3 - \mathbf{e}_1\right). \tag{5.5.15}$$

By equation (5.5.6), $\bar{\mathbf{g}}$ is constant. Equation (5.5.15) together with the inextensibility condition (2.5.6) and the definition (2.5.2) of $\bar{\mathbf{d}}_2$ can then be used to rewrite equation (5.5.14) as a second-order, constant-coefficient ordinary differential equation for $\bar{\phi}$,

$$(K_1 - c^2 I_1)\bar{\phi}'' = -c\bar{\mathbf{g}} \cdot \mathbf{e}_2 \cos\bar{\phi} + (c\bar{\mathbf{g}} \cdot \mathbf{e}_1 - c^2\rho)\sin\bar{\phi}. \tag{5.5.16}$$

Boundary conditions on the angle $\bar{\phi}$ require that $\bar{\phi}''(\xi) \to 0$, $\cos\bar{\phi}(\xi) \to 1$ and $\sin\bar{\phi}(\xi) \to 0$ as $\xi \to \pm\infty$, from which we infer by equation (5.5.16) that

$$\bar{\mathbf{g}} \cdot \mathbf{e}_2 = 0, \tag{5.5.17}$$

or

$$\bar{\mathbf{g}} = \bar{g}\mathbf{e}_1, \tag{5.5.18}$$

and so (5.5.16) can be rewritten as

$$(K_1 - c^2 I_1)\bar{\phi}'' = (c\bar{g}_1 - c^2\rho)\sin\bar{\phi}. \tag{5.5.19}$$

Notice that (5.5.19) has the same form as the equation for equilibria of the rod. The first integral of (5.5.19) is

$$\tfrac{1}{2}(K_1 - c^2 I_1)\bar{\phi}'(\xi)^2 = (c\bar{g}_1 - c^2\rho)[1 - \cos\bar{\phi}(\xi)]. \tag{5.5.20}$$

Integration of (5.5.20) over the entire rod yields

$$\tfrac{1}{2}(K_1 - c^2 I_1) \int_{-\infty}^{\infty} \bar{\phi}'(\xi)^2 \, d\xi = 4(c\bar{g}_1 - c^2\rho)a, \tag{5.5.21}$$

where we have used the integral condition (5.5.12). Equation (5.5.21) implies that $(K_1 - c^2 I_1)$ and $(c\bar{g}_1 - c^2\rho)$ have the same sign. Thus, equation (5.5.19) also describes the motion of an inverted pendulum.

A special case arises when $K_1 - c^2 I_1 = 0$, so that $c = \pm c_1$ where $c_1 \equiv \sqrt{K_1/I_1}$. (The critical speed c_1 is the propagation speed for an infinitesimal wave in the linearization of the equations of motion about the zero solution.) When $K_1 - c^2 I_1 = 0$, (5.5.21) implies that $(c\bar{g}_1 - c^2\rho) = 0$ for $a \neq 0$. Therefore from (5.5.19), it is apparent that when $K_1 - c^2 I_1 = 0$, any function $\bar{\phi}$ satisfying the specified constraints is a travelling wave profile. This degenerate behavior of our nonlinear system is directly analogous to the degenerate behavior of the linear wave equation. Henceforth we shall assume $K_1 - c^2 I_1 \neq 0$.

We seek a solitary wave profile such that $\bar{\phi}$ is monotonically increasing from 0 to 2π as ξ goes from $-\infty$ to ∞. The existence of this separatrix solution to the pendulum equation follows immediately from a simple phase plane analysis. (Such a function $\bar{\phi}$ is called a bore in the context of water waves, and a kink or topological soliton in the context of field theory.) Only $\bar{g}_1$ in (5.5.19) remains to be determined. By solving (5.5.20) for $\bar{\phi}'$, the integral in (5.5.12) can be computed in a second way, as

$$\begin{aligned}
\int_0^{2\pi} \frac{[1 - \cos\bar{\phi}(u)]}{\bar{\phi}'} \, d\bar{\phi} &= \sqrt{\frac{K_1 - c^2 I_1}{2(c\bar{g}_1 - c^2\rho)}} \int_0^{2\pi} \sqrt{1 - \cos\bar{\phi}} \, d\bar{\phi} \\
&= 4\sqrt{\frac{K_1 - c^2 I_1}{2(c\bar{g}_1 - c^2\rho)}}.
\end{aligned} \tag{5.5.22}$$

Equations (5.5.17), (5.5.12) and (5.5.22) together can be solved to obtain an explicit formula for $\bar{\mathbf{g}}$, namely

$$\bar{\mathbf{g}} = \bar{g}_1 \mathbf{e}_1 = \left[\frac{(K_1 - c^2 I_1)}{a^2 c} + c\rho \right] \mathbf{e}_1. \tag{5.5.23}$$

Therefore, a solitary wave profile is governed by the ordinary differential equation

$$\bar{\phi}''(\xi) = \frac{1}{a^2} \sin\bar{\phi}(\xi) \tag{5.5.24}$$

and its first integral

$$\tfrac{1}{2}\bar{\phi}'(\xi)^2 = \tfrac{1}{a^2}[1 - \cos\bar{\phi}(\xi)], \tag{5.5.25}$$

which may be rewritten as

$$\tfrac{1}{2}\bar{\phi}'(\xi) = \tfrac{1}{a} \sin\left[\tfrac{1}{2}\bar{\phi}(\xi)\right]. \tag{5.5.26}$$

An important consequence of (5.5.24) is that, for fixed a, all solitary waves share the same profile for $\bar{\phi}$. The explicit formulæ for the angular profile for the solitary wave,

$$\begin{aligned}
\bar{\phi}(\xi) &= 4\arctan\left[\exp\left(\frac{\xi}{a}\right)\right], && (5.5.27)\\
\sin\left[\tfrac{1}{2}\bar{\phi}(\xi)\right] &= \operatorname{sech}\left(\frac{\xi}{a}\right), && (5.5.28)\\
\cos\left[\tfrac{1}{2}\bar{\phi}(\xi)\right] &= -\tanh\left(\frac{\xi}{a}\right), && (5.5.29)\\
\sin\bar{\phi}(\xi) &= -2\operatorname{sech}\left(\frac{\xi}{a}\right)\tanh\left(\frac{\xi}{a}\right), && (5.5.30)\\
1-\cos\bar{\phi}(\xi) &= 2\operatorname{sech}^2\left(\frac{\xi}{a}\right), && (5.5.31)\\
\bar{\phi}'(\xi) &= \frac{2}{a}\operatorname{sech}\left(\frac{\xi}{a}\right), && (5.5.32)
\end{aligned}$$

were known to Euler and can be found, for example, in [13]. The above formulæ are valid up to translation in ξ. The directors are determined by equations (5.5.30) and (5.5.31):

$$\bar{\mathbf{d}}_1(\xi) = -\mathbf{e}_3, \tag{5.5.33}$$

$$\bar{\mathbf{d}}_2(\xi) = 2\operatorname{sech}\left(\frac{\xi}{a}\right)\tanh\left(\frac{\xi}{a}\right)\mathbf{e}_1 + \left[1-2\operatorname{sech}^2\left(\frac{\xi}{a}\right)\right]\mathbf{e}_2, \tag{5.5.34}$$

and

$$\bar{\mathbf{d}}_3(\xi) = \left[1-2\operatorname{sech}^2\left(\frac{\xi}{a}\right)\right]\mathbf{e}_1 - 2\operatorname{sech}\left(\frac{\xi}{a}\right)\tanh\left(\frac{\xi}{a}\right)\mathbf{e}_2. \tag{5.5.35}$$

Equation (5.5.35), together with the inextensibility condition (2.5.6) and the boundary conditions (2.5.14) and (2.5.17), allow us to determine the displacement profile:

$$\begin{aligned}
\bar{\mathbf{h}}(\xi) &= -2a\tanh\left(\frac{\xi}{a}\right)\mathbf{e}_1 + 2a\operatorname{sech}\left(\frac{\xi}{a}\right)\mathbf{e}_2\\
&= -2a\tanh\left(\frac{\xi}{a}\right)\bar{\mathbf{d}}_3(\xi) - 2a\operatorname{sech}\left(\frac{\xi}{a}\right)\bar{\mathbf{d}}_2(\xi). && (5.5.36)
\end{aligned}$$

We note for future reference that equations (5.5.32) and (5.5.36) together imply

$$\bar{y}(s) = \bar{h}_2(s) = a^2\bar{\phi}'(s). \tag{5.5.37}$$

Remarkably, as the solitary wave evolves, a material point traces out a semicircle, since

$$[\mathbf{e}_1\cdot\bar{\mathbf{h}}(\xi)]^2 + [\mathbf{e}_2\cdot\bar{\mathbf{h}}(\xi)]^2 = 4a^2 \tag{5.5.38}$$

as is pointed out in [13] and [19]. The profile of the linear impetus has already been given in equation (5.5.23). In order to determine the profile of the angular impetus, we must integrate equation (5.5.15) and insert the expressions for $\bar{g}_1$ (5.5.23) and $\bar{\mathbf{h}}$ (5.5.36) to determine the striction

$$\begin{aligned}
\bar{\mathbf{\Lambda}}(\xi) &= -\bar{g}_1\mathbf{e}_1\xi - c\rho\bar{\mathbf{h}}(\xi) + \bar{\mathbf{\Lambda}}^*,\\
&= \left\{-\bar{g}_1\xi + 2c\rho a\tanh\left(\frac{\xi}{a}\right)\right\}\mathbf{e}_1 - 2c\rho a\operatorname{sech}\left(\frac{\xi}{a}\right)\mathbf{e}_2 + \bar{\mathbf{\Lambda}}^*, && (5.5.39)
\end{aligned}$$

where $\bar{\mathbf{\Lambda}}^*$ is an undetermined constant. Note that the striction (5.5.39) grows linearly in ξ. Using equation (5.5.36), the components of the striction along the directors are

$$\bar{\mathbf{\Lambda}}(\xi) \cdot \bar{\mathbf{d}}_2 = \left(\bar{g}_1 \xi - \bar{\mathbf{\Lambda}}_1^*\right) \sin \bar{\phi}(\xi) + 2c\rho a \,\mathrm{sech}\left(\frac{\xi}{a}\right) \tag{5.5.40}$$

and

$$\bar{\mathbf{\Lambda}}(\xi) \cdot \bar{\mathbf{d}}_3 = \left(-\bar{g}_1 \xi + \bar{\mathbf{\Lambda}}_1^*\right) \cos \bar{\phi}(\xi) + 2c\rho a \,\tanh\left(\frac{\xi}{a}\right). \tag{5.5.41}$$

We can also determine the force field $\bar{\boldsymbol{\lambda}}$ as the time derivative of the striction $\bar{\mathbf{\Lambda}}$ through equations (5.5.23) and (5.5.39), together with the fact that $\bar{\mathbf{h}}$ is a travelling wave:

$$\bar{\boldsymbol{\lambda}} = (\bar{\mathbf{\Lambda}})_t = c\bar{\mathbf{g}} + c^2\rho[\bar{\mathbf{d}}_3 - \mathbf{e}_1] = \frac{(K_1 - c^2 I_1)}{a^2}\mathbf{e}_1 + c^2\rho\bar{\mathbf{d}}_3. \tag{5.5.42}$$

Equation (5.5.42) implies that the end loads are constant and equal, and are completely determined by the parameters a and c:

$$\bar{\boldsymbol{\lambda}}(\pm\infty) = c\bar{\mathbf{g}} = \left[\frac{(K_1 - c^2 I_1)}{a^2} + c^2\rho\right]\mathbf{e}_1. \tag{5.5.43}$$

We then introduce the expressions (5.5.35) for $\bar{\mathbf{d}}_3$ and (5.5.39) for $\bar{\mathbf{\Lambda}}$ into equation (5.5.13) for $\bar{\chi}$ and simplify to obtain the profile of the angular impetus,

$$\begin{aligned}\bar{\chi}(\xi) \quad = \quad & -2c\rho a\left(1 + \frac{I_1}{\rho a^2}\right)\,\mathrm{sech}\left(\frac{\xi}{a}\right) + 2\mathbf{e}_1 \cdot (\bar{g}_1 \mathbf{e}_1 \xi - \bar{\mathbf{\Lambda}}^*)\,\mathrm{sech}\left(\frac{\xi}{a}\right)\tanh\left(\frac{\xi}{a}\right) \\ & + \mathbf{e}_2 \cdot \bar{\mathbf{\Lambda}}^*\left[1 - 2\,\mathrm{sech}^2\left(\frac{\xi}{a}\right)\right].\end{aligned} \tag{5.5.44}$$

The presence of the two undetermined components of $\bar{\mathbf{\Lambda}}^*$ in (5.5.44) reflects the presence of the gauge symmetry (5.2.36) noted in the previous section. However, the gauge freedom described in the previous section is required to be a function of s only, while for a solitary wave motion of $\mathbf{z}$, $\bar{\mathbf{\Lambda}}^*$ must be a function of $s - ct$. For both of these conditions to be satisfied, $\bar{\mathbf{\Lambda}}^*$ must be a constant, as we have just shown. If we only required that the observable components of $\mathbf{z}$ exhibit a solitary wave motion, then $\bar{\mathbf{\Lambda}}^*$ can again be an arbitrary absolutely continuous function of s. But for each observable solitary wave motion, there exists a representative solitary wave motion of the entire phase $\mathbf{z}$. Thus, without loss of generality, we can restrict our attention to solitary wave motions of $\mathbf{z}$.

We have remarked that when $\mathbf{g}$ is chosen to be a constant in s, the reduced Hamiltonian system is invariant under translation in the arc-length s. Equation (5.5.23) shows that on a travelling wave in the unreduced system, $\mathbf{g}$ must in fact be the constant $\bar{\mathbf{g}}$. We therefore select $\mathbf{g} = \bar{\mathbf{g}}$ for our analysis of solitary waves in the reduced system. Under this assumption, the analysis presented above carries over to show that a solitary wave profile in the reduced Hamiltonian system can be characterized as an extremal of the variational principle

$$\min_{(\phi,\chi)} F_{\mathrm{red}}(\phi, \chi), \tag{5.5.45}$$

where $F_{\rm red}$ is given by (5.5.2).

5.6 Stability Analysis of Solitary Waves

Now suppose that $K_1 - c^2 I_1 > 0$; such waves are called subsonic since the wave speed satisfies $|c| < c_1$, where c_1 is the critical propagation speed of infinitesimal waves in an undeformed rod. In the remainder of this section we prove that a subsonic solitary wave profile $(\bar{\phi}, \bar{\chi})$ actually realizes a minimizer of the invariant (5.5.2). Provided that one is able to prove appropriate existence and continuation results, one may deduce the nonlinear, orbital stability of subsonic solitary waves.

We now consider the second variation of the functional $F_{\rm red}$. Because $F_{\rm red}$ is invariant under translation in s, minimizers cannot be isolated, so the second variation can at best be positive semidefinite. In order to facilitate the computation of the second variation, we first decompose $F_{\rm red}$ so as to isolate the contributions of $\mathbf{g}$, χ and $\tilde{\mathbf{\Lambda}}$ as much as possible. We are thus able to focus our attention on the influence of the configuration ϕ only. The calculations we perform are equivalent to a reformulation of our dynamics in a steadily translating frame, in which the travelling wave is an equilibrium. The explicit form of $F_{\rm red}(\phi, \chi)$ is

$$\begin{aligned} F_{\rm red}(\phi,\chi) \quad &= \quad \int_{-\infty}^{\infty} \tfrac{1}{2} \left\{ \rho^{-1} |\bar{\mathbf{g}} + \tilde{\mathbf{\Lambda}}'(\phi,\chi)|^2 + I_1^{-1} [\chi + \mathbf{d}_2 \cdot \tilde{\mathbf{\Lambda}}(\phi,\chi)]^2 + K_1 \phi'^2 \right\} ds \\ &\qquad + c \int_{-\infty}^{\infty} \chi \cdot \phi' ds. \end{aligned} \tag{5.6.1}$$

$F_{\rm red}$ can be rewritten by completing the squares and integrating by parts to obtain

$$F_{\rm red}(\phi,\chi) = F_1(\phi,\chi) + F_2(\phi), \tag{5.6.2}$$

where

$$\begin{aligned} F_1(\phi,\chi) \quad &\equiv \quad \tfrac{1}{2}\rho^{-1} \|\mathbf{g}_c\|^2 + \tfrac{1}{2} I_1^{-1} \|\chi_c\|^2, && (5.6.3) \\ \mathbf{g}_c(\phi,\chi) \quad &\equiv \quad \bar{\mathbf{g}} + \tilde{\mathbf{\Lambda}}'(\phi,\chi) + c\rho(\mathbf{d}_3 - \mathbf{e}_1), && (5.6.4) \\ \chi_c(\phi,\chi) \quad &\equiv \quad \chi + \mathbf{d}_2 \cdot \tilde{\mathbf{\Lambda}}(\phi,\chi) + cI_1\phi', && (5.6.5) \end{aligned}$$

and

$$F_2(\phi) = \int_{-\infty}^{\infty} \left\{ \tfrac{1}{2}(K_1 - c^2 I_1)\phi'(s)^2 - \tfrac{1}{2}c^2\rho|\mathbf{d}_3 - \mathbf{e}_1|^2 - c\bar{\mathbf{g}} \cdot (\mathbf{d}_3 - \mathbf{e}_1) \right\} ds. \tag{5.6.6}$$

Because a solitary wave profile $(\bar{\phi}, \bar{\chi})$ satisfies the extremal conditions (5.5.8) and (5.5.9) and the inextensibility condition (5.5.10), we see that $\mathbf{g}_c(\bar{\phi}, \bar{\chi}) = \mathbf{0}$ and $\chi_c(\bar{\phi}, \bar{\chi}) = 0$. Therefore $(\bar{\phi}, \bar{\chi})$ minimizes F_1.

We now show that $\bar{\phi}$ minimizes F_2. First observe that because $(\bar{\phi}, \bar{\chi})$ is by definition an extremal of $F_{\rm red}$, and it was just shown that $(\bar{\phi}, \bar{\chi})$ is an extremal of F_1, it follows from equation (5.6.2) that $\bar{\phi}$ must also be an extremal of F_2. The expression (2.5.1) and (5.5.23) for $\mathbf{d}_3$ and $\bar{\mathbf{g}}$ allow us to rewrite F_2 more simply as

$$\begin{aligned} F_2(\phi) &= (K_1 - c^2 I_1) \int_{-\infty}^{\infty} \left\{ \tfrac{1}{2}\phi'(s)^2 + \tfrac{1}{a^2}[1 - \mathbf{e}_1 \cdot \mathbf{d}_3(s)] \right\} ds \\ &= (K_1 - c^2 I_1) \int_{-\infty}^{\infty} \left\{ \tfrac{1}{2}\phi'(s)^2 + \tfrac{1}{a^2}[1 - \cos\phi(s)] \right\} ds. \end{aligned} \tag{5.6.7}$$

Using the Taylor series expansion of $\cos\phi$, F_2 can now be expanded, to order ϵ^2 terms, as

$$F_2(\bar{\phi} + \epsilon\delta\phi) = F_2(\bar{\phi}) + \tfrac{1}{2}\epsilon^2 \langle \delta\phi, \delta^2 F_2(\bar{\phi})\, \delta\phi \rangle + \epsilon^3 r(\delta\phi), \tag{5.6.8}$$

where we have taken $\|\delta\phi\|_{H^1} = 1$. (Here and in the remainder of this section, $\|\cdot\|$ denotes the standard $L^2(\mathbf{R})$ norm, and $\|\delta\phi\|_{H^1} \equiv (\|\delta\phi\|^2 + \|\delta\phi'\|^2)^{1/2}$). In (5.6.8),

$$\langle \delta\phi, \delta^2 F_2(\bar{\phi})\, \delta\phi \rangle = (K_1 - c^2 I_1) \int_{-\infty}^{\infty} \left\{ \delta\phi'(s)^2 + \frac{1}{a^2} \cos\bar{\phi}(s) \delta\phi(s)^2 \right\} ds, \tag{5.6.9}$$

and

$$r(\delta\phi) \equiv -\frac{(K_1 - c^2 I_1)}{6a^2} \int_{-\infty}^{\infty} \sin\tilde{\phi}(s) \delta\phi(s)^3\, ds, \tag{5.6.10}$$

where $\tilde{\phi}(s) = \bar{\phi}(s) + \epsilon\tau(s)\delta\phi(s)$ for some $\tau(s) \in (0,1)$. To analyze the second variation, we first perform an integration by parts to obtain

$$\langle \delta\phi, \delta^2 F_2(\bar{\phi})\, \delta\phi \rangle = (K_1 - c^2 I_1) \int_{-\infty}^{\infty} \delta\phi(s) \left\{ -\delta\phi''(s) + \frac{1}{a^2} \cos\bar{\phi}(s) \delta\phi(s) \right\} ds. \tag{5.6.11}$$

Now, to determine the definiteness of $\delta^2 F_2(\bar{\phi})$, we consider the eigenvalue problem

$$[P\delta\phi](s) \equiv -\delta\phi''(s) + \frac{1}{a^2} \cos\bar{\phi}(s) \delta\phi(s) = \mu\delta\phi(s). \tag{5.6.12}$$

Since $\cos\bar{\phi} \to 1$ exponentially as $s \to \pm\infty$, we associate with equation (5.6.12) the constant coefficient eigenvalue problem

$$[P_0\delta\phi](s) \equiv -\delta\phi''(s) + \frac{1}{a^2} \delta\phi(s) = \mu\delta\phi(s). \tag{5.6.13}$$

From standard theory (cf. [27]), the linear operator P_0 has continuous spectrum $[\frac{1}{a^2}, \infty)$. Moreover, due to the exponential decay of the solitary wave profile $\bar{\phi}$, the operator $P - P_0$ is a relatively compact perturbation of P_0. It follows (cf. [54], p. 113) that P has the same continuous spectrum as P_0, and P has at most a countable set of simple eigenvalues $\mu_0 < \mu_1 < \ldots < 1/a^2$, with the property that the eigenfunction corresponding to μ_k has k interior zeroes. Differentiation of the ordinary differential equation for the solitary wave (5.5.19) shows that $\bar{\phi}'$ is an eigenfunction corresponding to eigenvalue *zero*, and $\bar{\phi}'$ has no interior zeroes. Thus $\mu_0 = 0$, and so

$$\int_{-\infty}^{\infty} \delta\phi(s) \left[-\delta\phi''(s) + \frac{1}{a^2} \cos\bar{\phi}(s) \delta\phi(s) \right] ds \tag{5.6.14}$$

is positive semidefinite. Therefore when $K_1 - c^2 I_1 > 0$, the second variation of $\delta^2 F_2(\bar{\phi})$ is positive semidefinite. In fact we can draw a stronger conclusion. The eigenfunction $\bar{\phi}'$ associated with $\mu_0 = 0$ is simple, and corresponds to an infinitesimal translation of the solitary wave profile. Therefore, for any perturbation $\delta\phi$ that is orthogonal to $\bar{\phi}'$,

$$\langle \delta\phi, P\,\delta\phi \rangle \geq \mu_1 \|\delta\phi\|^2, \tag{5.6.15}$$

where μ_1 is the first positive eigenvalue of the operator P (5.6.12). Hence when $K_1 - c^2 I_1 > 0$, the second variation of F_2 is positive definite on variations orthogonal to $\bar{\phi}'$, with

$$\langle \delta\phi, \delta^2 F_2(\bar{\phi})\,\delta\phi \rangle \geq \tfrac{1}{2}(K_1 - c^2 I_1)\mu_1 \|\delta\phi\|^2. \tag{5.6.16}$$

Equation (5.6.16) can also be used to bound the second variation of F_2 below with respect to $\|\delta\phi\|_{H^1}$. From (5.6.12) and (5.6.15), we find that

$$\begin{aligned} \langle \delta\phi, P\,\delta\phi \rangle - C\|\delta\phi\|_{H^1}^2 &= (1-C)\int_{-\infty}^{\infty} \langle \delta\phi, P\,\delta\phi \rangle + C\int_{-\infty}^{\infty} \left[\frac{1}{a^2}\cos\bar{\phi}(s) - 1\right] \delta\phi(s)^2\,ds \\ &\geq \left[(1-C)\mu_1 - C\left(\frac{1}{a^2} + 1\right)\right] \|\delta\phi\|^2. \end{aligned} \tag{5.6.17}$$

Thus for $C \equiv \left[\mu_1 + a^{-2} + 1\right]^{-1}$, $\langle \delta\phi, P\,\delta\phi \rangle \geq C\|\delta\phi\|_{H^1}^2$, and

$$\langle \delta\phi, \delta^2 F_2(\bar{\phi})\,\delta\phi \rangle \geq \tfrac{1}{2}(K_1 - c^2 I_1) C \|\delta\phi\|_{H^1}^2. \tag{5.6.18}$$

Moreover, the remainder term r (5.6.10) can be uniformly bounded as

$$|r(\delta\phi)| \leq \frac{(K_1 - c^2 I_1)}{6a^2} \|\delta\phi\|_{L^\infty} \|\delta\phi\|^2. \tag{5.6.19}$$

Through the inequalities $\|\delta\phi\| \leq \|\delta\phi\|_{H^1}$ and $\|\delta\phi\|_{L^\infty} \leq \|\delta\phi\|_{H^1}$, the bound (5.6.19) can be rewritten as

$$|r(\delta\phi)| \leq \frac{(K_1 - c^2 I_1)}{6a^2} \|\delta\phi\|_{H^1}^3. \tag{5.6.20}$$

The comparison of equations (5.6.18) and (5.6.20) shows that, if $\epsilon < \epsilon_1 \equiv \frac{3}{2}a^2 C$, then $\epsilon^3 r(\delta\phi)$ is bounded as

$$|\epsilon^3 r(\delta\phi)| \leq \frac{1}{4}\epsilon^2 \langle \delta\phi, \delta^2 F_2(\bar{\phi})\,\delta\phi \rangle. \tag{5.6.21}$$

The combination of equations (5.6.7) and (5.6.21) implies that

$$F_2(\bar{\phi} + \epsilon\delta\phi) - F_2(\bar{\phi}) \geq \frac{1}{4}\epsilon^2 \langle \delta\phi, \delta^2 F_2(\bar{\phi})\,\delta\phi \rangle \geq \frac{1}{4}\epsilon^2 (K_1 - c^2 I_1) C \|\delta\phi\|_{H^1}^2, \tag{5.6.22}$$

provided that $\delta\phi$ is orthogonal to $\bar{\phi}'$ and $\|\epsilon\delta\phi\|_{H^1} < \epsilon_1$.

We can dispense with the restriction that $\delta\phi$ be orthogonal to $\bar{\phi}'$ by noting that for any function ϕ such that $\phi - \bar{\phi} \in L^2$ there is a translate of $\bar{\phi}$, which we denote by $T_\tau\bar{\phi}(s) \equiv$

$\phi(s+\tau)$, such that $\phi - T_\tau\bar{\phi}$ is orthogonal to $T_\tau\bar{\phi}'$. (This fact is proven by noting that $\phi - \bar{\phi} \in L^2$ implies there is a minimizer τ of

$$\int_{-\infty}^{\infty} \left[\phi(s) - \bar{\phi}(s+\tau)\right]^2 \, ds, \tag{5.6.23}$$

and then observing that at the minimizer, the derivative of (5.6.23) with respect to τ vanishes.) Therefore, provided that $\|\phi - T_\tau\bar{\phi}\| < \epsilon_1$,

$$\begin{aligned} F_2(\phi) - F_2(\bar{\phi}) &= F_2(\phi) - F_2(T_\tau\bar{\phi}) \\ &\geq \tfrac{1}{4}\langle(\phi - T_\tau\bar{\phi}), \delta^2 F_2(T_\tau\bar{\phi})\,(\phi - T_\tau\bar{\phi})\rangle \\ &\geq \tfrac{1}{4}(K_1 - c^2 I_1)C\|\phi - T_\tau\bar{\phi}\|^2_{H^1}. \end{aligned} \tag{5.6.24}$$

Thus $\bar{\phi}$ is a local minimizer of F_2 amongst all competitors ϕ for which $\phi - \bar{\phi} \in H^1$.

We can now combine (5.6.24) with equations (5.6.2) and (5.6.3) to show that if $\|\phi - T_\tau\bar{\phi}\|_{H^1} < \epsilon_1$ then

$$\begin{aligned} \Delta F_{\mathrm{red}} &\equiv F_{\mathrm{red}}(\phi,\chi) - F_{\mathrm{red}}(\bar{\phi},\bar{\chi}) \\ &\geq \tfrac{1}{2}\rho^{-1}\|\mathbf{P}(\phi,\chi)\|^2 + \tfrac{1}{2}I_1^{-1}\|\Pi(\phi,\chi)\|^2 + \tfrac{1}{4}(K_1 - c^2 I_1)C\|\phi - T_\tau\bar{\phi}\|^2_{H^1}. \end{aligned} \tag{5.6.25}$$

The inequality (5.6.25) implies several useful error bounds. We first observe that

$$\|\phi - T_\tau\bar{\phi}\|_{H^1} \leq C_1\sqrt{\Delta F_{\mathrm{red}}} \tag{5.6.26}$$

where $C_1 \equiv 2/\sqrt{(K_1 - c^2 I_1)C}$, and by applying Taylor's theorem to $\mathbf{d}_3(\phi)$ we further observe that

$$\|\mathbf{d}_3(\phi) - T_\tau\mathbf{d}_3(\bar{\phi})\| \leq \sqrt{2}C_1\sqrt{\Delta F_{\mathrm{red}}}. \tag{5.6.27}$$

From (5.6.26) and (5.6.27), the definitions (5.6.4) and (5.6.5) of $\mathbf{P}$ and Π, the extremal conditions (5.5.8) and (5.5.9), and the inequality (5.6.25) we then obtain the additional inequalities

$$\|I_1^{-1}(\chi + \mathbf{d}_2\cdot\tilde{\mathbf{\Lambda}}) - T_\tau I_1^{-1}(\bar{\chi} + \bar{\mathbf{d}}_2\cdot\bar{\mathbf{\Lambda}})\| \leq I_1^{-1}\|\Pi\| + c\|\phi' - T_\tau\bar{\phi}'\| \leq C_2\sqrt{\Delta F_{\mathrm{red}}}, \tag{5.6.28}$$

where $C_2 \equiv \sqrt{2/I_1} + cC_1$, and

$$\|\rho^{-1}(\bar{\mathbf{g}} + \tilde{\mathbf{\Lambda}}') - T_\tau\rho^{-1}(\bar{\mathbf{g}} + \bar{\mathbf{\Lambda}}')\| \leq \rho^{-1}\|\mathbf{P}\| + \sqrt{2}c\|\phi' - T_\tau\bar{\phi}'\| \leq C_3\sqrt{\Delta F_{\mathrm{red}}}, \tag{5.6.29}$$

where $C_3 \equiv \sqrt{2/\rho} + \sqrt{2}c\rho C_1$. It should be emphasized that the translation τ appearing in these inequalities is determined entirely by the function ϕ.

We summarize this variational analysis in the following theorem.

Theorem 5.2. *Assume that c satisfies $K_1 - c^2 I_1 > 0$. Then the planar solitary wave profile $\bar{\mathbf{z}}$ given by equations* (5.5.23), (5.5.27), (5.5.36) *and* (5.5.44) *is a nonisolated minimizer* *of the invariant of motion* F_{red} (5.5.2)*. Moreover any pair of functions* (ϕ, χ) *such that* $\|\phi - T_\tau\bar{\phi}\|_{H^1} < \epsilon_1$ *satisfies the inequalities* (5.6.25), (5.6.26), (5.6.28) *and* (5.6.29) .

Provided that we assume an appropriate existence result for the Cauchy problem of the time dynamics, and that the initial data for (ϕ, χ) are near (in the space $X \equiv H^1(\mathbf{R}) \times L^2(\mathbf{R})$) a translate of the solitary wave profile, our minimization result allows us to deduce the nonlinear, orbital stability of the subsonic solitary waves via a Lyapunov-type argument. To obtain such a stability theorem one requires a local existence result, which need not be in the space X, but which should be accompanied by a continuation result with respect to the norm in X. That is, for all initial data within a certain distance in X from a translate of the solitary wave profile, local existence should be proved on a fixed time interval. The time-invariance of the functional F_{red} is then used with (5.6.25) to extend the existence time to infinity.

To prove orbital stability of the planar solitary wave in the function space X we assume that at time $t = 0$ the pair (ϕ, ϕ_t) is near $(\bar{\phi}, (\bar{\phi})_t)$ so that $F_{\text{red}}(\phi, \chi) - F_{\text{red}}(\bar{\phi}, \bar{\chi})$ is small at $t = 0$. Specifically, assume

$$\Delta F_{\text{red}} \equiv F_{\text{red}}(\phi, \chi) - F_{\text{red}}(\bar{\phi}, \bar{\chi}) \leq C_1^2 \epsilon^2, \tag{5.6.30}$$

with $\epsilon < \epsilon_1$. Because F_{red} is a conserved quantity, equation (5.6.30) holds for all $t > 0$. It then follows from the inequalities (5.6.30), (5.6.26) and (5.6.28) equation (5.2.23) for ϕ_t that

$$\|\phi - T_\tau\bar{\phi}\|_{H^1} \leq \epsilon \tag{5.6.31}$$

and

$$\|\phi_t - T_\tau\bar{\phi}_t\| \leq C_2 C_1 \epsilon \tag{5.6.32}$$

for all $t > 0$. If we further assume that $\mathbf{h}$ satisfies the inextensibility condition at time $t = 0$, then the bounds (5.6.27) and (5.6.29) and equation (5.2.22) of $\mathbf{h}_t$ imply

$$\|\mathbf{h}_s - T_\tau\bar{\mathbf{h}}_s\| \leq \sqrt{2}\epsilon \tag{5.6.33}$$

and

$$\|\mathbf{h}_t - T_\tau\bar{\mathbf{h}}_t\| \leq C_3 C_1 \epsilon, \tag{5.6.34}$$

5.7 Validity of Assumptions

To conclude this chapter we reflect on the physical appropriateness of our model and the validity of our assumptions of inextensibility and unshearability of an elastic rod for the solitary wave motions we have analyzed. The analysis here is rigorous only for linearly elastic, homogeneous materials, yet even for nonlinearly elastic materials it gives an indication of the ranges of parameter values for which our model provides an accurate approximation.

To address this question we consider a uniform, isotropic rod with linear stress-strain laws for stretching and shearing, for which the axial stress is given by

$$\boldsymbol{\lambda} \cdot \mathbf{t} = EAv_3, \tag{5.7.1}$$

and the shear stress is

$$\boldsymbol{\lambda} \cdot \mathbf{n} = GAv_2. \tag{5.7.2}$$

Here A is the cross-sectional area, E is Young modulus, and G is the shear modulus $E/[2(1+\nu)]$, where ν is the Poisson ratio satisfying $0 \le \nu < 1/2$. For simplicity we will take $\nu = 0$.

Equation (5.5.42) yields the forces:

$$\begin{aligned} \boldsymbol{\lambda} \cdot \mathbf{t} &= \frac{K_1 - c^2 I_1}{a^2}\cos\phi + c^2 \mu A, & (5.7.3)\\ \boldsymbol{\lambda} \cdot \mathbf{n} &= \frac{K_1 - c^2 I_1}{a^2}\sin\phi, & (5.7.4) \end{aligned}$$

where μ is the mass/unit volume of the rod. The material constant in our model can be expressed as $\rho = \mu A$, $K_1 = EI$ and $I_1 = \mu I$, where I is the geometric moment of inertia of the rod cross-section. If we let d be the cross-sectional radius then to order-of-magnitude $A \sim d^2$ and $I \sim d^4$. Also note that the critical wave speed $c_1 = \sqrt{K_1/I_1} = \sqrt{E/\mu}$. (In steel for example the Young modulus is approximately $2 \cdot 10^{11}$ N/m^2 and the density is approximately 7850 kg/m^3, so the critical wave speed is $c_1 \approx 5000$ m/sec.)

In the above notation

$$v_3 \sim \left(\frac{d}{a}\right)^2 \left[1 - \left(\frac{c}{c_1}\right)^2\right] \cos\phi + \left(\frac{c}{c_1}\right)^2 \tag{5.7.5}$$

and

$$v_2 \sim 2\left(\frac{d}{a}\right)^2 \left[1 - \left(\frac{c}{c_1}\right)^2\right] \sin\phi. \tag{5.7.6}$$

There are three cases to consider.

For subsonic wave speeds $c/c_1 << 1$, so

$$\max v_3 \sim \left(\frac{d}{a}\right)^2 + \left(\frac{c}{c_1}\right)^2 \tag{5.7.7}$$

and

$$\max v_2 \sim \left(\frac{d}{a}\right)^2. \tag{5.7.8}$$

Thus for low wave speeds in a relatively thin rod ($d/a << 1$), inextensibility and unshearability both seem to be reasonable approximations.

For near-sonic wave speeds $c/c_1 \sim 1$, so

$$\max v_3 \sim 1, \tag{5.7.9}$$

(independent of d/a) while

$$\max v_2 \sim 0. \tag{5.7.10}$$

Thus inextensibility is not a reasonable approximation for near-sonic wave speeds, although unshearability remains reasonable.

For supersonic wave speeds $c/c_1 >> 1$, so

$$\max v_3 \sim \left[\left(\frac{d}{a}\right)^2 + 1\right]\left(\frac{c}{c_1}\right)^2 \tag{5.7.11}$$

and

$$\max v_2 \sim \left(\frac{d}{a}\right)^2 \left(\frac{c}{c_1}\right)^2. \tag{5.7.12}$$

Thus for extremely high wave speeds both inextensibility and unshearability are unreasonable approximations.

It must be remembered that the conclusions we have drawn apply only to a uniform, isotropic rod. By contrast, one can imagine a rod composed of rubber with Kevlar fibers (which are nearly inextensible) along its length and around its perimeter. Such a rod is distinctly anisotropic and could be considered inextensible and unshearable over a much broader range of values for a and c.

There is a second issue to be considered. In our model we have expressed the total energy as

$$E(\phi, \phi_t) = \tfrac{1}{2}\textstyle\int_{-\infty}^{\infty} \left\{\rho|\mathbf{r}_t(s)|^2 + I_1\phi_t(s)^2 + K_1\phi_s(s)^2\right\} ds, \tag{5.7.13}$$

in which the three terms in the integrand are respectively the densities of the translational kinetic energy, the rotational kinetic energy and the potential energy. We define the dimensionless variables $\bar{\mathbf{r}} \equiv \mathbf{r}/a$ (which is on the order of 1 in magnitude), $\bar{s} \equiv s/L$ and $\bar{t} \equiv t/T$, in which L and T are a length scale and a time scale chosen such that $\bar{\mathbf{r}}_{\bar{t}}$, $\phi_{\bar{t}}$ and $\phi_{\bar{s}}$ are also on the order of 1 in magnitude. Then the energy expressed in terms of these dimensionless variables behaves as

$$E(\phi, \phi_t) \sim \tfrac{1}{2}\tfrac{\mu a^4}{T^2}\textstyle\int_{-\infty}^{\infty} \left\{|\bar{\mathbf{r}}_{\bar{t}}(s)|^2 + \left(\frac{\delta}{a}\right)^2 \phi_{\bar{t}}(s)^2 + \left(\frac{c_1 T}{L}\right)^2 \left(\frac{\delta}{a}\right)^2 \phi_{\bar{s}}(s)^2\right\} ds. \tag{5.7.14}$$

It is clear from equation (5.7.14) that, for a thin rod, the rotational kinetic energy is small compared to the translational kinetic energy. It is common in linearized models of rod dynamics to use this argument to neglect the rotational kinetic energy. (Cf. [41].) For example Beliaev and Il'ichev [7] neglect the rotational kinetic energy in their model of inextensible and unshearable rods. As a consequence, while they obtained exactly the same solitary waves as we do, the stability analysis presented in [7] is much simpler than the stability analysis presented in this dissertation. In spite this fact, we have instead chosen to retain the rotational kinetic energy in our model for two reasons. First, for a solitary wave motion with speed c the rotational kinetic energy is small compared to the potential energy only when $c << c_1$. Thus the model in [7] is not valid for solitary waves near the sonic wave speed. More importantly, simply because a term appears small does not necessarily imply that it is negligible, as is apparent in a singular perturbation analysis. The rotational kinetic energy term may in particular be valuable to the proof of existence of solutions for inextensible rod dynamics, which neither this dissertation nor [7] presents. Caflisch and Maddocks [12] present an existence proof for dynamics of a finite rod, and their argument depends heavily on the presence of a nonzero rotary inertia. This suggests that the rotational kinetic energy term is at least beneficial, and possibly essential to an existence proof for the dynamics of infinite rods as well.

CHAPTER 6

THE SPATIAL ELASTICA: EULER PARAMETERS

6.1 Introduction

We now extend the previous analysis to obtain a Hamiltonian formulation describing the three-dimensional motion of an inextensible and unshearable rod, represented in terms of Euler parameters. Models of the dynamics of rigid bodies have often been formulated in terms of Euler parameters. (Cf. e.g. [8], [24], [36].) Their use is especially common in modelling the dynamics of rigid spacecraft. (Cf. [29], [32], [33], [61].) In rod dynamics, Simo & Vu-Quoc [59] used Euler parameters to model three-dimensional motion with finite strain. An unconstrained Hamiltonian formulation in terms of Euler parameters has already been developed by Morton [51] to model the dynamics of a rigid body. Motivated in part by Morton's work, we here apply the impetus-striction formulation to obtain a Hamiltonian formulation of the three-dimensional motion of an inextensible, unshearable rod. We begin in section 6.2 by deriving a Hamiltonian formulation. Then in section 6.3 the invariants of motion and their associated symmetries are presented, some of which were trivial for planar motion. In section 6.4 it is shown that the planar solitary wave profiles are extrema of an invariant of motion F that is a natural spatial generalization of that given in section 5.5. Finally in section 6.5 we consider the second variation of F with respect to admissible variations, i.e. those variations that preserve the invariants of motion. It is shown that for wave speeds less than a certain critical value $\bar{c}$, $\delta^2 F_{red}$ has exactly one negative eigenvalue in the space of admissible variations. We conclude by extending the result of our analysis of an isotropic rod to two related rod problems.

6.2 Hamiltonian Formulation Using Euler Parameters

In this section we obtain a Hamiltonian system describing the three-dimensional motion of an inextensible rod in terms of Euler parameters.

For the three-dimensional dynamics the time derivative of the inextensibility constraint is

$$\mathbf{h}_{st} = (\mathbf{d}_3)_t = \mathsf{D}_3[\boldsymbol{\beta}]^T \boldsymbol{\beta}_t \tag{6.2.1}$$

where the matrix $\mathsf{D}_3[\boldsymbol{\beta}]$ is defined in equation (2.6.2.21). In addition to the inextensibility constraint, when employing Euler parameters we must also enforce the time derivative of the constraint (2.6.2.1), namely

$$\boldsymbol{\beta} \cdot \boldsymbol{\beta}_t = 0. \tag{6.2.2}$$

Therefore the Lagrangian we consider is

$$\begin{aligned}
\mathcal{L} &= T(\mathbf{h}, \boldsymbol{\omega}) - V(\mathbf{u}) + \int_{-L}^{L} \{\Lambda \cdot (\mathbf{h}_{st} - \boldsymbol{\omega} \times \mathbf{d}_3) + \kappa \boldsymbol{\beta} \cdot \boldsymbol{\beta}_t\}\, ds \\
&= \int_{-L}^{L} \left\{ \tfrac{1}{2}\rho |\mathbf{h}_t|^2 + \tfrac{2}{|\boldsymbol{\beta}|^4} \boldsymbol{\beta}_t \cdot \mathsf{B}[\boldsymbol{\beta}]^T \,\mathsf{I}\, \mathsf{B}[\boldsymbol{\beta}] \boldsymbol{\beta}_t - \tfrac{2}{|\boldsymbol{\beta}|^4} \boldsymbol{\beta}_s \cdot \mathsf{B}[\boldsymbol{\beta}]^T \,\mathsf{K}\, \mathsf{B}[\boldsymbol{\beta}] \boldsymbol{\beta}_s \right\} ds \\
&\quad + \int_{-L}^{L} \left\{ \boldsymbol{\Lambda} \cdot \left(\mathbf{h}_{st} - \mathsf{D}_3[\boldsymbol{\beta}]^T \boldsymbol{\beta}_t \right) + \kappa \boldsymbol{\beta} \cdot \boldsymbol{\beta}_t \right\} ds
\end{aligned} \tag{6.2.3}$$

where the kinetic energy T is given by equation (2.6.2.76) and the potential energy V is given by equation (2.6.2.77). Proceeding as before, we define the impetuses

$$\mathbf{g} \equiv \frac{\delta \mathcal{L}}{\delta \mathbf{h}_t} = \rho \mathbf{h}_t - \boldsymbol{\Lambda}_s \tag{6.2.4}$$

and

$$\boldsymbol{\gamma} \equiv \frac{\delta \mathcal{L}}{\delta \boldsymbol{\beta}_t} = \frac{4}{|\boldsymbol{\beta}|^4} \mathsf{B}[\boldsymbol{\beta}]^T \,\mathsf{I}\, \mathsf{B}[\boldsymbol{\beta}] \boldsymbol{\beta}_t - \mathsf{D}_3[\boldsymbol{\beta}] \boldsymbol{\Lambda} + \kappa \boldsymbol{\beta}. \tag{6.2.5}$$

The inverse of equation (6.2.4) is

$$\mathbf{h}_t = \rho^{-1} (\mathbf{g} + \boldsymbol{\Lambda}_s). \tag{6.2.6}$$

In contrast with the previous formulations, the presence of the norm constraint on $\boldsymbol{\beta}$ presents the problem of determining an additional multiplier κ. There are various ways of proceeding. It is certainly possible to determine κ using an impetus-striction formulation. However there is a complication because the kinetic energy as it is currently formulated is not strictly convex in $\boldsymbol{\beta}_t$. One way around this difficulty is to artificially replace the 3-by-3 matrix I with a 4-by-4 invertible matrix, much as Morton [50] has done in his Hamiltonian formulation of rigid body dynamics. We will avoid this complication and simply solve equations (6.2.2) and (6.2.5) simultaneously for $\boldsymbol{\beta}_t$ and κ. We first rewrite the pair of equations as

$$\begin{pmatrix} \boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}] \boldsymbol{\Lambda} \\ 0 \end{pmatrix} = \begin{pmatrix} 4\mathsf{B}[\boldsymbol{\beta}]^T \,\mathsf{I}\, \mathsf{B}[\boldsymbol{\beta}]/|\boldsymbol{\beta}|^4 & \boldsymbol{\beta} \\ \boldsymbol{\beta}^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_t \\ \kappa \end{pmatrix}. \tag{6.2.7}$$

The solution of this equation is

$$\begin{pmatrix} \boldsymbol{\beta}_t \\ \kappa \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\mathsf{B}[\boldsymbol{\beta}]^T \,\mathsf{I}^{-1}\, \mathsf{B}[\boldsymbol{\beta}] & \boldsymbol{\beta}/|\boldsymbol{\beta}|^2 \\ \boldsymbol{\beta}^T/|\boldsymbol{\beta}|^2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}] \boldsymbol{\Lambda} \\ 0 \end{pmatrix}, \tag{6.2.8}$$

which is to say

$$\boldsymbol{\beta}_t = \tfrac{1}{4}\mathsf{B}[\boldsymbol{\beta}]^T \,\mathsf{I}^{-1}\, \mathsf{B}[\boldsymbol{\beta}] \left(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}] \boldsymbol{\Lambda} \right) \tag{6.2.9}$$

and

$$\kappa = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|^2} \cdot (\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\boldsymbol{\Lambda}) = \frac{\boldsymbol{\beta}\cdot\boldsymbol{\gamma}}{|\boldsymbol{\beta}|^2}, \tag{6.2.10}$$

where the second equality in (6.2.10) follows from equation (2.6.2.24). By introducing (6.2.10) back into the expression (6.2.5) for $\boldsymbol{\gamma}$ and rearranging terms, we observe that

$$\boldsymbol{\Pi}[\boldsymbol{\beta}]\boldsymbol{\gamma} = \frac{4}{|\boldsymbol{\beta}|^4}\mathsf{B}[\boldsymbol{\beta}]^T\,\mathsf{I}\,\mathsf{B}[\boldsymbol{\beta}]\,\boldsymbol{\beta}_t - \mathsf{D}_3[\boldsymbol{\beta}]\boldsymbol{\Lambda}, \tag{6.2.11}$$

where the orthogonal projection operator $\boldsymbol{\Pi}$ is given by equation (2.6.2.32). Equation (6.2.11) illustrates a symmetry of the Hamiltonian system, whereby any multiple of $\boldsymbol{\beta}$ may be added to $\boldsymbol{\gamma}$ without changing the dynamics. In section 6.2 we confirm this symmetry and discuss its consequences.

Using equation (6.2.9) we can express the angular momentum in body coordinates as

$$\boldsymbol{\pi} = \mathsf{I}\,\boldsymbol{\omega} = \tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\,(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\boldsymbol{\Lambda}) = \tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\gamma} + \tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}. \tag{6.2.12}$$

Using the definition (2.6.2.21) of $\mathsf{D}_3[\boldsymbol{\beta}]$ one can show that the product $\mathsf{B}[\boldsymbol{\beta}]\,\mathsf{D}_3[\boldsymbol{\beta}]$ has an especially simple form, namely

$$\tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\,\mathsf{D}_3[\boldsymbol{\beta}] = \tfrac{1}{|\boldsymbol{\beta}|^2}\begin{pmatrix} -\mathbf{d}_2(\boldsymbol{\beta})^T \\ \mathbf{d}_1(\boldsymbol{\beta})^T \\ \mathbf{0}^T \end{pmatrix}. \tag{6.2.13}$$

With respect to the observer frame the angular momentum is

$$\boldsymbol{\pi}^o = \mathsf{A}^T\boldsymbol{\pi} = \tfrac{1}{2}\mathsf{B}^o[\boldsymbol{\beta}]\,(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\boldsymbol{\Lambda}) = \tfrac{1}{2}\mathsf{B}^o[\boldsymbol{\beta}]\boldsymbol{\gamma} + \mathbf{d}_3(\boldsymbol{\beta}) \times \boldsymbol{\Lambda}. \tag{6.2.14}$$

Equations (6.2.6) and (6.2.9) allow us to formulate the pre-Hamiltonian

$$\begin{aligned} M(\boldsymbol{\beta},\mathbf{g},\boldsymbol{\gamma},\boldsymbol{\Lambda}) \;=\;& \int_{-L}^{L} \tfrac{1}{2}\rho^{-1}|\mathbf{g}+\boldsymbol{\Lambda}_s|^2\,ds \\ &+ \int_{-L}^{L} \tfrac{1}{8}\mathsf{B}[\boldsymbol{\beta}](\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\boldsymbol{\Lambda})\cdot\mathsf{I}^{-1}\,\mathsf{B}[\boldsymbol{\beta}](\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\boldsymbol{\Lambda})\,ds \qquad (6.2.15) \\ &+ \int_{-L}^{L} \frac{2}{|\boldsymbol{\beta}|^4}\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_s\cdot\mathsf{K}\,\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_s\,ds. \qquad (6.2.16) \end{aligned}$$

In light of equation (6.2.13) we may also express M as

$$\begin{aligned} M(\boldsymbol{\beta},\mathbf{g},\boldsymbol{\gamma},\boldsymbol{\Lambda}) \;=\;& \int_{-L}^{L} \tfrac{1}{2}\rho^{-1}|\mathbf{g}+\boldsymbol{\Lambda}_s|^2\,ds \\ &+ \int_{-L}^{L} \tfrac{1}{2}\left(\tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\gamma} + \tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\mathsf{D}_3[\boldsymbol{\beta}]\boldsymbol{\Lambda}\right)\cdot\,\mathsf{I}^{-1}\,\left(\tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\gamma} + \tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\mathsf{D}_3[\boldsymbol{\beta}]\boldsymbol{\Lambda}\right)\,ds \\ &+ \int_{-L}^{L} 2\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_s\cdot\,\mathsf{K}\,\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_s/|\boldsymbol{\beta}|^4\,ds \qquad (6.2.17) \end{aligned}$$

or

$$\begin{aligned} M &= \int_{-L}^{L} \tfrac{1}{2}\rho^{-1}|\mathbf{g}+\mathbf{\Lambda}_s|^2\,ds \\ &\quad+\int_{-L}^{L} \tfrac{1}{2}I_j^{-1}\left[\tfrac{1}{2}\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\gamma}+\mathbf{d}_j\cdot(\mathbf{d}_3\times\mathbf{\Lambda})\right]^2\,ds \\ &\quad+\int_{-L}^{L} 2K_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)^2/|\boldsymbol{\beta}|^4\,ds. \end{aligned} \tag{6.2.18}$$

We require that the striction $\mathbf{\Lambda}=\tilde{\mathbf{\Lambda}}(\mathbf{z})$ minimize M for given phase $\mathbf{z}=(\mathbf{h},\boldsymbol{\beta},\mathbf{g},\boldsymbol{\gamma})$, so the striction causes the first variation of M with respect to $\mathbf{\Lambda}$ to vanish. Therefore the striction satisfies

$$\begin{aligned} \mathbf{0}=\frac{\delta M}{\delta\mathbf{\Lambda}} &= -\rho^{-1}(\mathbf{g}+\mathbf{\Lambda}_s)_s+\tfrac{1}{4}\mathsf{D}_3[\boldsymbol{\beta}]^T\mathsf{B}[\boldsymbol{\beta}]^T\,\mathsf{I}^{-1}\mathsf{B}[\boldsymbol{\beta}](\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\mathbf{\Lambda}) \\ &= -\rho^{-1}(\mathbf{g}+\mathbf{\Lambda}_s)_s-I_1^{-1}\left(\tfrac{1}{2}\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma}-\mathbf{d}_2\cdot\mathbf{\Lambda}\right)\mathbf{d}_2 \\ &\quad+I_2^{-1}\left(\tfrac{1}{2}\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\gamma}+\mathbf{d}_1\cdot\mathbf{\Lambda}\right)\mathbf{d}_1, \end{aligned} \tag{6.2.19}$$

with the boundary conditions

$$\rho^{-1}(\mathbf{g}+\mathbf{\Lambda}_s)=\mathbf{0} \qquad \text{at} \quad s=\pm L. \tag{6.2.20}$$

Here

$$\rho^{-1}(\mathbf{g}+\mathbf{\Lambda}_s)_s=\mathbf{h}_{st}, \tag{6.2.21}$$

and from equations (2.6.2.21) and (6.2.9) we see

$$\tfrac{1}{4}\mathsf{D}_3[\boldsymbol{\beta}]^T\mathsf{B}[\boldsymbol{\beta}]^T\,\mathsf{I}^{-1}\mathsf{B}[\boldsymbol{\beta}](\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\mathbf{\Lambda})=\mathsf{D}_3[\boldsymbol{\beta}]^T\boldsymbol{\beta}_t=(\mathbf{d}_3)_t\,, \tag{6.2.22}$$

so the condition (6.2.19) on the striction is simply an enforcement of the inextensibility constraint

$$-\mathbf{h}_{st}+(\mathbf{d}_3)_t=\mathbf{0}. \tag{6.2.23}$$

The proof that the striction $\tilde{\mathbf{\Lambda}}$ exists and is unique in the case of an infinite rod is essentially the same as that presented in section 5.3. (As in the planar case, the proof for the finite rod is much easier.) To show existence we first fix the phase $(\mathbf{h},\boldsymbol{\beta},\mathbf{g},\boldsymbol{\gamma})$ and define the sequences

$$\mathbf{u}^j(s)\equiv\mathbf{g}(s)+\mathbf{\Lambda}_s^j(s), \tag{6.2.24}$$

$$\mathbf{v}^j(s)\equiv\boldsymbol{\gamma}(s)+\mathsf{D}_3[\boldsymbol{\beta}(s)]\mathbf{\Lambda}^j(s), \tag{6.2.25}$$

and

$$\mathbf{w}^j(s)\equiv\int_0^s\mathbf{u}^j(\sigma)\,d\sigma. \tag{6.2.26}$$

The arguments in section 5.3 can be used to show that $\{\mathbf{u}^j\}$ and $\{\mathbf{v}^j\}$ converge weakly in L^2, and $\{\mathbf{w}^j\}$ has a convergent subsequence that converges uniformly on bounded sets. If we use equation (6.2.24) to express $\boldsymbol{\Lambda}^j$ as

$$\boldsymbol{\Lambda}^j(s) = \boldsymbol{\Lambda}^j(0) - \int_0^s \mathbf{g}(\sigma)\, d\sigma + \mathbf{w}^j(s), \tag{6.2.27}$$

then it remains only to show that the sequence $\{\boldsymbol{\Lambda}^j(0)\}$ converges. We again assume that there exist two points s_k for $k = 1$ and 2 such that the two vectors $\{\mathbf{d}_3(s_k)\}$ are linearly independent. We can therefore choose smooth scalar functions $\{\eta_k\}$ such that η_k has compact support on a neighborhood of s_k, and the vectors

$$\mathbf{t}_k \equiv \int_{-\infty}^{\infty} \eta_k(s)\mathbf{d}_3(s)\, ds \qquad \text{for } k = 1 \text{ and } 2 \tag{6.2.28}$$

are linearly independent. Equations (6.2.24) - (6.2.27) together with the definition (2.6.2.21) of D_3 can be used to solve for $\mathsf{D}_3[\boldsymbol{\beta}]\boldsymbol{\Lambda}^j(0)$ and then premultiply by $\frac{1}{2}|\boldsymbol{\beta}|^2\mathsf{B}^o[\boldsymbol{\beta}]$ in order to write

$$\mathbf{d}_3 \times \boldsymbol{\Lambda}^j(0) = \tfrac{1}{2}\mathsf{B}^o[\boldsymbol{\beta}]\mathbf{v}^j - \tfrac{1}{2}\mathsf{B}^o[\boldsymbol{\beta}]\boldsymbol{\gamma} + \mathbf{d}_3 \times \textstyle\int_0^s \mathbf{g}(\sigma)\, d\sigma - \mathbf{d}_3 \times \mathbf{w}^j, \tag{6.2.29}$$

where we have used the identity

$$\mathsf{B}^o[\boldsymbol{\beta}]\ \left(\mathsf{B}^o[\boldsymbol{\beta}]\right)^T = |\boldsymbol{\beta}|^2\,\mathbf{1}. \tag{6.2.30}$$

If we then multiply equation (6.2.29) by η_k and integrate we obtain

$$\mathbf{t}_k \times \boldsymbol{\Lambda}^j(0) = \mathbf{c}_k^j \qquad \text{for } k = 1 \text{ and } 2, \tag{6.2.31}$$

where

$$\begin{aligned} \mathbf{c}_k^j \ \equiv\ & \int_{-L}^{L} \tfrac{1}{2}\eta_k(s)\mathsf{B}^o[\boldsymbol{\beta}(s)]\mathbf{v}^j(s)\, ds - \textstyle\int_{-L}^{L} \tfrac{1}{2}\eta_k(s)\mathsf{B}^o[\boldsymbol{\beta}(s)]\boldsymbol{\gamma}(s)\, ds \\ & + \mathbf{t}_k \times \int_0^s \mathbf{g}(\sigma)\, d\sigma - \mathbf{t}_k \times \mathbf{w}^j. \end{aligned} \tag{6.2.32}$$

The linear independence of the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ implies that the pair of vector equations given by (6.2.31) can be solved for $\boldsymbol{\Lambda}^j(0)$. For example, the vectors $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3 \equiv \mathbf{t}_1 \times \mathbf{t}_2$ form a basis for $\mathbf{R}^3$, with a dual basis we denote by $\{\mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3\}$. We can then express $\boldsymbol{\Lambda}^j(0)$ as $\alpha_k^j\mathbf{t}_k$, so that equation (6.2.31) is equivalent to the system of scalar equations

$$\alpha_1^j = -\mathbf{c}_2^j \cdot \mathbf{t}^3, \tag{6.2.33}$$

$$\alpha_2^j = \mathbf{c}_1^j \cdot \mathbf{t}^3, \tag{6.2.34}$$

$$\alpha_3^j = -\mathbf{c}_1^j \cdot \mathbf{t}^2. \tag{6.2.35}$$

The argument used in section 5.4 shows that the sequence $\{\mathbf{c}_k\}$ converges. Equations (6.2.33) - (6.2.35) therefore show that the sequence $\boldsymbol{\Lambda}^j(0)$ converges, so a minimizer of the pre-Hamiltonian M exists. The proof of uniqueness is then essentially as before.

The Hamiltonian then is

$$\begin{aligned} H(\boldsymbol{\beta},\mathbf{g},\boldsymbol{\gamma}) &= \int_{-L}^{L} \tfrac{1}{2}\rho^{-1}|\mathbf{g}+\tilde{\boldsymbol{\Lambda}}_s|^2\,ds \\ &\quad+\int_{-L}^{L} \tfrac{1}{8}\mathsf{B}[\boldsymbol{\beta}](\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}})\cdot\mathsf{I}^{-1}\,\mathsf{B}[\boldsymbol{\beta}](\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}})\,ds \qquad (6.2.36)\\ &\quad+\int_{-L}^{L} \frac{2}{|\boldsymbol{\beta}|^4}\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_s\cdot\mathsf{K}\,\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_s\,ds, \qquad (6.2.37)\end{aligned}$$

or

$$\begin{aligned} H(\boldsymbol{\beta},\mathbf{g},\boldsymbol{\gamma}) &= \int_{-L}^{L} \tfrac{1}{2}\rho^{-1}|\mathbf{g}+\tilde{\boldsymbol{\Lambda}}_s|^2\,ds \\ &\quad+\int_{-L}^{L} \tfrac{1}{2}\left(\tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\gamma}+\tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\cdot\mathsf{I}^{-1}\left(\tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\gamma}+\tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)ds \\ &\quad+\int_{-L}^{L} 2\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_s\cdot\mathsf{K}\,\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_s/|\boldsymbol{\beta}|^4\,ds \\ &= \int_{-L}^{L} \tfrac{1}{2}\rho^{-1}|\mathbf{g}+\tilde{\boldsymbol{\Lambda}}_s|^2\,ds \\ &\quad+\int_{-L}^{L} \tfrac{1}{2}I_j^{-1}\left[\tfrac{1}{2}\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\gamma}+\mathbf{d}_j\cdot\left(\mathbf{d}_3\times\tilde{\boldsymbol{\Lambda}}\right)\right]^2\,ds \\ &\quad+\int_{-L}^{L} 2K_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)^2/|\boldsymbol{\beta}|^4\,ds. \qquad (6.2.38)\end{aligned}$$

It follows from equation (6.2.38) that

$$\mathbf{h}_t=\frac{\delta H}{\delta\mathbf{g}}=\rho^{-1}(\mathbf{g}+\tilde{\boldsymbol{\Lambda}}_s), \qquad (6.2.39)$$

$$\mathbf{g}_t=-\frac{\delta H}{\delta\mathbf{h}}=\mathbf{0}, \qquad (6.2.40)$$

and

$$\begin{aligned}\boldsymbol{\beta}_t=\frac{\delta H}{\delta\boldsymbol{\gamma}} &= \tfrac{1}{4}\mathsf{B}[\boldsymbol{\beta}]^T\,\mathsf{I}^{-1}\mathsf{B}[\boldsymbol{\beta}](\boldsymbol{\gamma}+\mathsf{D}_3\tilde{\boldsymbol{\Lambda}}) \\ &= \tfrac{1}{4}I_1^{-1}(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma}-2\mathbf{d}_2\cdot\tilde{\boldsymbol{\Lambda}})\mathsf{B}_1\boldsymbol{\beta} \\ &\quad+\tfrac{1}{4}I_2^{-1}(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\gamma}+2\mathbf{d}_1\cdot\tilde{\boldsymbol{\Lambda}})\mathsf{B}_2\boldsymbol{\beta} \\ &\quad+\tfrac{1}{4}I_3^{-1}(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma})\mathsf{B}_3\boldsymbol{\beta}, \qquad (6.2.41)\end{aligned}$$

as was found in equation (6.2.9).

To obtain an equation for $\boldsymbol{\gamma}_t$ we first note that the first variation of the kinetic energy with respect to $\boldsymbol{\beta}$ is

$$\begin{aligned}
\frac{\delta T}{\delta \boldsymbol{\beta}} &= -\tfrac{1}{4}I_1^{-1}(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma} - 2\mathbf{d}_2\cdot\tilde{\boldsymbol{\Lambda}})(\mathsf{B}_1\boldsymbol{\gamma} + 2\mathsf{D}_2\,\tilde{\boldsymbol{\Lambda}}) \\
&\quad -\tfrac{1}{4}I_2^{-1}(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\gamma} + 2\mathbf{d}_1\cdot\tilde{\boldsymbol{\Lambda}})(\mathsf{B}_2\boldsymbol{\gamma} - 2\mathsf{D}_1\,\tilde{\boldsymbol{\Lambda}}) \\
&\quad -\tfrac{1}{4}I_3^{-1}(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma})(\mathsf{B}_3\boldsymbol{\gamma}),
\end{aligned} \tag{6.2.42}$$

where D_1 and D_2 are given by equations (2.6.2.19) and (2.6.2.20). It will prove convenient to use the explicit forms of D_1 and D_2 to decompose $\delta T/\delta\boldsymbol{\beta}$ into its components with respect to the basis vectors $\boldsymbol{\beta}$, $\mathsf{B}_1\boldsymbol{\beta}$, $\mathsf{B}_2\boldsymbol{\beta}$, and $\mathsf{B}_3\boldsymbol{\beta}$, from which we obtain

$$\begin{aligned}
\frac{\delta T}{\delta \boldsymbol{\beta}}\cdot\boldsymbol{\beta} &= \tfrac{1}{4}I_1^{-1}(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma} - 2\mathbf{d}_2\cdot\tilde{\boldsymbol{\Lambda}})(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma}) \\
&\quad +\tfrac{1}{4}I_2^{-1}(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\gamma} + 2\mathbf{d}_1\cdot\tilde{\boldsymbol{\Lambda}})(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\gamma}) \\
&\quad +\tfrac{1}{4}I_3^{-1}(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma})(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma}),
\end{aligned} \tag{6.2.43}$$

$$\begin{aligned}
\frac{\delta T}{\delta \boldsymbol{\beta}}\cdot\mathsf{B}_1\boldsymbol{\beta} &= -\tfrac{1}{4}I_1^{-1}(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma} - 2\mathbf{d}_2\cdot\tilde{\boldsymbol{\Lambda}})(\boldsymbol{\beta}\cdot\boldsymbol{\gamma} + 4\mathbf{d}_3\cdot\tilde{\boldsymbol{\Lambda}}) \\
&\quad +\tfrac{1}{4}I_2^{-1}(\mathsf{B}_2\boldsymbol{\beta} + 2\mathbf{d}_1\cdot\tilde{\boldsymbol{\Lambda}})(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma}) \\
&\quad -\tfrac{1}{4}I_3^{-1}(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma})(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma}),
\end{aligned} \tag{6.2.44}$$

$$\begin{aligned}
\frac{\delta T}{\delta \boldsymbol{\beta}}\cdot\mathsf{B}_2\boldsymbol{\beta} &= -\tfrac{1}{4}I_1^{-1}(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma} - 2\mathbf{d}_2\cdot\tilde{\boldsymbol{\Lambda}})(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma}) \\
&\quad -\tfrac{1}{4}I_2^{-1}(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\gamma} + 2\mathbf{d}_1\cdot\tilde{\boldsymbol{\Lambda}})(\boldsymbol{\beta}\cdot\boldsymbol{\gamma} + 4\mathbf{d}_3\cdot\tilde{\boldsymbol{\Lambda}}) \\
&\quad +\tfrac{1}{4}I_3^{-1}(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma})(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma}),
\end{aligned} \tag{6.2.45}$$

$$\begin{aligned}
\frac{\delta T}{\delta \boldsymbol{\beta}}\cdot\mathsf{B}_3\boldsymbol{\beta} &= \tfrac{1}{4}I_1^{-1}(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma} - 2\mathbf{d}_2\cdot\tilde{\boldsymbol{\Lambda}})(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\gamma} + 4\mathbf{d}_1\cdot\tilde{\boldsymbol{\Lambda}}) \\
&\quad -\tfrac{1}{4}I_2^{-1}(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\gamma} - 2\mathbf{d}_1\cdot\tilde{\boldsymbol{\Lambda}})(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma} - 4\mathbf{d}_2\cdot\tilde{\boldsymbol{\Lambda}}) \\
&\quad -\tfrac{1}{4}I_3^{-1}(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma})(\boldsymbol{\beta}\cdot\boldsymbol{\gamma}).
\end{aligned} \tag{6.2.46}$$

Through an integration by parts we find the first variation of the potential energy to be

$$\begin{aligned}
\frac{\delta V}{\delta \boldsymbol{\beta}} &= -\frac{4}{|\boldsymbol{\beta}|^4}K_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{ss})\mathsf{B}_j\boldsymbol{\beta} - \frac{8}{|\boldsymbol{\beta}|^4}K_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)\mathsf{B}_j\boldsymbol{\beta}_s \\
&\quad -\frac{8}{|\boldsymbol{\beta}|^6}(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)^2\boldsymbol{\beta} + \frac{16}{|\boldsymbol{\beta}|^6}K_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)(\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)\mathsf{B}_j\boldsymbol{\beta}.
\end{aligned} \tag{6.2.47}$$

Using the commutation relation (2.6.2.41) to decompose (6.2.47) into components produces

$$\frac{\delta V}{\delta \boldsymbol{\beta}}\cdot\boldsymbol{\beta} = 0, \tag{6.2.48}$$

as a consequence of the homogeneity of V in $\boldsymbol{\beta}$, and

$$\begin{aligned}
\frac{\delta V}{\delta \boldsymbol{\beta}}\cdot\mathsf{B}_1\boldsymbol{\beta} &= -\frac{4}{|\boldsymbol{\beta}|^2}K_1(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{ss}) + \frac{8}{|\boldsymbol{\beta}|^4}(K_2 - K_3)(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s) \\
&\quad +\frac{8}{|\boldsymbol{\beta}|^4}K_1(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)(\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s),
\end{aligned} \tag{6.2.49}$$

$$
\begin{aligned}
\frac{\delta V}{\delta \boldsymbol{\beta}} \cdot \mathsf{B}_2\boldsymbol{\beta} &= -\frac{4}{|\boldsymbol{\beta}|^2} K_2(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{ss}) + \frac{8}{|\boldsymbol{\beta}|^4}(K_3 - K_1)(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s) \\
&\quad + \frac{8}{|\boldsymbol{\beta}|^4} K_2(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)(\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s),
\end{aligned} \tag{6.2.50}
$$

and

$$
\begin{aligned}
\frac{\delta V}{\delta \boldsymbol{\beta}} \cdot \mathsf{B}_3\boldsymbol{\beta} &= \frac{4}{|\boldsymbol{\beta}|^2} K_3(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{ss}) + \frac{8}{|\boldsymbol{\beta}|^4}(K_1 - K_2)(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s) \\
&\quad - \frac{16}{|\boldsymbol{\beta}|^4} K_3(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)(\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s).
\end{aligned} \tag{6.2.51}
$$

Combining equation (6.2.42) and (6.2.47), the remaining evolution equation for the Hamiltonian system is therefore

$$
\begin{aligned}
\boldsymbol{\gamma}_t = -\frac{\delta H}{\delta \boldsymbol{\beta}} &= \tfrac{1}{4} I_1^{-1}(\mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma} - 2\mathbf{d}_2\cdot\tilde{\boldsymbol{\Lambda}})(\mathsf{B}_1\boldsymbol{\gamma} + 2\mathsf{D}_2\,\tilde{\boldsymbol{\Lambda}}) \\
&\quad + \tfrac{1}{4} I_2^{-1}(\mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\gamma} + 2\mathbf{d}_1\cdot\tilde{\boldsymbol{\Lambda}})(\mathsf{B}_2\boldsymbol{\gamma} - 2\mathsf{D}_1\,\tilde{\boldsymbol{\Lambda}}) \\
&\quad + \tfrac{1}{4} I_3^{-1}(\mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma})(\mathsf{B}_3\boldsymbol{\gamma}) \\
&\quad + \frac{4}{|\boldsymbol{\beta}|^4} K_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_{ss})\mathsf{B}_j\boldsymbol{\beta} + \frac{8}{|\boldsymbol{\beta}|^4} K_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)\mathsf{B}_j\boldsymbol{\beta}_s \\
&\quad + \frac{8}{|\boldsymbol{\beta}|^6} K_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)^2\boldsymbol{\beta} - \frac{16}{|\boldsymbol{\beta}|^6} K_j(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)(\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)\mathsf{B}_j\boldsymbol{\beta}.
\end{aligned} \tag{6.2.52}
$$

The Hamiltonian system can be expressed compactly as

$$
\mathbf{z}_t = \mathbf{J}\,\frac{\delta H}{\delta \mathbf{z}}, \tag{6.2.53}
$$

in which $\mathbf{J}$ is the 14-by-14 skew symmetric matrix given by

$$
\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{6.2.54}
$$

where $\mathbf{0}$ is the 7-by-7 zero matrix, and $\mathbf{1}$ is the 7-by-7 identity matrix.

6.3 Hamiltonian Form of Conservation Laws

Henceforth in this chapter we assume the rod is infinite and transversely isotropic. In terms of the potential energy (2.6.2.77) this means $I_2 = I_1$ and $K_2 = K_1$.

In analogy with the invariant (5.4.12), we again have an infinite set of invariants

$$
C(\mathbf{z}) \equiv \int_{-\infty}^{\infty} \boldsymbol{\Lambda}^*(s)\cdot[\mathbf{h}_s(s,t) - \mathbf{d}_3(s,t) + \mathbf{e}_1]\,ds \tag{6.3.1}
$$

associated with the inextensibility condition. The first variation of C is

$$
\frac{\delta C}{\delta \mathbf{z}} = (-(\boldsymbol{\Lambda}^*)_s, -\left(\frac{\partial \mathbf{d}_3}{\partial \boldsymbol{\beta}}\right)^T \boldsymbol{\Lambda}^*, \mathbf{0}, \mathbf{0}) = (-(\boldsymbol{\Lambda}^*)_s, -\mathsf{D}_3[\boldsymbol{\beta}]\,\boldsymbol{\Lambda}^*, \mathbf{0}, \mathbf{0}), \tag{6.3.2}
$$

and the generator of the associated symmetry is

$$\mathbf{J}\frac{\delta C}{\delta \mathbf{z}} = (\mathbf{0}, \mathbf{0}, (\mathbf{\Lambda}^*)_s, \mathsf{D}_3[\boldsymbol{\beta}]\,\mathbf{\Lambda}^*). \tag{6.3.3}$$

As in the planar case, the translational symmetry of the Hamiltonian with respect to arc-length implies the invariance of the linear impulse, given by

$$I \equiv \int_{-\infty}^{\infty} \{\mathbf{g}\cdot\mathbf{h}_s + \boldsymbol{\gamma}\cdot\boldsymbol{\beta}_s\}\, ds. \tag{6.3.4}$$

Using the inextensibility condition (2.5.6) and the definitions of $\mathbf{g}$ (6.2.4) and $\boldsymbol{\gamma}$ (6.2.5) we can rewrite the impulse given by (6.3.4) in the form given in Chapter 3:

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \left\{\left(\mathbf{g} + \tilde{\mathbf{\Lambda}}_s\right)\cdot\mathbf{h}_s + \left(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\,\tilde{\mathbf{\Lambda}}\right)\cdot\boldsymbol{\beta}_s\right\}\, ds \\ &= \int_{-\infty}^{\infty} \left\{\rho\mathbf{h}_t\cdot\mathbf{h}_s + 4\mathsf{B}[\boldsymbol{\beta}]^T\,\mathsf{I}\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}_t\cdot\boldsymbol{\beta}_s\right\}\, ds \\ &= \int_{-\infty}^{\infty} \{\mathbf{p}\cdot\mathbf{h}_s + \boldsymbol{\pi}\cdot\mathbf{u}\}\, ds \end{aligned} \tag{6.3.5}$$

The first variation of I is

$$\frac{\delta I}{\delta \mathbf{z}} = (-\mathbf{g}_s, -\gamma_s, \mathbf{h}_s, \boldsymbol{\beta}_s). \tag{6.3.6}$$

As was remarked in Chapter 5, the invariance of the linear impulse is associated with the translational symmetry of the rod with respect to the arc-length s. The infinitesimal generator of this symmetry is

$$\mathbf{J}\frac{\delta I}{\delta \mathbf{z}} = (\mathbf{h}_s, \boldsymbol{\beta}_s, \mathbf{g}_s, \boldsymbol{\gamma}_s). \tag{6.3.7}$$

In addition, we obtain several other invariants which were discussed in Chapter 3 for the isotropic three-dimensional Hamiltonian system governing the dynamics of the rod, namely the isotropic angular momentum A, the total angular momentum, and the transverse components of linear momentum. These invariants were not discussed in the planar case because they were not pertinent to the development of the stability result. Also in the planar case we assumed zero twist, which implies that the isotropic angular momentum is identically zero.

Using the expression (6.2.12) for the angular momentum we find that the isotropic angular momentum A takes a simple form in terms of the Hamiltonian state variables, namely

$$\begin{aligned} A \equiv \int_{-\infty}^{\infty} \mathbf{e}_3\cdot\mathsf{I}\boldsymbol{\omega}\, ds &= \tfrac{1}{2}\textstyle\int_{-\infty}^{\infty} \mathbf{e}_3\cdot\mathsf{B}[\boldsymbol{\beta}]\left(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\,\tilde{\mathbf{\Lambda}}\right)\, ds \\ &= \tfrac{1}{2}\textstyle\int_{-\infty}^{\infty} \mathsf{B}_3\boldsymbol{\beta}\cdot\left(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\tilde{\mathbf{\Lambda}}\right)\, ds \\ &= \tfrac{1}{2}\textstyle\int_{-\infty}^{\infty} \mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma}\, ds, \end{aligned} \tag{6.3.8}$$

where the last equality is a consequence of the identity (2.6.2.23). The first variation of A is

$$\frac{\delta A}{\delta \mathbf{z}} = (\mathbf{0},\ -\tfrac{1}{2}\mathsf{B}_3\boldsymbol{\gamma},\ \mathbf{0},\ \tfrac{1}{2}\mathsf{B}_3\boldsymbol{\beta}). \tag{6.3.9}$$

The invariance of the isotropic angular momentum A is related to the symmetry of an isotropic rod with respect to rotation about the tangent vector $\mathbf{d}_3$. This can be seen in several ways. First, the infinitesimal generator of the symmetry associated with A is

$$\mathbf{J}\,\frac{\delta A}{\delta \mathbf{z}} = (\mathbf{0},\ \tfrac{1}{2}\mathsf{B}_3\boldsymbol{\beta},\ \mathbf{0},\ \tfrac{1}{2}\mathsf{B}_3\boldsymbol{\gamma}), \tag{6.3.10}$$

which generates the transformation group $(\mathbf{h}, \mathsf{Q}_3\,\boldsymbol{\beta}, \mathbf{g}, \mathsf{Q}_3\,\boldsymbol{\gamma})$ where $\mathsf{Q}_3 = \exp(\frac{1}{2}\epsilon\mathsf{B}_3)$. Moreover, one can exploit equations (2.6.2.42) and (2.6.2.43) to show that under this transformation group

$$\begin{pmatrix} \mathsf{B}_1\mathsf{Q}_3\boldsymbol{\beta}\cdot\mathsf{Q}_3\boldsymbol{\gamma} - 2\mathbf{d}_2[\mathsf{Q}_3\boldsymbol{\beta}]\cdot\tilde{\boldsymbol{\Lambda}} \\ \mathsf{B}_2\mathsf{Q}_3\boldsymbol{\beta}\cdot\mathsf{Q}_3\boldsymbol{\gamma} + 2\mathbf{d}_1[\mathsf{Q}_3\boldsymbol{\beta}]\cdot\tilde{\boldsymbol{\Lambda}} \\ \mathsf{B}_3\mathsf{Q}_3\boldsymbol{\beta}\cdot\mathsf{Q}_3\boldsymbol{\gamma} \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\epsilon) & \sin(\epsilon) & 0 \\ -\sin(\epsilon) & \cos(\epsilon) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\gamma} - 2\mathbf{d}_2[\boldsymbol{\beta}]\cdot\tilde{\boldsymbol{\Lambda}} \\ \mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\gamma} + 2\mathbf{d}_1[\boldsymbol{\beta}]\cdot\tilde{\boldsymbol{\Lambda}} \\ \mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\gamma} \end{pmatrix}, \tag{6.3.11}$$

and

$$\begin{pmatrix} \mathsf{B}_1\mathsf{Q}_3\boldsymbol{\beta}\cdot\mathsf{Q}_3\boldsymbol{\beta}_s \\ \mathsf{B}_2\mathsf{Q}_3\boldsymbol{\beta}\cdot\mathsf{Q}_3\boldsymbol{\beta}_s \\ \mathsf{B}_3\mathsf{Q}_3\boldsymbol{\beta}\cdot\mathsf{Q}_3\boldsymbol{\beta}_s \end{pmatrix} = \begin{pmatrix} \cos(\epsilon) & \sin(\epsilon) & 0 \\ -\sin(\epsilon) & \cos(\epsilon) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathsf{B}_1\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s \\ \mathsf{B}_2\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s \\ \mathsf{B}_3\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s \end{pmatrix}, \tag{6.3.12}$$

so the pre-Hamiltonian M (6.2.18) remains invariant. The striction $\tilde{\boldsymbol{\Lambda}}$, which is defined in terms of M, therefore remains invariant also, as does the Hamiltonian.

We denote the total angular momentum about the origin by

$$\boldsymbol{\Omega} \equiv \int_{-\infty}^{\infty} \{\boldsymbol{\pi} + \mathbf{r}\times\mathbf{p}\}\ ds \tag{6.3.13}$$

and let $\Omega_k^o \equiv \boldsymbol{\Omega}\cdot\mathbf{e}_k$. In terms of the Hamiltonian variables

$$\begin{aligned} \boldsymbol{\Omega}^o &= \int_{-\infty}^{\infty} \left\{\tfrac{1}{2}\mathsf{B}^o[\boldsymbol{\beta}]\left(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right) + (\mathbf{h} + s\mathbf{e}_1)\times(\mathbf{g} + \tilde{\boldsymbol{\Lambda}}_s)\right\}\ ds \\ &= \int_{-\infty}^{\infty} \left\{\tfrac{1}{2}\mathsf{B}^o[\boldsymbol{\beta}]\boldsymbol{\gamma} + \mathbf{d}_3\times\tilde{\boldsymbol{\Lambda}} + (\mathbf{h} + s\mathbf{e}_1)\times(\mathbf{g} + \tilde{\boldsymbol{\Lambda}}_s)\right\}\ ds, \end{aligned} \tag{6.3.14}$$

which may be rewritten through an integration by parts and the inextensibility condition as

$$\begin{aligned} \boldsymbol{\Omega}^o &= \int_{-\infty}^{\infty} \left\{\tfrac{1}{2}\mathsf{B}^o[\boldsymbol{\beta}]\boldsymbol{\gamma} + (\mathbf{h} + s\mathbf{e}_1)\times\mathbf{g} + (\mathbf{d}_3 - \mathbf{e}_1 - \mathbf{h}_s)\times\tilde{\boldsymbol{\Lambda}}\right\}\ ds \\ &= \int_{-\infty}^{\infty} \left\{\tfrac{1}{2}\mathsf{B}^o[\boldsymbol{\beta}]\boldsymbol{\gamma} + (\mathbf{h} + s\mathbf{e}_1)\times\mathbf{g}\right\}\ ds, \end{aligned} \tag{6.3.15}$$

so

$$\Omega_k^o = \int_{-\infty}^{\infty} \left\{ \tfrac{1}{2} \mathsf{B}_k^o \boldsymbol{\beta} \cdot \boldsymbol{\gamma} + [(\mathbf{h} + s\mathbf{e}_1) \times \mathbf{g}] \cdot \mathbf{e}_k \right\} ds. \tag{6.3.16}$$

The first variation of Ω_k^o is therefore

$$\frac{\delta \Omega_k^o}{\delta \mathbf{z}} = \left(-\mathbf{e}_k \times \mathbf{g},\ -\tfrac{1}{2} \mathsf{B}_k^o \boldsymbol{\gamma},\ \mathbf{e}_k \times (\mathbf{h} + s\mathbf{e}_1),\ \tfrac{1}{2} \mathsf{B}_k^o \boldsymbol{\beta} \right). \tag{6.3.17}$$

The associated infinitesimal generator is

$$\mathbf{J} \frac{\delta \Omega_k^o}{\delta \mathbf{z}} = \left(\mathbf{e}_k \times (\mathbf{h} + s\mathbf{e}_1),\ \tfrac{1}{2} \mathsf{B}_k^o \boldsymbol{\beta},\ \mathbf{e}_k \times \mathbf{g},\ \tfrac{1}{2} \mathsf{B}_k^o \boldsymbol{\gamma} \right), \tag{6.3.18}$$

which generates the transformation group $(\tilde{\mathsf{Q}}_k^o\,(\mathbf{h}+s\mathbf{e}_1), \mathsf{Q}_k^o \boldsymbol{\beta},\ \tilde{\mathsf{Q}}_k^o\, \mathbf{g},\ \mathsf{Q}_k^o \boldsymbol{\gamma})$, where $\tilde{\mathsf{Q}}_k^o$ is the rotation in $\mathbf{R}^3$ associated with the transformation Q_k^o which was described in section 2.6.2. The invariance property (2.6.2.54) of the B_k matrices is essential to the demonstration of this symmetry for the energy H. Under this transformation group the striction becomes $\mathsf{Q}_k^o\, \tilde{\boldsymbol{\Lambda}}$.

It is apparent from equation (6.2.40) that the three components of the total linear impetus

$$G_k \equiv \int_{-\infty}^{\infty} \mathbf{g} \cdot \mathbf{e}_k \, ds, \qquad k = 1, 2, 3 \tag{6.3.19}$$

are invariants of motion. Because the transverse forces vanish at the ends of the rod, it follows that the transverse components of linear momentum

$$P_2^o \equiv \int_{-\infty}^{\infty} \rho \mathbf{h}_t \cdot \mathbf{e}_2 \, ds \tag{6.3.20}$$

and

$$P_3^o \equiv \int_{-\infty}^{\infty} \rho \mathbf{h}_t \cdot \mathbf{e}_3 \, ds \tag{6.3.21}$$

are also conserved, because

$$\begin{aligned} P_2^o &= \int_{-\infty}^{\infty} (\mathbf{g} + \tilde{\boldsymbol{\Lambda}}_s) \cdot \mathbf{e}_2 \, ds \\ &= G_2 + \tilde{\boldsymbol{\Lambda}} \cdot \mathbf{e}_2 |_{-\infty}^{\infty} = G_2, \end{aligned} \tag{6.3.22}$$

and

$$P_3^o = G_3 + \tilde{\boldsymbol{\Lambda}} \cdot \mathbf{e}_3 |_{-\infty}^{\infty} = G_3. \tag{6.3.23}$$

However, the longitudinal component of linear momentum is not conserved in general, because the forces at $s = -\infty$ and $s = \infty$ may not be equal. The first variations of the invariants G_k are

$$\frac{\delta G_k}{\delta \mathbf{z}} = (\mathbf{0}, \mathbf{0}, \mathbf{e}_k, \mathbf{0}) \qquad \text{for } k = 2, 3, \tag{6.3.24}$$

which correspond to the generators

$$\mathbf{J} \frac{\delta G_k}{\delta \mathbf{z}} = (\mathbf{e}_k, \mathbf{0}, \mathbf{0}, \mathbf{0}) \tag{6.3.25}$$

of the translational symmetry groups $(\mathbf{h} + \epsilon \mathbf{e}_k, 0, 0, 0)$. The striction is unchanged under these symmetries.

All of the above invariants of motion have a physical interpretation in terms of Lagrangian variables. There are other invariants which arise in this Euler parameter formulation that have no such interpretation. In particular, as a consequence of the pointwise constraint $\boldsymbol{\beta}(s) \cdot \boldsymbol{\beta}(s) = 1$ on the Euler parameters there is an infinite set of invariants described by

$$N(\nu) \equiv \tfrac{1}{2} \textstyle\int_{-\infty}^{\infty} \nu(s)[\boldsymbol{\beta}(s) \cdot \boldsymbol{\beta}(s) - 1]\, ds, \tag{6.3.26}$$

where ν is an arbitrary smooth function. It was noted in the previous section that for each function ν, the Hamiltonian is invariant under to the transformation $\boldsymbol{\gamma} \to \boldsymbol{\gamma} + \nu\boldsymbol{\beta}$, whose infinitesimal generator is

$$\mathbf{J}\frac{\delta N}{\delta \mathbf{z}} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, -\nu\boldsymbol{\beta}). \tag{6.3.27}$$

This symmetry is directly analogous to that for the spherical pendulum observed in section 2.3

This Hamiltonian system has another infinite set of invariants, namely

$$Z(\zeta) \equiv \int_{-\infty}^{\infty} \zeta(s)\, \boldsymbol{\beta}(s) \cdot \boldsymbol{\gamma}(s)\, ds \tag{6.3.28}$$

where ζ is an arbitrary smooth function. The invariance of Z follows directly from equations (6.2.41) and (6.2.52) which imply

$$\boldsymbol{\beta}_t \cdot \boldsymbol{\gamma} + \boldsymbol{\gamma}_t \cdot \boldsymbol{\beta} = 0. \tag{6.3.29}$$

For equation (6.3.29) to hold it is essential that the potential energy be homogeneous with degree zero in the Euler parameters. In fact one of our primary reasons for homogenizing the kinetic and potential energies was to obtain this family of invariants. The symmetry associated with Z is given by the group of transformations $\boldsymbol{\beta}^\epsilon \equiv (1+\epsilon)\boldsymbol{\beta}$ and $\boldsymbol{\gamma}^\epsilon \equiv (1+\epsilon)^{-1}\boldsymbol{\gamma}$. To see that this is truly a symmetry of the Hamiltonian system, first observe that the products $\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\gamma}$ and $\mathsf{B}[\boldsymbol{\beta}]\mathsf{D}_3[\boldsymbol{\beta}]$ are unchanged under this transformation, and so the striction given by equation (6.2.19) is also unchanged this transformation. Therefore the Hamiltonian H is invariant under the transformation $\boldsymbol{\beta}^\epsilon, \boldsymbol{\gamma}^\epsilon$ as well. In order to simplify the analysis we assume henceforth that the initial data satisfy the orthogonality condition

$$\boldsymbol{\gamma}(s) \cdot \boldsymbol{\beta}(s) = 0. \tag{6.3.30}$$

In a Hamiltonian system, it is often useful to determine whether the invariants of motion commute. Two functions F and G are said to commute, or to be in involution, if the Poisson bracket $\{F, G\}$ vanishes, where

$$\{F, G\} \equiv \int_{-\infty}^{\infty} \frac{\delta F}{\delta \mathbf{z}} \cdot \mathbf{J} \, \frac{\delta G}{\delta \mathbf{z}} \, ds \tag{6.3.31}$$

and the skew symmetric operator $\mathbf{J}$ is given by equation (6.2.54). The Poisson bracket is a bilinear and skew-symmetric, and satisfies the Jacobi identity

$$\{\{F, G\}, K\} + \{\{G, K\}, F\} + \{\{K, F\}, G\} = 0. \tag{6.3.32}$$

In the case $K = H$, the Jacobi identity implies that the Poisson bracket of two invariants of motion F and G must again be an invariant of motion. In finite dimensional Hamiltonian systems the existence of k independent invariants in involution implies that the dimension of the system can in principle be reduced by $2k$. In an infinite dimensional system the significance of invariants in involution is less clear. For the purposes of this section it is most important to note a geometrical interpretation: two functions F and G commute if F is invariant under the transformation group generated by $\mathbf{J} \, \delta G / \delta \mathbf{z}$, and conversely G is invariant under the transformation group generated by $\mathbf{J} \, \delta F / \delta \mathbf{z}$. In particular, all invariants of motion commute with the Hamiltonian H.

Rather than explicitly calculate the Poisson brackets for all 55 pairings of the 11 invariants (C, I, A, Ω_1^o, Ω_2^o, Ω_3^o, G_1, G_2, G_3, N, and Z) that we have discussed in this section, I will summarize the results. The linear impulse I, the invariant C associated with inextensibility and the isotropic angular momentum A commute with all of the invariants. (That A commutes with all three components of the total angular momentum follows from the symmetry of the matrices $\mathsf{B}_3 \, \mathsf{B}_k^o = -\mathsf{D}_3^k$.)

However, the components of the total angular momentum $\boldsymbol{\Omega}^o$ do not commute with each other. A detailed calculation shows that

$$\{\Omega_j^o, \Omega_k^o\} = \epsilon_{ijk} \Omega_i^o. \tag{6.3.33}$$

In the particular case of the solitary wave $\Omega_3^o \neq 0$, so the pairs (Ω_2^o, Ω_3^o) does not commute. In addition the components of $\boldsymbol{\Omega}$ do not commute with the transverse components of linear impetus. Instead

$$\{\Omega_j^o, G_k\} = \epsilon_{ijk} G_i. \tag{6.3.34}$$

The transverse components of linear momentum do however commute with all of the other invariants.

Finally the invariants N and Z, associated respectively with the constancy of $|\boldsymbol{\beta}|$ and $\boldsymbol{\beta} \cdot \boldsymbol{\gamma}$, commute with all of the invariants of motion *except* each other, because

$$\{N(\nu), Z(\zeta)\} = \int_{-\infty}^{\infty} \nu(s) \zeta(s) |\boldsymbol{\beta}(s)|^2 \, ds \tag{6.3.35}$$

does not always vanish.

In closing we remark that, if as before we choose the linear impulse $\mathbf{g}$ to be constant in s as well as t, then we can go to a reduced Hamiltonian dynamical system involving only the variables $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. In the reduced system the associated linear impulse

$$I_{\text{red}} \equiv \int_{-\infty}^{\infty} \boldsymbol{\gamma} \cdot \boldsymbol{\beta}_s \, ds \tag{6.3.36}$$

is again a conserved quantity.

6.4 A Variational Characterization of Solitary Waves

In this section we show that each planar solitary wave profile is a critical point of the invariant

$$F = H + cI + \tfrac{1}{2} \textstyle\int_{-\infty}^{\infty} \mu(s) |\boldsymbol{\beta}(s)|^2 \, ds, \tag{6.4.1}$$

for an appropriate choice of the function μ. We begin by writing the general equations for a critical point, but then specialize to the planar case. We express the variations with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in terms of their components relative to $\boldsymbol{\beta}$, $\mathsf{B}_1\boldsymbol{\beta}$, $\mathsf{B}_2\boldsymbol{\beta}$, and $\mathsf{B}_3\boldsymbol{\beta}$ by exploiting equations (6.2.41), (6.2.43) – (6.2.46) and (6.2.48) – (6.2.51). The extremals of F are given by

$$\mathbf{0} = \frac{\delta F}{\delta \mathbf{h}} = -c\bar{\mathbf{g}}', \tag{6.4.2}$$

$$\mathbf{0} = \frac{\delta F}{\delta \mathbf{g}} = \rho^{-1}\left(\bar{\mathbf{g}} + \bar{\boldsymbol{\Lambda}}'\right) + c\bar{\mathbf{h}}', \tag{6.4.3}$$

$$\begin{aligned} 0 = \frac{\delta F}{\delta \boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}} \;=\;& \tfrac{1}{4} I_1^{-1} \left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} - 2\bar{\mathbf{d}}_2 \cdot \tilde{\boldsymbol{\Lambda}}\right)\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}}\right) \\ &+ \tfrac{1}{4} I_2^{-1} \left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} + 2\bar{\mathbf{d}}_1 \cdot \tilde{\boldsymbol{\Lambda}}\right)\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}}\right) \\ &+ \tfrac{1}{4} I_3^{-1} \left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}}\right)\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}}\right) \\ &- c\bar{\boldsymbol{\gamma}}' \cdot \bar{\boldsymbol{\beta}} + \mu|\bar{\boldsymbol{\beta}}|^2, \end{aligned} \tag{6.4.4}$$

$$
\begin{aligned}
0 = \frac{\delta F}{\delta \boldsymbol{\beta}} \cdot \mathsf{B}_1 \bar{\boldsymbol{\beta}} \quad = \quad & -\tfrac{1}{4} I_1^{-1} \left(\mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} - 2\bar{\mathbf{d}}_2 \cdot \tilde{\boldsymbol{\Lambda}} \right) \left(\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} + 4\mathbf{d}_3 \cdot \tilde{\boldsymbol{\Lambda}} \right) \\
& + \tfrac{1}{4} I_2^{-1} \left(\mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} + 2\bar{\mathbf{d}}_1 \cdot \tilde{\boldsymbol{\Lambda}} \right) \left(\mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} \right) \\
& - \tfrac{1}{4} I_3^{-1} \left(\mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} \right) \left(\mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} \right) \\
& - c \bar{\boldsymbol{\gamma}}' \cdot \mathsf{B}_1 \bar{\boldsymbol{\beta}} \\
& - \frac{4}{|\bar{\boldsymbol{\beta}}|^2} K_1 \left(\mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'' \right) \\
& + \frac{8}{|\bar{\boldsymbol{\beta}}|^4} (K_2 - K_3) \left(\mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right) \left(\mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right) \\
& + \frac{8}{|\bar{\boldsymbol{\beta}}|^4} K_1 \left(\mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right) \left(\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right), \qquad (6.4.5)
\end{aligned}
$$

$$
\begin{aligned}
0 = \frac{\delta F}{\delta \boldsymbol{\beta}} \cdot \mathsf{B}_2 \bar{\boldsymbol{\beta}} \quad = \quad & -\tfrac{1}{4} I_1^{-1} \left(\mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} - 2\bar{\mathbf{d}}_2 \cdot \tilde{\boldsymbol{\Lambda}} \right) \left(\mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} \right) \\
& - \tfrac{1}{4} I_2^{-1} \left(\mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} + 2\mathbf{d}_1 \cdot \tilde{\boldsymbol{\Lambda}} \right) \left(\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} + 4\bar{\mathbf{d}}_3 \cdot \tilde{\boldsymbol{\Lambda}} \right) \\
& + \tfrac{1}{4} I_3^{-1} \left(\mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} \right) \left(\mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} \right) \\
& - c \bar{\boldsymbol{\gamma}}' \cdot \mathsf{B}_2 \bar{\boldsymbol{\beta}} \\
& - \frac{4}{|\bar{\boldsymbol{\beta}}|^2} K_2 \left(\mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'' \right) \\
& + \frac{8}{|\bar{\boldsymbol{\beta}}|^4} (K_3 - K_1) \left(\mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right) \left(\mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right) \\
& + \frac{8}{|\bar{\boldsymbol{\beta}}|^4} K_2 \left(\mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right) \left(\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right), \qquad (6.4.6)
\end{aligned}
$$

$$
\begin{aligned}
0 = \frac{\delta F}{\delta \boldsymbol{\beta}} \cdot \mathsf{B}_3 \bar{\boldsymbol{\beta}} \quad = \quad & \tfrac{1}{4} I_1^{-1} \left(\mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} - 2\bar{\mathbf{d}}_2 \cdot \tilde{\boldsymbol{\Lambda}} \right) \left(\mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} + 4\bar{\mathbf{d}}_1 \cdot \tilde{\boldsymbol{\Lambda}} \right) \\
& - \tfrac{1}{4} I_2^{-1} \left(\mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} + 2\mathbf{d}_1 \cdot \tilde{\boldsymbol{\Lambda}} \right) \left(\mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} - 4\bar{\mathbf{d}}_2 \cdot \tilde{\boldsymbol{\Lambda}} \right) \\
& - \tfrac{1}{4} I_3^{-1} \left(\mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} \right) \left(\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} \right) \\
& - c \bar{\boldsymbol{\gamma}}' \cdot \mathsf{B}_3 \bar{\boldsymbol{\beta}} \\
& - \frac{4}{|\bar{\boldsymbol{\beta}}|^2} K_3 \left(\mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'' \right) \\
& + \frac{8}{|\bar{\boldsymbol{\beta}}|^4} (K_1 - K_2) \left(\mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right) \left(\mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right) \\
& + \frac{8}{|\bar{\boldsymbol{\beta}}|^4} K_3 \left(\mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right) \left(\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' \right), \qquad (6.4.7)
\end{aligned}
$$

$$
\begin{aligned}
0 = \frac{\delta F}{\delta \boldsymbol{\gamma}} \cdot \bar{\boldsymbol{\beta}} \quad &= \quad c \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}', && (6.4.8) \\
0 = \frac{\delta F}{\delta \boldsymbol{\gamma}} \cdot \mathsf{B}_1 \bar{\boldsymbol{\beta}} \quad &= \quad \tfrac{1}{4} I_1^{-1} \left(\mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} - 2\bar{\mathbf{d}}_2 \cdot \bar{\boldsymbol{\Lambda}} \right) |\bar{\boldsymbol{\beta}}|^2 + c \mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}', && (6.4.9) \\
0 = \frac{\delta F}{\delta \boldsymbol{\gamma}} \cdot \mathsf{B}_2 \bar{\boldsymbol{\beta}} \quad &= \quad \tfrac{1}{4} I_2^{-1} \left(\mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} + 2\bar{\mathbf{d}}_1 \cdot \bar{\boldsymbol{\Lambda}} \right) |\bar{\boldsymbol{\beta}}|^2 + c \mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}', && (6.4.10)
\end{aligned}
$$

$$0 = \frac{\delta F}{\delta \boldsymbol{\gamma}} \cdot \mathsf{B}_3\bar{\boldsymbol{\beta}} \quad = \quad \tfrac{1}{4} I_3^{-1} \left(\mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\gamma}} \right) |\bar{\boldsymbol{\beta}}|^2 + c \mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'. \tag{6.4.11}$$

Equations (6.4.8) - (6.4.11) can be expressed collectively as

$$\mathbf{0} = \frac{\delta F}{\delta \boldsymbol{\gamma}} = \tfrac{1}{4} \mathsf{B}[\boldsymbol{\beta}]^T \mathsf{I}^{-1} I_3^{-1} \mathsf{B}[\boldsymbol{\beta}] \left(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}] \tilde{\boldsymbol{\Lambda}} \right) + c \bar{\boldsymbol{\beta}}'. \tag{6.4.12}$$

Equation (6.4.2) implies $\mathbf{g}$ is constant, and one can solve the second equation (6.4.3) for $\bar{\boldsymbol{\Lambda}}'$ to obtain

$$\bar{\boldsymbol{\Lambda}}' = -\bar{\mathbf{g}} - c\rho \bar{\mathbf{h}}'. \tag{6.4.13}$$

In addition to the above first-order conditions we have the equation governing $\bar{\boldsymbol{\Lambda}}$, namely

$$\mathbf{0} = -\rho^{-1} (\bar{\mathbf{g}} + \bar{\boldsymbol{\Lambda}}')' + \tfrac{1}{4} \mathsf{D}_3[\bar{\boldsymbol{\beta}}]^T \mathsf{B}[\bar{\boldsymbol{\beta}}]^T \, \mathsf{I}^{-1} \mathsf{B}[\bar{\boldsymbol{\beta}}] (\bar{\boldsymbol{\gamma}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}] \bar{\boldsymbol{\Lambda}}). \tag{6.4.14}$$

From equation (6.4.2) it is apparent that $\bar{\mathbf{g}}$ is a constant. By combining equations (6.4.3), (6.4.9) – (6.4.11) and (A.4.9) we obtain

$$c \bar{\mathbf{h}}'' - c \mathsf{D}_3[\bar{\boldsymbol{\beta}}]^T \bar{\boldsymbol{\phi}}' = c(\bar{\mathbf{h}}' - \bar{\mathbf{d}}_3)' = 0, \tag{6.4.15}$$

which together with boundary conditions implies

$$\bar{\mathbf{h}}' = \bar{\mathbf{d}}_3 - \mathbf{e}_1, \tag{6.4.16}$$

Thus both $\bar{\mathbf{h}}$ and $\bar{\boldsymbol{\Lambda}}$ will be determined when $\bar{\mathbf{g}}$ and $\bar{\boldsymbol{\beta}}$ are found.

Equations (6.4.9) - (6.4.11) can be solved to obtain

$$\begin{aligned}
\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_1 \bar{\boldsymbol{\beta}} &= -4cI_1 \frac{1}{|\bar{\boldsymbol{\beta}}|^2} \mathsf{B}_1 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' + 2\bar{\mathbf{d}}_2 \cdot \bar{\boldsymbol{\Lambda}}, && (6.4.17)\\
\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_2 \bar{\boldsymbol{\beta}} &= -4cI_2 \frac{1}{|\bar{\boldsymbol{\beta}}|^2} \mathsf{B}_2 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}' - 2\bar{\mathbf{d}}_1 \cdot \bar{\boldsymbol{\Lambda}}, && (6.4.18)\\
\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_3 \bar{\boldsymbol{\beta}} &= -4cI_3 \frac{1}{|\bar{\boldsymbol{\beta}}|^2} \mathsf{B}_3 \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'. && (6.4.19)
\end{aligned}$$

Equations (6.4.8) and (6.4.17) – (6.4.19) can next be exploited to rewrite equations (6.4.4) – (6.4.7) more simply as

$$0 \quad = \quad -c\bar{\boldsymbol{\gamma}}' \cdot \bar{\boldsymbol{\beta}} - c \left(\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_j \bar{\boldsymbol{\beta}} \right) \frac{\mathsf{B}_j \bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'}{|\bar{\boldsymbol{\beta}}|^2} + \mu |\bar{\boldsymbol{\beta}}|^2, \tag{6.4.20}$$

$$
\begin{aligned}
0 \quad = \quad & -c\bar{\boldsymbol{\gamma}}' \cdot \mathsf{B}_1\bar{\boldsymbol{\beta}} + c\left(\bar{\boldsymbol{\gamma}} \cdot \bar{\boldsymbol{\beta}}\right) \frac{\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& -4K_1 \frac{\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}''\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& -4c^2(I_2 - I_3) \frac{\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \frac{\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& +8(K_2 - K_3) \frac{\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \frac{\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& +4c\tilde{\boldsymbol{\Lambda}} \cdot \bar{\mathbf{d}}_3 \frac{\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} - 2c\tilde{\boldsymbol{\Lambda}} \cdot \bar{\mathbf{d}}_1 \frac{\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2},
\end{aligned}
\tag{6.4.21}
$$

$$
\begin{aligned}
0 \quad = \quad & -c\bar{\boldsymbol{\gamma}}' \cdot \mathsf{B}_2\bar{\boldsymbol{\beta}} + c\left(\bar{\boldsymbol{\gamma}} \cdot \bar{\boldsymbol{\beta}}\right) \frac{\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& -4K_2 \frac{\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}''\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& -4c^2(I_3 - I_1) \frac{\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \frac{\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& +8(K_3 - K_1) \frac{\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \frac{\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& +4c\tilde{\boldsymbol{\Lambda}} \cdot \bar{\mathbf{d}}_3 \frac{\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} - 2c\tilde{\boldsymbol{\Lambda}} \cdot \bar{\mathbf{d}}_2 \frac{\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2},
\end{aligned}
\tag{6.4.22}
$$

and

$$
\begin{aligned}
0 \quad = \quad & -c\bar{\boldsymbol{\gamma}}' \cdot \mathsf{B}_3\bar{\boldsymbol{\beta}} + c\left(\bar{\boldsymbol{\gamma}} \cdot \bar{\boldsymbol{\beta}}\right) \frac{\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& -4K_3 \frac{\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}''\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& -4c^2(I_1 - I_2) \frac{\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \frac{\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \\
& +8(K_1 - K_2) \frac{\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} \frac{(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}')}{|\bar{\boldsymbol{\beta}}|^2} \\
& -4c\tilde{\boldsymbol{\Lambda}} \cdot \bar{\mathbf{d}}_1 \frac{\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2} - 4c\tilde{\boldsymbol{\Lambda}} \cdot \bar{\mathbf{d}}_2 \frac{\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)}{|\bar{\boldsymbol{\beta}}|^2}.
\end{aligned}
\tag{6.4.23}
$$

Equation (6.4.8) implies that $|\bar{\boldsymbol{\beta}}|$ is constant, which by the constraint (2.6.2.1) must be

$$
|\bar{\boldsymbol{\beta}}| = 1. \tag{6.4.24}
$$

It also follows from (6.4.8) that so

$$\bar{\boldsymbol{\beta}}' = \left(\mathsf{B}_k\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right) \mathsf{B}_k\bar{\boldsymbol{\beta}}. \tag{6.4.25}$$

Equation (6.4.20), (6.4.24) and (6.4.25) together yield the expression for the multiplier μ, namely

$$\mu = \bar{\boldsymbol{\gamma}} \cdot \bar{\boldsymbol{\beta}}' + \bar{\boldsymbol{\gamma}}' \cdot \bar{\boldsymbol{\beta}} = \left(\bar{\boldsymbol{\gamma}} \cdot \bar{\boldsymbol{\beta}}\right)'. \tag{6.4.26}$$

Equations (6.4.17) - (6.4.25) can be manipulated to produce a system of differential equations governing $\bar{\boldsymbol{\beta}}$. Through a long sequence of calculations we will derive an expression for $(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}')'$, from which the expressions for $(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}')'$ and $(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}')'$ can be obtained in a similar manner. We first note that equation (6.4.25) implies

$$\begin{aligned} c\left(\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_1\bar{\boldsymbol{\beta}}\right)' &= c\bar{\boldsymbol{\gamma}}' \cdot \mathsf{B}_1\bar{\boldsymbol{\beta}} + c\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_1\bar{\boldsymbol{\beta}}' \\ &= c\bar{\boldsymbol{\gamma}}' \cdot \mathsf{B}_1\bar{\boldsymbol{\beta}} - c\left(\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_1\bar{\boldsymbol{\beta}}\right)\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right) \\ &\quad -c\left(\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_3\bar{\boldsymbol{\beta}}\right)\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right) + c(\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_2\bar{\boldsymbol{\beta}})\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right), \end{aligned} \tag{6.4.27}$$

into which we can introduce equations (6.4.18), (6.4.19) and (6.4.21) to obtain

$$\begin{aligned} c(\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_1\bar{\boldsymbol{\beta}})' &= -4K_1\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}''\right) \\ &\quad +8\left[\left(K_2 - c^2I_2\right) - \left(K_3 - c^2I_3\right)\right]\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right) \\ &\quad +4c\tilde{\boldsymbol{\Lambda}} \cdot \bar{\mathbf{d}}_3\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right) - 4c\tilde{\boldsymbol{\Lambda}} \cdot \bar{\mathbf{d}}_1\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right). \end{aligned} \tag{6.4.28}$$

Alternatively, one can express $c\left(\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_1\bar{\boldsymbol{\beta}}\right)'$ by differentiating equation (6.4.17) and multiplying by c, which produces

$$c\left(\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_1\bar{\boldsymbol{\beta}}\right)' = -4c^2I_1\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}''\right) + 2c\bar{\mathbf{d}}_2 \cdot \bar{\boldsymbol{\Lambda}}' + 2c\left(\bar{\mathbf{d}}_2\right)' \cdot \bar{\boldsymbol{\Lambda}}. \tag{6.4.29}$$

Equation (6.4.13) and the identity

$$\left(\bar{\mathbf{d}}_2\right)' = 2\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)\bar{\mathbf{d}}_3 - 2\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)\bar{\mathbf{d}}_1, \tag{6.4.30}$$

which follows from the kinematic equation (2.2.2) and the expression (2.6.2.15) for the strain u, can be used to rewrite equation (6.4.29) as

$$\begin{aligned} c(\bar{\boldsymbol{\gamma}} \cdot \mathsf{B}_1\bar{\boldsymbol{\beta}})' &= -4c^2I_1\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}''\right) - 2c\bar{\mathbf{d}}_2 \cdot (\bar{\mathbf{g}} - c\rho\mathbf{e}_1) \\ &\quad +4c\tilde{\boldsymbol{\Lambda}} \cdot \bar{\mathbf{d}}_3\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right) - 4c\tilde{\boldsymbol{\Lambda}} \cdot \bar{\mathbf{d}}_1\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right). \end{aligned} \tag{6.4.31}$$

The comparison of equations (6.4.28) and (6.4.31) yields the differential equation

$$\begin{aligned} 2\left(K_1 - c^2I_1\right)\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'' &= 4\left[\left(K_2 - c^2I_2\right) - \left(K_3 - c^2I_3\right)\right]\left(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)\left(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right) \\ &\quad -c\bar{\mathbf{d}}_2 \cdot (\bar{\mathbf{g}} - c\rho\mathbf{e}_1). \end{aligned} \tag{6.4.32}$$

The analogous equations for $\mathsf{B}_2\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'$ and $\mathsf{B}_3\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'$ are

$$\begin{aligned} 2\left(K_2 - c^2 I_2\right)\mathsf{B}_2\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'' &= 4\left[\left(K_3 - c^2 I_3\right) - \left(K_1 - c^2 I_1\right)\right]\left(\mathsf{B}_3\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right)\left(\mathsf{B}_1\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right) \\ &\quad + c\bar{\mathbf{d}}_1\cdot\left(\bar{\mathbf{g}} - c\rho\mathbf{e}_1\right) \end{aligned} \tag{6.4.33}$$

and

$$2\left(K_3 - c^2 I_3\right)\mathsf{B}_3\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'' = 4\left[\left(K_1 - c^2 I_1\right) - \left(K_2 - c^2 I_2\right)\right]\left(\mathsf{B}_1\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right)\left(\mathsf{B}_2\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right). \tag{6.4.34}$$

Note that if we now multiply equations (6.4.32) - (6.4.34) by $\left(\mathsf{B}_1\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right)$, $\left(\mathsf{B}_2\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right)$, and $\left(\mathsf{B}_3\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right)$ respectively and sum, the cross terms cancel and we obtain

$$\begin{aligned} 2\left(K_j - c^2 I_j\right)\left(\mathsf{B}_j\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right)\left(\mathsf{B}_j\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}''\right) &= -c\left(\bar{\mathbf{g}} - c\rho\mathbf{e}_1\right)\left[\left(\mathsf{B}_1\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right)\bar{\mathbf{d}}_2 - \left(\mathsf{B}_2\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right)\bar{\mathbf{d}}_1\right] \\ &= -\tfrac{1}{2}c\left(\bar{\mathbf{g}} - c\rho\mathbf{e}_1\right)\mathsf{D}_3[\bar{\boldsymbol{\beta}}]^T\bar{\boldsymbol{\beta}}' \\ &= -\tfrac{1}{2}c\left(\bar{\mathbf{g}} - c\rho\mathbf{e}_1\right)\left(\bar{\mathbf{d}}_3\right)', \end{aligned} \tag{6.4.35}$$

where we have exploited the expression (2.6.2.21) for $\mathsf{D}_3[\boldsymbol{\beta}]$ to simplify the right hand side. The boundary conditions imply that

$$\bar{\mathbf{g}} = \bar{g}_1\mathbf{e}_1 \tag{6.4.36}$$

for some $\bar{g}_1$. Therefore equation (6.4.35) takes the form of an exact derivative with the first integral

$$\left(K_j - c^2 I_j\right)\left(2\mathsf{B}_j\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'\right)^2 = c\left(\bar{g}_1 - c\rho\right)\left(1 - \mathbf{e}_1\cdot\bar{\mathbf{d}}_3\right). \tag{6.4.37}$$

In the case of planar dynamics, the Euler parameters were found in equation (2.6.2.14) to be

$$\bar{\boldsymbol{\beta}} = \tfrac{1}{2}\left(-\sin\left(\tfrac{1}{2}\bar{\phi}\right), \cos\left(\tfrac{1}{2}\bar{\phi}\right), \sin\left(\tfrac{1}{2}\bar{\phi}\right), \cos\left(\tfrac{1}{2}\bar{\phi}\right)\right). \tag{6.4.38}$$

Consequently

$$\bar{\boldsymbol{\beta}}' = \tfrac{1}{\sqrt{2}}\left(-\cos\left(\tfrac{1}{2}\bar{\phi}\right), -\sin\left(\tfrac{1}{2}\bar{\phi}\right), \cos\left(\tfrac{1}{2}\bar{\phi}\right), -\sin\left(\tfrac{1}{2}\bar{\phi}\right)\right)\tfrac{1}{2}\bar{\phi}' \tag{6.4.39}$$

$$= -\tfrac{1}{2}\bar{\phi}'\mathsf{B}_1\bar{\boldsymbol{\beta}}, \tag{6.4.40}$$

which implies $\mathsf{B}_2\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}' = 0 = \mathsf{B}_3\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'$. The system (6.4.32) - (6.4.34) then reduces to the single differential equation

$$(K_1 - c^2 I_1)\mathsf{B}_1\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'' = c(\bar{g}_1 - c\rho)\left(\bar{\beta}_1\bar{\beta}_2 - \bar{\beta}_3\bar{\beta}_4\right). \tag{6.4.41}$$

Introducing equation (6.4.38) into (6.4.41) reproduces equation (5.5.19) from section 5.5. Using the calculations from that section, we then conclude that $\bar{\mathbf{g}}$ is given by equation (5.5.23),

$$\sin(\tfrac{1}{2}\bar{\phi}) = -\tanh(s/a) \tag{6.4.42}$$

and

$$\cos(\tfrac{1}{2}\bar{\phi}) = \operatorname{sech}(s/a), \tag{6.4.43}$$

up to translation in s, as was already seen in equations (5.5.28) and (5.5.29). As a consequence of equations (6.4.38) - (6.4.40),

$$\begin{aligned}
\mathsf{B}_1\bar{\boldsymbol{\beta}} &= \tfrac{1}{\sqrt{2}}\Big(\ \ \cos\left(\tfrac{1}{2}\phi\right), \quad \sin\left(\tfrac{1}{2}\phi\right), -\cos\left(\tfrac{1}{2}\phi\right), \quad \sin\left(\tfrac{1}{2}\phi\right)\Big), && (6.4.44)\\
\mathsf{B}_2\bar{\boldsymbol{\beta}} &= \tfrac{1}{\sqrt{2}}\Big(-\sin\left(\tfrac{1}{2}\phi\right), \quad \cos\left(\tfrac{1}{2}\phi\right), -\sin\left(\tfrac{1}{2}\phi\right), -\cos\left(\tfrac{1}{2}\phi\right)\Big), && (6.4.45)\\
\mathsf{B}_3\bar{\boldsymbol{\beta}} &= \tfrac{1}{\sqrt{2}}\Big(\ \ \cos\left(\tfrac{1}{2}\phi\right), \quad \sin\left(\tfrac{1}{2}\phi\right), \quad \cos\left(\tfrac{1}{2}\phi\right), -\sin\left(\tfrac{1}{2}\phi\right)\Big) && (6.4.46)
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{B}_1\bar{\boldsymbol{\beta}}' &= \tfrac{1}{2}\bar{\phi}'\bar{\boldsymbol{\beta}}, && (6.4.47)\\
\mathsf{B}_2\bar{\boldsymbol{\beta}}' &= -\tfrac{1}{2}\bar{\phi}'\mathsf{B}_3\bar{\boldsymbol{\beta}}, && (6.4.48)\\
\mathsf{B}_3\bar{\boldsymbol{\beta}}' &= \tfrac{1}{2}\bar{\phi}'\mathsf{B}_2\bar{\boldsymbol{\beta}}. && (6.4.49)
\end{aligned}$$

Equations (6.4.42) and (6.4.43) show that the trigonometric functions of $\bar{\phi}/2$ take a very simple form in this problem, which suggests that Euler parameters are a natural choice of coordinates here.

Finally, it follows from equations (6.4.17) - (6.4.19) that the impetus is given by

$$\begin{aligned}
\bar{\boldsymbol{\gamma}} &= 2cI_1\bar{\phi}'\mathsf{B}_1\bar{\boldsymbol{\beta}} - \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\bar{\boldsymbol{\Lambda}} + \kappa\bar{\boldsymbol{\beta}}\\
&= 2\left(cI_1\bar{\phi}' + \bar{\mathbf{d}}_2\cdot\bar{\boldsymbol{\Lambda}}\right)\mathsf{B}_1\bar{\boldsymbol{\beta}} - 2(\bar{\mathbf{d}}_1\cdot\bar{\boldsymbol{\Lambda}})\mathsf{B}_2\bar{\boldsymbol{\beta}} + \kappa\bar{\boldsymbol{\beta}}, && (6.4.50)
\end{aligned}$$

where $\kappa = \bar{\boldsymbol{\gamma}}\cdot\bar{\boldsymbol{\beta}}$ is an arbitrary scalar function.

6.5 Analysis of the Second Variation for Planar Solitary Waves

We now analyze the second variation of the invariant F_{red} on the planar solitary wave with respect to admissible variations. An admissible variation is defined to be a variation of the state variables which preserves to first order all the known invariants of motion. That is to say, for each invariant of motion the linearization about the solitary wave must vanish. A particular challenge is the choice of variations in the Euler parameters which satisfy the linearization of the normalization constraint (2.6.2.1). Fortunately the algebra of the quaternions provides an elegant means to handle this constraint. In contrast with the results of section 5.6, it is demonstrated here that the planar solitary wave with sufficiently samll wave speed is not a minimizer of the functional F_{red} amongst admissible variations. Rather, for wave speeds less than a critical value $\bar{c}$ in an isotropic rod the second variation of F_{red} has a single negative eigenvalue. We close with some remarks on the implications of our analysis upon the second variation of F_{red} for an isotropic rod under different boundary conditions and for anisotropic rods.

6.5.1 Parametrization of Variations

We begin by presenting a parametrization of the admissible variations $(\delta\mathbf{h}, \delta\boldsymbol{\beta}, \delta\mathbf{g}, \delta\boldsymbol{\gamma})$. The variations $\delta\boldsymbol{\beta}$ in the Euler parameters must satisfy the linearized constraint

$$\boldsymbol{\beta} \cdot \delta\boldsymbol{\beta} = 0. \tag{6.5.1.1}$$

The fact that the rows $\{\boldsymbol{\beta}, \mathsf{B}_1\boldsymbol{\beta}, \mathsf{B}_2\boldsymbol{\beta}, \mathsf{B}_3\boldsymbol{\beta}\}$ of the matrix $\mathsf{R}[\boldsymbol{\beta}]$ (2.6.2.28) form an orthogonal basis[1] for $\mathbf{R}^4$ suggests that we parametrize $\delta\boldsymbol{\beta}$ by the triple of functions $\boldsymbol{\zeta} \equiv (\zeta_1, \zeta_2, \zeta_3)$ such that

$$\delta\boldsymbol{\beta} = \tfrac{1}{2}\mathsf{B}[\boldsymbol{\beta}]^T\boldsymbol{\zeta} = \tfrac{1}{2}\zeta_1\mathsf{B}_1\bar{\boldsymbol{\beta}} + \tfrac{1}{2}\zeta_2\mathsf{B}_2\bar{\boldsymbol{\beta}} + \tfrac{1}{2}\zeta_3\mathsf{B}_3\bar{\boldsymbol{\beta}}. \tag{6.5.1.2}$$

The parametrization (6.5.1.2) is closely associated with the constraint that the directors $\{\mathbf{d}_k\}$ have unit length. Specifically equation (6.5.1.2) can be rewritten as

$$\boldsymbol{\zeta} = 2\mathsf{B}[\boldsymbol{\beta}]\delta\boldsymbol{\beta}, \tag{6.5.1.3}$$

which in direct analogy with equations (2.2.2) and (2.6.2.15) can also be expressed as

$$\delta\mathbf{d}_k = \mathbf{z} \times \mathbf{d}_k, \tag{6.5.1.4}$$

where $\mathbf{z} \equiv \zeta_k\mathbf{d}_k$. Thus the vector $\mathbf{z}$ represents an infinitesimal rotation of the directors. The parameters ζ_j have an especially simple interpretation for the planar solitary wave. Equation (6.5.1.4) is equivalent to the system of three equations

$$\begin{aligned}
\delta\mathbf{d}_1 &= -\zeta_2\mathbf{d}_3 + \zeta_3\mathbf{d}_2, && (6.5.1.5)\\
\delta\mathbf{d}_2 &= \zeta_1\mathbf{d}_3 - \zeta_3\mathbf{d}_1, && (6.5.1.6)\\
\delta\mathbf{d}_3 &= -\zeta_1\mathbf{d}_2 + \zeta_2\mathbf{d}_1. && (6.5.1.7)
\end{aligned}$$

If instead the variations in the directors are computed in terms of Euler angles using equation (2.6.1.1), taking advantage of the facts that $\theta = \pi/2$ and $\psi = 0$ on the planar solitary wave, one obtains

$$\begin{aligned}
\delta\mathbf{d}_1 &= -\delta\theta\mathbf{d}_3 + \delta\psi\mathbf{d}_2, && (6.5.1.8)\\
\delta\mathbf{d}_2 &= -\delta\phi\mathbf{d}_3 - \delta\psi\mathbf{d}_1, && (6.5.1.9)\\
\delta\mathbf{d}_3 &= \delta\phi\mathbf{d}_2 + \delta\theta\mathbf{d}_1. && (6.5.1.10)
\end{aligned}$$

[1] The existence of this orthonormal basis demonstrates that the sphere $\mathbf{S}^3$ is parallelizable, i.e. there exist three linearly independent, nonvanishing vector fields tangent to $\mathbf{S}^3$. It is a remarkable theorem proven by Milnor in 1958 [48] that $\mathbf{S}^1$, $\mathbf{S}^3$, and $\mathbf{S}^7$ are the only spheres which are parallelizable. The parallelizability of these spheres is closely associated with the algebra of the complex numbers, the quaternions and the Cayley numbers in $\mathbf{R}^2$, $\mathbf{R}^4$ and $\mathbf{R}^8$ respectively. (Cf. [6].)

Comparing equations (6.5.1.5) - (6.5.1.7) with equations (6.5.1.8) - (6.5.1.10) it is apparent that for our geometry

$$\zeta_1 = -\delta\phi, \tag{6.5.1.11}$$

$$\zeta_2 = \delta\theta, \tag{6.5.1.12}$$

$$\zeta_3 = \delta\psi. \tag{6.5.1.13}$$

We make use of this correspondence to relate the variational analysis in this section with the analysis in terms of Euler angles in section A.5. However for a more complex, nonplanar configuration in which all three of the Euler angles vary, equations (6.5.1.8) – (6.5.1.10) no longer hold, and must be replaced by the more general equations (A.5.24) – (A.5.26). The difficulty is that the three Euler angles describe a sequence of rotations about three distinct sets of axes. By contrast the Euler parameters describe a single net rotation to the basis $\{\mathbf{d}_k\}$, so equations (6.5.1.5) – (6.5.1.7) always hold. Thus a representation of the directors in terms of Euler parameters is no more complicated for variational analysis than the analogous representation in terms of Euler angles, and is potentially much simpler.

Because the boundary conditions (2.5.14) – (2.5.16) on $\mathbf{h}$ must hold for all configurations, the variations in the configuration variables must satisfy

$$\delta\mathbf{h}|_{-\infty}^{\infty} = \int_{-\infty}^{\infty} \delta\mathbf{h}'(s)\, ds = \mathbf{0}. \tag{6.5.1.14}$$

In addition, the linearized constraint on the inextensibility condition

$$\int_{-\infty}^{\infty} \mathbf{\Lambda}^*(s) \cdot (\delta\mathbf{h}'(s) - \delta\mathbf{d}_3(s))\, ds = 0, \tag{6.5.1.15}$$

or equivalently

$$\delta\mathbf{h}' = \delta\mathbf{d}_3 = \mathsf{D}_3[\boldsymbol{\beta}]^T \delta\boldsymbol{\beta}. \tag{6.5.1.16}$$

From equation (6.5.1.7) and the form of the solitary wave the perturbation in $\mathbf{d}_3$ takes the form

$$\delta\mathbf{d}_3 = -\zeta_1\bar{\mathbf{d}}_2 - \zeta_2\mathbf{e}_3 = \begin{pmatrix} \sin\bar{\phi}\,\zeta_1 \\ -\cos\bar{\phi}\,\zeta_1 \\ -\zeta_2 \end{pmatrix}, \tag{6.5.1.17}$$

so from equation (6.5.1.16)

$$\delta\mathbf{h}(s) = \int_{-\infty}^{s} -\zeta_1(u)\bar{\mathbf{d}}_2(u)\, du - \mathbf{e}_3 \int_{-\infty}^{s} \zeta_2(u)\, du = \begin{pmatrix} \int_{-\infty}^{s} \sin\bar{\phi}(u)\zeta_1(u)\, du \\ -\int_{-\infty}^{s} \cos\bar{\phi}(u)\zeta_1(u)\, du \\ -\int_{-\infty}^{s} \zeta_2(u)\, du \end{pmatrix}. \tag{6.5.1.18}$$

Letting $s \to \infty$ in equation (6.5.1.18) we see that the linearized constraint (6.5.1.14) implies three linearized constraints, namely

$$\int_{-\infty}^{\infty} \sin\bar{\phi}(s)\, \zeta_1(s)\, ds = 0, \tag{6.5.1.19}$$

$$\int_{-\infty}^{\infty} \cos\bar{\phi}(s)\,\zeta_1(s)\,ds = 0, \tag{6.5.1.20}$$

and

$$\int_{-\infty}^{\infty} \zeta_2(s)\,ds = 0. \tag{6.5.1.21}$$

The remaining invariants involve the impetuses $\mathbf{g}$ and $\boldsymbol{\gamma}$ and the striction $\tilde{\boldsymbol{\Lambda}}$. The variations in these variables are related by the linearization of the elliptic equation (6.2.19) governing $\tilde{\boldsymbol{\Lambda}}$, namely

$$\begin{aligned}
\mathbf{0} \;=\;& -\rho^{-1}(\delta\mathbf{g} + \delta\tilde{\boldsymbol{\Lambda}}')' \\
&+\tfrac{1}{4}\delta\mathsf{D}_3[\boldsymbol{\beta}]^T\mathsf{B}[\bar{\boldsymbol{\beta}}]^T\,\mathsf{I}^{-1}\mathsf{B}[\bar{\boldsymbol{\beta}}](\bar{\boldsymbol{\gamma}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\bar{\boldsymbol{\Lambda}}) \\
&+\tfrac{1}{4}\mathsf{D}_3[\bar{\boldsymbol{\beta}}]^T\mathsf{B}[\delta\boldsymbol{\beta}]^T\,\mathsf{I}^{-1}\mathsf{B}[\bar{\boldsymbol{\beta}}](\bar{\boldsymbol{\gamma}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\bar{\boldsymbol{\Lambda}}) \\
&+\tfrac{1}{4}\mathsf{D}_3[\bar{\boldsymbol{\beta}}]^T\mathsf{B}[\bar{\boldsymbol{\beta}}]^T\,\mathsf{I}^{-1}\mathsf{B}[\delta\boldsymbol{\beta}](\bar{\boldsymbol{\gamma}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\bar{\boldsymbol{\Lambda}}) \\
&+\tfrac{1}{4}\mathsf{D}_3[\bar{\boldsymbol{\beta}}]^T\mathsf{B}[\bar{\boldsymbol{\beta}}]^T\,\mathsf{I}^{-1}\mathsf{B}[\bar{\boldsymbol{\beta}}](\delta\boldsymbol{\gamma} + \delta\mathsf{D}_3[\boldsymbol{\beta}]\bar{\boldsymbol{\Lambda}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\delta\tilde{\boldsymbol{\Lambda}}),
\end{aligned} \tag{6.5.1.22}$$

where

$$\begin{aligned}
\delta\mathsf{D}_3[\boldsymbol{\beta}] \;=\;& 2\mathsf{B}_2\delta\boldsymbol{\beta}\otimes\bar{\mathbf{d}}_1 - 2\mathsf{B}_1\delta\boldsymbol{\beta}\otimes\bar{\mathbf{d}}_2 \\
&+2\mathsf{B}_2\bar{\boldsymbol{\beta}}\otimes\delta\mathbf{d}_1 - 2\mathsf{B}_1\bar{\boldsymbol{\beta}}\otimes\delta\mathbf{d}_2
\end{aligned} \tag{6.5.1.23}$$

is the variation of $\mathsf{D}_3[\boldsymbol{\beta}]$ (2.6.2.21). In addition $\delta\boldsymbol{\gamma}$ must satisfy the linearization of equation (6.3.30), namely

$$\delta\boldsymbol{\gamma}\cdot\bar{\boldsymbol{\beta}} + \bar{\boldsymbol{\gamma}}\cdot\delta\boldsymbol{\beta} = 0. \tag{6.5.1.24}$$

Note that the variations in $\mathbf{g}$ and $\boldsymbol{\gamma}$ appear only in the combinations $\delta\mathbf{g} + \delta\tilde{\boldsymbol{\Lambda}}'$ and $\delta\boldsymbol{\gamma} + \delta\mathsf{D}_3[\boldsymbol{\beta}]\bar{\boldsymbol{\Lambda}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\delta\tilde{\boldsymbol{\Lambda}}$, because these variations exhibit the same gauge freedom in $\delta\tilde{\boldsymbol{\Lambda}}$ that arises for the striction. To be specific we set $\delta\mathbf{g} = \mathbf{0}$. Note also that the term $\delta\boldsymbol{\gamma}$ is premultipled by $\mathsf{B}[\bar{\boldsymbol{\beta}}]$ in equation (6.5.1.22). Thus $\delta\boldsymbol{\gamma}\cdot\bar{\boldsymbol{\beta}}$ may be chosen to satisfy (6.5.1.24) by letting

$$\begin{aligned}
\left(\delta\boldsymbol{\gamma} + \delta\mathsf{D}_3[\boldsymbol{\beta}]\bar{\boldsymbol{\Lambda}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\delta\tilde{\boldsymbol{\Lambda}}\right)\cdot\bar{\boldsymbol{\beta}} \;&=\; -\left(\bar{\boldsymbol{\gamma}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\bar{\boldsymbol{\Lambda}}\right)\cdot\delta\boldsymbol{\beta} \\
&=\; -\zeta_1 c I_1\bar{\phi}'\bar{\boldsymbol{\beta}},
\end{aligned} \tag{6.5.1.25}$$

where we have used equation (6.4.50) for $\bar{\boldsymbol{\gamma}}$.

For our analysis we represent $\delta\mathbf{g} + \delta\tilde{\boldsymbol{\Lambda}}'$ and $\delta\boldsymbol{\gamma} + \delta\mathsf{D}_3[\boldsymbol{\beta}]\bar{\boldsymbol{\Lambda}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\delta\tilde{\boldsymbol{\Lambda}}$ by

$$\begin{aligned}
\delta\mathbf{g}(s) + \delta\tilde{\boldsymbol{\Lambda}}'(s) \;=\;& \rho c\zeta_1(s)\bar{\mathbf{d}}_2(s) - \rho\int_{-\infty}^{s}\left(\eta_1(u) + c\zeta_1'(u)\right)\bar{\mathbf{d}}_2(u)\,du \\
&+\rho\int_{-\infty}^{s}\eta_2(u)\bar{\mathbf{d}}_1\,du
\end{aligned} \tag{6.5.1.26}$$

and

$$\begin{aligned}
&\delta\boldsymbol{\gamma}(s) + \delta\mathsf{D}_3[\boldsymbol{\beta}(s)]\bar{\boldsymbol{\Lambda}}(s) + \mathsf{D}_3[\bar{\boldsymbol{\beta}}(s)]\delta\tilde{\boldsymbol{\Lambda}}(s) \\
&\quad = 2I_1\eta_1(s)\mathsf{B}_1\bar{\boldsymbol{\beta}}(s) + \left\{2I_1\left[\eta_2(s) - c\bar{\phi}'(s)\zeta_3(s)\right] + cI_1\bar{\phi}'(s)\zeta_3(s)\right\}\mathsf{B}_2\bar{\boldsymbol{\beta}}(s) \\
&\qquad + \left\{2I_3\left[\eta_3(s) + c\bar{\phi}'(s)\zeta_2(s)\right] - cI_1\bar{\phi}'(s)\zeta_2(s)\right\}\mathsf{B}_3\bar{\boldsymbol{\beta}}(s) \\
&\qquad - cI_1\bar{\phi}'(s)\zeta_1(s)\bar{\boldsymbol{\beta}}(s). \qquad (6.5.1.27)
\end{aligned}$$

In equations (6.5.1.26) and (6.5.1.27) the parameters η_1, η_2 and η_3 correspond to $(\zeta_1)_t$, $(\zeta_2)_t$ and $(\zeta_3)_t$ respectively. The motivation for these representations is discussed in Appendix C. Here we simply remark that equations (6.5.1.26) and (6.5.1.27) define a completely general representation for solutions of equations (6.5.1.22) and (6.5.1.24).

By inserting these representations into the linearizations of the invariants, one characterize the admissible variations. These rather involved calculations are relegated to Appendix C. The ten linearized constraints are given by equations (6.5.1.19) – (6.5.1.21) involving the configuration variables only, and equations (C.3.12), (C.3.17), (C.3.30) – (C.3.32), (C.3.35), and (C.3.36).

6.5.2 Analysis of Second Variation

We now proceed to analyze the second variation of the reduced Lyapunov function

$$F_{\text{red}} = H + cI_{\text{red}} + \tfrac{1}{2}\textstyle\int_{-\infty}^{\infty}\mu(s)|\boldsymbol{\beta}|^2, \qquad (6.5.2.1)$$

where the reduced impulse can be rewritten as

$$\begin{aligned}
I_{\text{red}} = \int_{-\infty}^{\infty}\boldsymbol{\gamma}\cdot\boldsymbol{\beta}_s\,ds &= \int_{-\infty}^{\infty}\boldsymbol{\gamma}\cdot\frac{1}{|\boldsymbol{\beta}|^2}(\mathsf{B}[\boldsymbol{\beta}]^T\mathsf{B}[\boldsymbol{\beta}] + \boldsymbol{\beta}\otimes\boldsymbol{\beta})\bar{\boldsymbol{\beta}}'\,ds \\
&= \int_{-\infty}^{\infty}\left\{\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\gamma}\cdot\frac{\mathsf{B}[\boldsymbol{\beta}]\bar{\boldsymbol{\beta}}'}{|\boldsymbol{\beta}|^2} + (\boldsymbol{\gamma}\cdot\boldsymbol{\beta})\frac{(\boldsymbol{\beta}\cdot\boldsymbol{\beta}')}{|\boldsymbol{\beta}|^2}\right\}ds. \qquad (6.5.2.2)
\end{aligned}$$

We begin by calculating the second variation of F_{red}. Through an integration by parts, the third term in F_{red} can also be rewritten as

$$\tfrac{1}{2}\textstyle\int_{-\infty}^{\infty}\mu(s)|\boldsymbol{\beta}(s)|^2\,ds = -\int_{-\infty}^{\infty}\left(\bar{\boldsymbol{\gamma}}\cdot\bar{\boldsymbol{\beta}}\right)(\boldsymbol{\beta}\cdot\boldsymbol{\beta}_s)\,ds. \qquad (6.5.2.3)$$

Using equations (6.5.2.2) and (6.5.2.3) and a completion of squares, F_{red} can then be decomposed as

$$F_{\text{red}} = F_1 + F_2, \qquad (6.5.2.4)$$

where

$$\begin{aligned}
F_1 &\equiv \tfrac{1}{2}\textstyle\int_{-\infty}^{\infty}\left\{\rho^{-1}|\mathbf{g}_c(s)|^2 + \frac{1}{4}I_j^{-1}\left(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\gamma}_c\right)^2\right\}ds, && (6.5.2.5)\\
\mathbf{g}_c &\equiv \mathbf{g} + \tilde{\boldsymbol{\Lambda}}' + \rho c\left(\mathbf{d}_3 - \mathbf{e}_1\right), && (6.5.2.6)\\
\boldsymbol{\gamma}_c &\equiv \boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}} + 4c\mathsf{B}[\boldsymbol{\beta}]^T\mathsf{I}\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}'/|\boldsymbol{\beta}|^2, && (6.5.2.7)
\end{aligned}$$

and

$$
\begin{aligned}
F_2 \equiv\; & 2\int_{-\infty}^{\infty} \frac{\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}'}{|\boldsymbol{\beta}|^2} \cdot (\mathsf{K} - c^2\mathsf{I}) \frac{\mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}'}{|\boldsymbol{\beta}|^2}\, ds \\
& + \int_{-\infty}^{\infty} \left\{ -c\bar{\mathbf{g}} \cdot (\mathbf{d}_3 - \mathbf{e}_1) - \tfrac{1}{2} c^2 \rho |\mathbf{d}_3 - \mathbf{e}_1|^2 \right\} ds \\
& \quad + \int_{-\infty}^{\infty} \left\{ -c\tilde{\boldsymbol{\Lambda}}' \cdot (\mathbf{d}_3 - \mathbf{e}_1) - c\mathsf{B}[\boldsymbol{\beta}]\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}} \cdot \mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}'/|\boldsymbol{\beta}|^2 \right\} ds \\
& \quad + c\int_{-\infty}^{\infty} \left\{ \frac{(\boldsymbol{\beta}\cdot\boldsymbol{\gamma})}{|\boldsymbol{\beta}|^2} - \left(\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\gamma}}\right) \right\} (\boldsymbol{\beta}\cdot\boldsymbol{\beta}')\, ds. \qquad (6.5.2.8)
\end{aligned}
$$

The expression for F_2 can be greatly simplified. From the identities $|\mathbf{d}_3| = 1$ and $|\mathbf{e}_1| = 1$ and the expression for $\bar{\mathbf{g}}$ we find that

$$
\begin{aligned}
-c\bar{\mathbf{g}} \cdot (\mathbf{d}_3 - \mathbf{e}_1) - \tfrac{1}{2} c^2 \rho |\mathbf{d}_3 - \mathbf{e}_1|^2 &= -c\bar{\mathbf{g}} \cdot (\mathbf{d}_3 - \mathbf{e}_1) + c^2 \rho \mathbf{e}_1 \cdot (\mathbf{d}_3 - \mathbf{e}_1) \\
&= -\frac{K_1 - c^2 I_1}{a^2} \mathbf{e}_1 \cdot (\mathbf{d}_3 - \mathbf{e}_1) \\
&= \frac{K_1 - c^2 I_1}{a^2} (1 - \mathbf{e}_1 \cdot \mathbf{d}_3). \qquad (6.5.2.9)
\end{aligned}
$$

Using the definition (2.6.2.9) of $\mathbf{d}_3$ one can rewrite

$$
\begin{aligned}
1 - \mathbf{e}_1 \cdot \mathbf{d}_3 &= \left\{ \beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 - 2(\beta_1\beta_3 - \beta_2\beta_4) \right\} / |\boldsymbol{\beta}|^2 \\
&= \left[(\beta_1 - \beta_3)^2 + (\beta_2 - \beta_4)^2 \right] / |\boldsymbol{\beta}|^2. \qquad (6.5.2.10)
\end{aligned}
$$

Through an integration by parts we can rewrite

$$
-\int_{-\infty}^{\infty} \tilde{\boldsymbol{\Lambda}}' \cdot (\mathbf{d}_3 - \mathbf{e}_1)\, ds = \int_{-\infty}^{\infty} \tilde{\boldsymbol{\Lambda}} \cdot \mathbf{d}_3'\, ds, \qquad (6.5.2.11)
$$

and through the properties of B and D_3 we also find that

$$
\begin{aligned}
\mathsf{B}[\boldsymbol{\beta}]\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}} \cdot \mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}'/|\boldsymbol{\beta}|^2 &= \tilde{\boldsymbol{\Lambda}} \cdot \mathsf{D}_3[\boldsymbol{\beta}]^T \mathsf{B}[\boldsymbol{\beta}]^T \mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}'/|\boldsymbol{\beta}|^2 \\
&= \tilde{\boldsymbol{\Lambda}} \cdot \mathsf{D}_3[\boldsymbol{\beta}]^T \boldsymbol{\Pi}[\boldsymbol{\beta}]\boldsymbol{\beta}' \\
&= \tilde{\boldsymbol{\Lambda}} \cdot \mathsf{D}_3[\boldsymbol{\beta}]^T \boldsymbol{\beta}' \\
&= \tilde{\boldsymbol{\Lambda}} \cdot \frac{\partial \mathbf{d}_3}{\partial \boldsymbol{\beta}} \boldsymbol{\beta}' \\
&= \tilde{\boldsymbol{\Lambda}} \cdot \mathbf{d}_3'. \qquad (6.5.2.12)
\end{aligned}
$$

Equations (6.5.2.11) and (6.5.2.9) imply that

$$
\int_{-\infty}^{\infty} \left\{ -c\tilde{\boldsymbol{\Lambda}}' \cdot (\mathbf{d}_3 - \mathbf{e}_1) - c\mathsf{B}[\boldsymbol{\beta}]\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}} \cdot \mathsf{B}[\boldsymbol{\beta}]\boldsymbol{\beta}'/|\boldsymbol{\beta}|^2 \right\} ds = 0. \qquad (6.5.2.13)
$$

Inserting equations (6.5.2.9), (6.5.2.10) and (6.5.2.13) into equation (6.5.2.8), we can rewrite the functional F_2 more simply as

$$
\begin{aligned}
F_2 \equiv\ & 2\int_{-\infty}^{\infty}(K_j - c^2 I_j)\frac{1}{|\boldsymbol{\beta}|^4}\left(\mathsf{B}_j\boldsymbol{\beta}\cdot\boldsymbol{\beta}'\right)^2 ds \\
& + \frac{K_1 - c^2 I_1}{a^2}\int_{-\infty}^{\infty}\frac{1}{|\boldsymbol{\beta}|^2}\left[(\beta_1-\beta_3)^2 + (\beta_2-\beta_4)^2\right] ds \\
& + c\int_{-\infty}^{\infty}\left\{\frac{(\boldsymbol{\beta}\cdot\boldsymbol{\gamma})}{|\boldsymbol{\beta}|^2} - \left(\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\gamma}}\right)\right\}(\boldsymbol{\beta}\cdot\boldsymbol{\beta}')\, ds. \qquad (6.5.2.14)
\end{aligned}
$$

The second variation $\delta^2 F_2$ is computed in section C.4. The result, which appears in equation (C.4.23), can be rewritten as

$$
\begin{aligned}
\delta^2 F_2 =\ & (K_1 - c^2 I_1)\langle\zeta_1, P(\zeta_1)\rangle && (6.5.2.15)\\
& + (K_1 - c^2 I_1)\langle\zeta_2, S(\zeta_2)\rangle && (6.5.2.16)\\
& + \left\{(K_2 - c^2 I_2) - (K_1 - c^2 I_1)\right\}\int_{-\infty}^{\infty}(\zeta_2' + \bar{\phi}'\zeta_3)^2\, ds && (6.5.2.17)\\
& + (K_3 - c^2 I_3)\int_{-\infty}^{\infty}(\zeta_3' - \bar{\phi}'\zeta_2)^2\, ds, && (6.5.2.18)
\end{aligned}
$$

where P is the operator defined in equation (5.6.12) and S is given by

$$
S(f) \equiv -f'' + \left(\frac{1}{a^2}\cos\bar{\phi} - \bar{\phi}'^2\right) f. \qquad (6.5.2.19)
$$

Using the equation (5.5.25) for $\bar{\phi}'$, S may be rewritten as

$$
S(f) \equiv -f'' + \frac{1}{a^2}\left(3\cos\bar{\phi} - 2\right) f. \qquad (6.5.2.20)
$$

In the expression (6.5.2.15) – (6.5.2.18) for the second variation of F_2, one can identify the term (6.5.2.16) with the effect of planar variations, the term (6.5.2.16) with the effect of out of plane variations, the term (6.5.2.17) with the effect of anisotropy, and the term (6.5.2.18) with the effect of twist variations.

The main goal of this section is to show that, for sufficiently small wave speeds, the second variation of F_{red} has exactly one negative eigenvalue amongst admissible variations. Specifically we assume the wave speed c satisfies $|c| < \bar{c}$ where

$$
\bar{c} \equiv \min\left(c_1, c_3\right), \qquad (6.5.2.21)
$$

$$
c_1 = \sqrt{\frac{K_1}{I_1}} \qquad (6.5.2.22)
$$

is the critical propagation speed of infinitesimal transverse vibrations which was defined earlier, and

$$
c_3 \equiv \sqrt{\frac{K_3}{I_3}} \qquad (6.5.2.23)
$$

is the critical propagation speed of infinitesimal torsional vibrations. Before proceeding, a remark about the magnitude of c_3 is appropriate. If the rod is homogeneous, then from section 5.7 we have $K_1 = EI$ and $I_1 = \mu I$, where μ is the mass/unit volume of the rod, I is the geometric moment of inertia of the rod cross-section, and E is Young modulus. From equation (2.3.4) we also have the relation $I_3 = I_1 + I_2 = 2I_1$. Finally, the resistance to twist is given by $K_3 = GI_3$, in which the shear modulus $G = E/[2(1+\nu)]$, and ν is the Poisson ratio satisfying $0 \leq \nu < 1/2$. Therefore $c_1 = \sqrt{E/\mu}$ and

$$c_3 = \sqrt{\frac{G}{\mu}} = \sqrt{\frac{E/\mu}{2(1+\nu)}} < c_0. \tag{6.5.2.24}$$

Thus in the special case of a homogeneous rod, the critical wave speed $\bar{c}$ (6.5.2.21) is less than the speed of sound c_1. More precisely, because $0 \leq \nu < 1/2$ the critical speed $\bar{c}$ is bounded by $c_1/\sqrt{3} < \bar{c} \leq c_1/\sqrt{2}$.

From equations (6.4.3), (6.4.12) and (6.4.16) we observe that the terms $\mathbf{g}_c$ (6.5.2.6) and $\boldsymbol{\gamma}_c$ (6.5.2.7) in F_1 vanish on a solitary wave profile. Thus F_1 is minimized at the solitary wave profile, and so $\delta^2 F_1$ is positive semidefinite. In the second variation of F_2 given by (6.5.2.15) – (6.5.2.18), we have restricted our attention to the case of an isotropic rod, so the anisotropy term (6.5.2.17) vanishes identically. Moreover the term (6.5.2.15) representing the effect of planar perturbations on the second variation of F_2 was shown in the previous chapter to be nonnegative for all subsonic wave speeds. Under the assumption $|c| < \bar{c}$, the term (6.5.2.18) is also nonnegative. (By contrast, if $|c| > \bar{c}$ then the term (6.5.2.18) is negative semidefinite and the second variation of F_{red} is infinitely indefinite.) Thus the term (6.5.2.16) is the only component which could make $\delta^2 F_{\text{red}}$ negative. For this reason our analysis focuses on the behavior of the operator S. It is first shown that S has one negative eigenvalue, so for $|c| < \bar{c}$ the second variation $\delta^2 F_{\text{red}}$ has at most one negative eigenvalue. Then by a particular choice of variations, constructed to satisfy the linearized constraints, we show that $\delta^2 F_{\text{red}}$ has at least one negative eigenvalue. It follows that $\delta^2 F_{\text{red}}$ has at exactly one negative eigenvalue amongst the admissible variations for wave speeds satisfying $|c| < \bar{c}$.

We begin by establishing some information about the spectrum of the operator S. Maddocks [42] observed that

$$\zeta_2 = \epsilon \sin \bar{\phi} \tag{6.5.2.25}$$

is an eigenfunction of S, with

$$S(\sin \bar{\phi}) = 0. \tag{6.5.2.26}$$

For the variation (6.5.2.25) about the solitary wave we use equations (5.5.37) and (5.5.24) to write

$$y(s) = \bar{y}(s) + O(\epsilon^2) = a^2 \bar{\phi}'(s) + O(\epsilon^2) \tag{6.5.2.27}$$

and

$$\begin{aligned} z(s) &= -\int_{-\infty}^{s} \zeta_2(u)\, du \\ &= \epsilon \int_{-\infty}^{s} \sin \bar{\phi}(u)\, ds + O(\epsilon^2) \\ &= -\epsilon a^2 \bar{\phi}'(s) + O(\epsilon^2) \end{aligned} \tag{6.5.2.28}$$

Thus to first order in ϵ the rod lies in the plane $z = -\epsilon y$, so geometrically the variation (6.5.2.25) corresponds to a rotation of the rod about the x-axis.

Since $\sin \bar{\phi}$ has exactly one zero, the spatial operator S must have exactly one negative eigenvalue which has no interior zeroes. (Cf. [46], p. 877, Lemma 2.2.) Specifically,

$$S(1 - \cos \bar{\phi}) = -\frac{3}{a^2}(1 - \cos \bar{\phi}). \tag{6.5.2.29}$$

Consequently $\delta^2 F_{\mathrm{red}}$ can have at most one negative eigenvalue.

We now construct an admissible set of variations for which $\delta^2 F_{\mathrm{red}}$ is negative. From the existence of a negative direction one can infer through the Rayleigh quotient the existence of at least one negative eigenvalue, which together with our analysis of the operator S implies that $\delta^2 F_{\mathrm{red}}$ has constrained index one.

In order to make the planar term (6.5.2.15) in $\delta^2 F_2$ vanish, we set

$$\zeta_1 \equiv 0. \tag{6.5.2.30}$$

By also setting

$$\eta_1 \equiv 0, \tag{6.5.2.31}$$

the six linearized constraints on x (6.5.1.19), y (6.5.1.20), the impulse (C.3.12), the z-component of angular momentum (C.3.32) and the y-component of linear momentum (C.3.35) are all satisfied. In order to make the twist term (6.5.2.18) also vanish we set

$$\zeta_3(s) = \int_{-\infty}^{s} \bar{\phi}'(u)\zeta_2(u)\, du. \tag{6.5.2.32}$$

By simultaneously choosing

$$\eta_3 = -c\zeta_3' = -c\bar{\phi}'\zeta_2, \tag{6.5.2.33}$$

the linearized constraint (C.3.17) on isotropic angular momentum is satisfied. Finally if we set

$$\eta_2 = -c\zeta_2', \tag{6.5.2.34}$$

then the constraint (C.3.30) becomes identical to constraint (6.5.1.21), and the second variation of F_1 (C.4.3) vanishes.

The remaining three linearized constraints (6.5.1.21), (C.3.30) and (C.3.31) can be written as

$$\int_{-\infty}^{s} \zeta_2(s)\, ds = 0, \tag{6.5.2.35}$$

$$2cI_1 \int_{-\infty}^{\infty} \zeta_2'(s) \sin\bar{\phi}(s)\, ds + 2c\rho \int_{-\infty}^{\infty} \bar{y}(s)\zeta_2(s)\, ds = 0 \tag{6.5.2.36}$$

and

$$\begin{aligned} 2cI_1 \int_{-\infty}^{\infty} \zeta_2'(s) \cos\bar{\phi}(s)\, ds + c\rho \int_{-\infty}^{\infty} \int_{-\infty}^{s} \zeta_2(u)\, du\, ds & \\ +2c\rho \int_{-\infty}^{\infty} \bar{x}(s)\zeta_2(s)\, ds &= 0. \end{aligned} \tag{6.5.2.37}$$

Using equation (5.5.37), the identity $\bar{x}'(s) = \cos\bar{\phi}(s)$ and integration by parts, equations (6.5.2.36) and (6.5.2.37) may be rewritten as

$$\int_{-\infty}^{\infty} \zeta_2(s) \left[I_1 a \cos\bar{\phi}(s)\bar{\phi}'(s) - \rho a^2 a\bar{\phi}'(s) \right] ds = 0 \tag{6.5.2.38}$$

and

$$\begin{aligned} 2cI_1 \int_{-\infty}^{\infty} \zeta_2(s) \sin\bar{\phi}(s)\bar{\phi}'(s)\, ds + c\rho \int_{-\infty}^{\infty} \int_{-\infty}^{s} \zeta_2(u)\, du\, ds & \\ -2c\rho \int_{-\infty}^{\infty} \cos\bar{\phi}(s) \int_{-\infty}^{s} \zeta_2(u)\, du\, ds &= 0. \end{aligned} \tag{6.5.2.39}$$

For the particular choice of variations (6.5.2.30) - (6.5.2.34),

$$\delta^2 F_1 = 0 \tag{6.5.2.40}$$

and

$$\delta^2 F_2 = (K_1 - c^2 I_1)\langle \zeta_2, S(\zeta_2)\rangle. \tag{6.5.2.41}$$

We now show that there exist a function ζ_2 such that the constraints (6.5.2.35), (6.5.2.38) and (6.5.2.39) are satisfied and

$$\langle \zeta_2, S(\zeta_2)\rangle < 0. \tag{6.5.2.42}$$

It will then follow that $\delta^2 F_{\text{red}} < 0$ for all wave speeds satisfying $|c| < \bar{c}$. It is apparent from equation (6.5.2.29) that the eigenfunction $1 - \cos\bar{\phi}$ satisfies the inequality (6.5.2.42), but this function violates the linearized constraint (6.5.1.21). A search for a function ζ_2 which is both admissible and satisfies (6.5.2.42) led us to consider linear combinations of the functions

$$f_1 = a \cos\bar{\phi}\, \bar{\phi}' \tag{6.5.2.43}$$

and

$$f_2 = 2a\bar{\phi}' - \pi \left(1 - \cos\bar{\phi}\right). \tag{6.5.2.44}$$

These functions are of interest because

$$\langle f_1, S(f_1)\rangle = -\frac{64}{35a} \approx -\frac{1.83}{a} \tag{6.5.2.45}$$

and

$$\langle f_2, S(f_2)\rangle = -\frac{256 - 24\pi^2}{3a} \approx -\frac{6.38}{a}, \tag{6.5.2.46}$$

so f_1 and f_2 define two independent directions along which S is negative. Because

$$\int_{-\infty}^{s} f_1(u)\,du = a\sin\bar{\phi}(s) \tag{6.5.2.47}$$

and

$$\int_{-\infty}^{s} f_2(u)\,du = 2a\bar{\phi}(s) - 2\pi\left[1 - \cos(\bar{\phi}(s)/2)\right], \tag{6.5.2.48}$$

we see that f_1 and f_2 satisfy the constraint (6.5.2.35). Moreover both f_1 and f_2 are even functions while $\sin\bar{\phi}\,\bar{\phi}'$ is odd, so

$$\int_{-\infty}^{\infty} f_j(s)\sin\bar{\phi}(s)\bar{\phi}'(s)\,ds = 0, \tag{6.5.2.49}$$

and similarly both $\int_{-\infty}^{s} f_1$ and $\int_{-\infty}^{s} f_2$ are odd functions while $\cos\bar{\phi}$ is even, so

$$\int_{-\infty}^{\infty}\int_{-\infty}^{s} f_j(u)\,du\,ds = 0 \tag{6.5.2.50}$$

and

$$\int_{-\infty}^{\infty} \cos\bar{\phi}(s)\int_{-\infty}^{s} f_j(u)\,du\,ds = 0. \tag{6.5.2.51}$$

It follows from equations (6.5.2.49) – (6.5.2.51) that f_1 and f_2 also satisfy the constraint (6.5.2.39). And while neither f_1 nor f_2 satisfies the remaining constraint (6.5.2.38), there is a linear combination of f_1 and f_2 which does satisfy (6.5.2.38), namely

$$\zeta_2 = f_2 + m f_1 \tag{6.5.2.52}$$

in which m is given by

$$m = -\frac{\rho a^2\langle f_1, f_3\rangle - I_1\langle f_1, f_1\rangle}{\rho a^2\langle f_2, f_3\rangle - I_1\langle f_2, f_1\rangle}, \tag{6.5.2.53}$$

and $f_3 \equiv a\bar{\phi}'$. The inner products appearing in equation (6.5.2.53) are

$$\langle f_1, f_1\rangle = \frac{56a}{15} \approx 3.73a, \tag{6.5.2.54}$$

$$\langle f_1, f_3\rangle = -\frac{8a}{3} \approx -2.67a, \tag{6.5.2.55}$$

$$\langle f_2, f_1\rangle = \left(\pi^2 - \frac{16}{3}\right) \approx 4.54a, \tag{6.5.2.56}$$

$$\langle f_2, f_3\rangle = -2\left(\pi^2 - 8\right)a \approx -3.74a. \tag{6.5.2.57}$$

Because I_1 and ρa^2 are positive, the ratio m (6.5.2.53) lies between

$$-\frac{\langle f_1, f_3 \rangle}{\langle f_2, f_3 \rangle} \approx -0.475 \qquad (6.5.2.58)$$

and

$$-\frac{\langle f_1, f_1 \rangle}{\langle f_2, f_1 \rangle} \approx -0.823 \qquad (6.5.2.59)$$

It remains to show that $\langle \zeta_2, S(\zeta_2) \rangle < 0$ for the variation ζ_2 defined in equation (6.5.2.52). To do so we note that the quadratic form

$$\langle \zeta_2, S(\zeta_2) \rangle = \langle f_2, S(f_2) \rangle + 2m\langle f_1, S(f_2) \rangle + m^2\langle f_1, S(f_1) \rangle \qquad (6.5.2.60)$$

is positive only if m lies between the roots

$$m_1 = \frac{-\langle f_1, S(f_2) \rangle + \sqrt{\langle f_1, S(f_2) \rangle^2 - \langle f_1, S(f_1) \rangle \langle f_2, S(f_2) \rangle}}{\langle f_1, S(f_1) \rangle} \qquad (6.5.2.61)$$

and

$$m_2 = \frac{-\langle f_1, S(f_2) \rangle - \sqrt{\langle f_1, S(f_2) \rangle^2 - \langle f_1, S(f_1) \rangle \langle f_2, S(f_2) \rangle}}{\langle f_1, S(f_1) \rangle}. \qquad (6.5.2.62)$$

Utilizing equations (6.5.2.45) and (6.5.2.46) together with

$$\langle f_1, S(f_2) \rangle = -\frac{15\pi^2 - 128}{5a} \approx -\frac{4.01}{a}, \qquad (6.5.2.63)$$

we find that $m_1 \approx -1.048$ and $m_2 \approx -3.326$. Therefore the inequality (6.5.2.42) holds for the variation ζ_2 defined by (6.5.2.52), so the second variation $\delta^2 F_{\text{red}} < 0$ at wave speeds for the variations specified above. Based on our ealier remarks, we conclude that $\delta^2 F_{\text{red}}$ has exactly one negative eigenvalue amongst admissible variations.

It is enlightening to consider the physical deformation to which the variation (6.5.2.52) corresponds. In view of equations (6.5.1.18), (6.5.2.47) and (6.5.2.48), the out-of-plane deformation is

$$z(s) \quad = \quad -\int_{-\infty}^{s} \zeta_2 \, du = -2a\bar{\phi} + 2\pi \left[1 - \cos(\bar{\phi}/2)\right] + ma \sin \bar{\phi}. \qquad (6.5.2.64)$$

Because the deformation z is an odd function of s, we see that ζ_2 corresponds to an out of plane twisting of the solitary wave about its axis of symmetry of the loop.

It closing we consider the implications of our variational analysis upon some closely related rod problems. Notice first that equation (6.5.2.29) implies a lower bound on $\langle \zeta_2, S\zeta_2 \rangle$, namely

$$\langle \zeta_2, S\zeta_2 \rangle > -\frac{3}{a}\|\zeta_2\|^2. \qquad (6.5.2.65)$$

In our analysis we have assumed boundary conditions which allow free rotation of the rod about the $\mathbf{e}_1$ axis at its ends, as if it were in a sleeve. Thus we were able to assume the variation (6.5.2.32) in ζ_3 which eliminates the twist term (6.5.2.18) in the second variation of F_2. If instead a boundary condition were specified on the twist so that $\zeta_3 = 0$ at the ends, then the integral appearing in the twist term (6.5.2.18) will in general be positive. It follows that for K_3 sufficiently large, there is a range of wave speeds c for which the total second variation of F_2 is positive, so that the planar solitary wave is a minimizer of F_{red}.

Alternatively, suppose the rod is anisotropic with $I_2 > I_1$ and $K_2 > K_1$. (For example, think of a rod with elliptic cross-section for which the semimajor axis is orthogonal to the plane of the solitary wave.) The inequality $K_2 > K_1$ means that the solitary wave lies in the plane of minimum stiffness of the rod. If $K_2 - K_1$ is sufficiently large, then for small wave speeds c the total second variation of F_2 must be positive, so the planar solitary wave is a minimizer of F_{red}. If on the other hand $I_2 < I_1$ and $K_2 < K_1$, then the solitary wave lies in the plane of maximum stiffness of the rod, and the anisotropic term in the second variation is negative for wave speed c such that $(K_2 - c^2 I_2) < (K_1 - c^2 I_1)$. For example, for the variations $\zeta_1 = 0$, $\zeta_2 = \sin\bar{\phi}$, with $\zeta_3 = 1 - \cos\bar{\phi}$ in accordance with (6.5.2.32), we obtain $\zeta_2' + \bar{\phi}'\zeta_3 = \bar{\phi}'$ and $\zeta_3' - \bar{\phi}'\zeta_2 = 0$, and so

$$\delta^2 F_2 = \tfrac{1}{2}\left\{(K_2 - c^2 I_2) - (K_1 - c^2 I_1)\right\} \int_{-\infty}^{\infty} \bar{\phi}'^2 \, ds \qquad (6.5.2.66)$$

That is to say, the variation $\zeta_2 = \sin\bar{\phi}$ which describes an out of plane rotation of the solitary wave tends to decrease F_2. In this case the second variation of F_2 is infinitely indefinite, so the the planar solitary wave is not a minimizer of F_2.

CHAPTER 7

CONCLUSION AND DISCUSSION

In this dissertation I have described a new and quite general procedure, called the impetus-striction formulation, for the derivation of an unconstrained Hamiltonian system describing constrained Lagrangian dynamics. This formulation has some unusual features, such as a gauge freedom and consequent partial unobservability of the state variables, but it has proved useful in analyzing infinite dimensional systems governed by partial differential equations. In Chapter 5 the formulation was used to obtain a Hamiltonian evolutionary system governing the planar twist-free motion of an inextensible and unshearable elastic rod. For this planar model the directors $\mathbf{d}_k(s,t)$ at time t and arclength s are parameterized by a single angle $\phi(s,t)$, and the displacement of each material point at s from the reference configuration is given by $\mathbf{h}(s,t)$. These variables are illustrated in Figure 1. The Hamiltonian for this model (in the case of a rod with finite length L) is

$$H(\mathbf{h},\phi,\mathbf{g},\chi) = \int_{-L}^{L} \tfrac{1}{2}\left\{\rho^{-1}|\mathbf{g}+\tilde{\mathbf{\Lambda}}_s|^2 + I_1^{-1}[\chi+\mathbf{d}_2\cdot\tilde{\mathbf{\Lambda}}]^2 + K_1\phi_s^2\right\}\,ds, \tag{7.1}$$

where $\mathbf{g}$ and χ are the impetuses conjugate to $\mathbf{h}$ and ϕ, respectively. The two components of the striction $\tilde{\mathbf{\Lambda}}$ are determined through a minimization by which one obtains the elliptic equation

$$\rho^{-1}(\mathbf{g}+\mathbf{\Lambda}_s)_s - I_1^{-1}(\chi+\mathbf{d}_2\cdot\mathbf{\Lambda})\mathbf{d}_2 = \mathbf{0}, \tag{7.2}$$

subject to the boundary conditions

$$\rho^{-1}(\mathbf{g}+\mathbf{\Lambda}_s) = \mathbf{0} \quad \text{at } s=\pm L. \tag{7.3}$$

The vector equation (7.2) governing the striction is equivalent to the inextensibility and unshearability conditions, which are therefore invariants of motion under the impetus-striction formulation.

Because the displacement $\mathbf{h}$ does not appear explicitly in the Hamiltonian H, the associated impetus $\mathbf{g}$ is constant in time. We were therefore able to fix $\mathbf{g}$, eliminating the gauge freedom, and go to a rather simple reduced system involving only the two variables

ϕ and χ, namely

$$\phi_t = \delta_\chi H = I_1^{-1}\left(\chi + \mathbf{d}_2 \cdot \tilde{\mathbf{\Lambda}}\right), \tag{7.4}$$

$$\chi_t = -\delta_\phi H = I_1^{-1}\left(\chi + \mathbf{d}_2 \cdot \tilde{\mathbf{\Lambda}}\right)(\mathbf{d}_3 \cdot \tilde{\mathbf{\Lambda}}) + K_1 \phi_{ss}, \tag{7.5}$$

where again $\tilde{\mathbf{\Lambda}}$ is defined by equation (7.2).

The rod model was then extended in Chapter 6 to describe general three-dimensional motions. The directors were parameterized there using a set of four Euler parameters, whose properties are outlined in section 2.6. In this model the associated Hamiltonian is given by equation (6.2.38), in which three components of the striction $\tilde{\mathbf{\Lambda}}$ are determined by the elliptic equation (6.2.19) with boundary conditions (6.2.20). Once again one can go to a reduced system (6.2.41) and (6.2.52), in this case involving the four Euler parameters and their four associated impetuses. In Appendix A an analogous Hamiltonian system involving a set of three Euler angles is also presented. A comparison of the two models demonstrates the utility and in many respects the superiority of Euler parameters, which involve only polynomial and rational functions and do not exhibit the singularities inherent to an Euler angle representation.

The Hamiltonian formulation was developed in order to analyze the stability properties of planar solitary waves in an infinite, isotropic rod. This analysis employed a new conservation law for rod dynamics, which together with the specified boundary conditions imply the invariance of the linear impulse I (3.2.14). The solitary wave profile was first characterized as a critical point of the invariant $F = H + cI$, where c is the wave speed, and the solitary wave profile (5.5.27) for arbitrary wave speed was constructed explicitly. Among planar, twist-free variations with subsonic wave speed ($K_1 - c^2 I_1 > 0$), it was demonstrated that the solitary wave profile is actually a minimizer of the invariant of motion F_{red} (5.6.1) associated with the reduced dynamical system. Provided that we assume appropriate existence and continuation results for the associated Cauchy problem for the full time dynamics, we may conclude nonlinear, orbital stability of the subsonic solitary waves via a Lyapunov-type argument in the planar rod model. The analysis in [12] does include an existence result for the Cauchy problem for the Lagrangian dynamics of a finite rod subject to certain sets of boundary conditions, but I have not yet attempted to adapt that approach to the precise boundary-value problem of interest here in either the finite or infinite arc-length cases.

In addition to the inextensibility and unshearability conditions and the linear impulse discussed in the context of the planar model, several other invariants and their associated symmetries were described for the three-dimensional motion of a rod. In the particular case of an isotropic rod, the isotropic angular momentum A (6.3.8) was shown to be conserved.

The other invariants presented which have physical significance are the three components of angular momentum and the two transverse components of linear momentum. Because the longitudinal forces at the two ends of the rods need not be equal, the longitudinal component of linear momentum is in general not conserved. Nevertheless, the longitudinal component of the linear impetus $\mathbf{g}$ is conserved, corresponding to the translational symmetry of the rod along the $\mathbf{e}_1$ axis. Finally there are two invariants of motion, namely $|\boldsymbol{\beta}|^2$ and $\boldsymbol{\gamma}\cdot\boldsymbol{\beta}$, which have no physical interpretations but instead arise through the parameterization chosen for the directors.

In a variational analysis with respect to spatial perturbations, the examination of the special case of an isotropic rod whose ends are free to rotate provided the essential information. For this model, amongst the class of all spatial variations which preserve the invariants of motion, the subsonic wave profile was shown not to be a minimizer of F_{red}. In fact, for the spatial analysis the critical speed is in general not the speed of sound c_1, but rather the speed $\bar{c}$ given by equation (6.5.2.21) which is in general less than c_1. For wave speeds c less than $\bar{c}$, the second variation of F_{red} has index one; otherwise the second variation of F_{red} is infinitely indefinite. The variational analysis focused on a study of the behavior of the differential operators P and S given by (5.6.12) and (6.5.2.20) which respectively describe the influence of planar and nonplanar perturbations upon the second variation of F_{red}. For our analysis only partial information regarding of the spectra of these operators was required in order to determine the index of the second variation of F_{red} at the planar solitary wave profile. In fact the expression (5.5.31) for $\cos\bar{\phi}$ may be used to rewrite these operators as

$$P(f)(s) \equiv -f''(s) + \frac{1}{a^2}\left[1 - 2\,\mathrm{sech}^2\left(\frac{s}{a}\right)\right] f(s) \tag{7.6}$$

and

$$S(f)(s) \equiv -f''(s) + \frac{1}{a^2}\left[1 - 6\,\mathrm{sech}^2\left(\frac{s}{a}\right)\right] f(s). \tag{7.7}$$

It is possible to determine the complete spectra of operators in the above form through an appropriate change of variables presented by Morse & Feshbach ([49], p. 768).

Finally, the variational analysis of an isotropic rod with freely rotating ends may be used to show that the planar solitary wave profile is a minimizer of F_{red} for two closely related rod problems, namely where the solitary waves lies in the plane of minimum stiffness of a sufficiently anisotropic rod, and where the twist is fixed at the ends of a rod with sufficient resistance to twist deformations.

The results in this dissertation suggest several directions for further research. For example, the fact that the second variation of F_{red} has odd index suggests that the subsonic solitary wave may be unstable. Although I have not attempted to prove an instability result in this case, there are several promising approaches. For example, Pego & Weinstein [53] have developed a technique using Evans functions to prove instability which might be applicable.

Similarly, it is apparent from (5.6.2), (5.6.3) and (5.6.9) that for planar motions the second variation of F_{red} switches from positive semidefiniteness for subsonic wave speeds to complete indefiniteness for supersonic wave speeds. Thus supersonic solitary waves are not minima of any obvious variational principle. This leaves open the question of whether or not a supersonic solitary wave is dynamically stable. Preliminary numerical studies by Xu [63] suggest that supersonic solitary waves may actually be stable. These results are based upon a numerical scheme and associated error estimates which Xu [62] and Falk & Xu [23] have developed for the integration of the planar rod dynamics considered in [12] and [13] (i.e. the Lagrangian version of the dynamics considered here) subject to periodic boundary conditions (of possibly very large period). Attempts to prove or disprove instability for supersonic waves have been unsuccessful so far.

For the boundary conditions appropriate to the infinite rod there can be no component of shear force at infinity, i.e. the end load must be tangential. Accordingly, while the focus here has been on the distinction of subsonic versus supersonic waves, it is also reasonable to consider the separate distinction between waves for which the rod is in tension or compression at its ends. The expression (5.5.43) for the end load in a solitary wave motion shows that the transition from the tensile to compressive case does not correspond to the transition from subsonic to supersonic wave speeds. For subsonic and slightly supersonic waves the rod must be in tension at its ends, but for sufficiently fast supersonic waves the rod is in compression at its ends. It is perhaps of interest to note from equation (5.5.42) that at the critical speed c_1, the force in the solitary wave profile is everywhere tangential so there is no shear force at any arc-length s.

Using essentially the same analysis as above, one can also consider the existence and stability of solitary wave solutions when a nonlinear constitutive law for bending is assumed. If a strictly convex bending energy is assumed, then solitary wave solutions exist, but in general the amplitude of the wave depends upon the wave speed. Moreover, there exists a critical speed, c_0 say, such that a solitary wave whose speed satisfies $|c| \leq c_0$ can be characterized as a minimizer. However, any associated dynamical stability result must be interpreted carefully. For with a nonlinear bending law, the governing equation is in general replaced by a fully nonlinear hyperbolic equation, so shocks must be anticipated.

We conclude with a discussion of the relations between the results presented in this dissertation and prior work.

Chapter 2 contains an essentially standard review of the director theory of rods. The one unusual feature is the representation of the directors in terms of Euler parameters. While Euler parameters have often been utilized in models of rigid body mechanics, particularly in spacecraft dynamics, the only prior application we are aware of in the context of rod dynamics is in the work of Simo & Vu-Quoc [59].

In Chapter 3 I have discussed conservation laws in rod dynamics. Each conservation law necessarily generates an integral of the corresponding system of ordinary differential equations describing the equilibrium configurations. Thus it is not coincidental that the symmetries that are here used to generate conservation laws for the dynamics are exactly cases in which integrals of the static equations are known. Similarly the special cases of the conservation laws (3.2.9) and (3.3.19) that arise for the system of ordinary differential equations describing travelling waves in a rod were previously found by Antman & Liu ([4], eqs. (3.8) and (3.13)). Because of these relations to known special cases, it was comparatively easy to conjecture the appropriate form of the conservation laws for the case of general dynamics. Accordingly it was simpler to couch the presentation in terms of the *a posteriori* verification of various identities. Nevertheless it should be recognized that one can exploit Noether's Theorem to find the conservation laws (3.2.9) and (3.3.19) constructively, as was done in Chapter 5.

Despite the above remarks, to my knowledge neither of the conservation laws (3.2.9) and (3.3.19) have been described previously for the generality of dynamic rod models that are adopted here. However each conservation law certainly has its antecedents in the literature. For a hyperelastic, shearable, extensible rod, Simo et al. ([58], eq. (6.26)) found the conserved quantity (3.3.25) as a consequence of their Hamiltonian formulation of rod dynamics. The direct derivation employed here for the conservation law (3.3.19) and associated integral (3.3.25) is slightly more general because we need not assume hyperelasticity, which is required for the Hamiltonian formulation to be valid. Perhaps more relevantly, this derivation also implies (3.3.19) in the case of an inextensible, unshearable rod. It is a somewhat surprising fact that while the system governing the statics of an inextensible, unshearable rod is a considerable simplification of the system governing the statics of an extensible, shearable rod, the dynamics in the supposedly special case are in many ways much more complicated. The difficulty is associated with the fact that the net force $\boldsymbol{\lambda}$ is no longer determined via a constitutive relation, but instead plays the role of an additional dependent variable (or Lagrange multiplier maintaining the constraints of inextensibility and absence of shear), with no explicit equation governing its time evolution. (Cf. [12], [13], [14], [17], [23], and [62].) We also record the fact that Coleman et al. [14] independently discovered conservation of isotropic angular momentum for an inextensible, unshearable, linearly elastic rod using a formulation of the rod dynamics in terms of a set of Euler angles different from the one we employ in this dissertation.

The terminology 'impulse' for the conserved quantity associated with translation in arc-length is motivated by analogy with similar problems arising in the analysis of one-dimensional wave motions in fluid mechanics. (Cf. [10].) Subsequently the particular form (3.4.33) of the impulse conservation law for three-dimensional motion within the context of a certain string model was discovered by Healey [26] and was exploited in the analysis of relative equilibria. In the case of these classic string dynamics some simplifications arise, cf. the two expressions (3.2.14) and (3.4.34) for the impulse. Generalization from the planar case also led Coleman et al. [14] (independently of this work) to discovery of the impulse for three-dimensional motions of an inextensible, unshearable, linearly elastic rod, again using their analysis involving Euler angles. It seems almost certain that the impulse (3.2.14) will prove useful in the variational characterization and stability analysis of steady motions of rods in three dimensions, (such as the three-dimensional travelling waves that have been discussed in [4] and more recently in [15]).

The development in Chapter 4 of the impetus-striction formulation has its roots in the the vakonomics mechanics of Kozlov [5]. However vakonomics deals primarily with non-holonomic constraints, where vakonomic mechanics gives rise to dynamics distinct from those produced by the D'Alembert-Lagrange principle. The impetus-striction formulation treats only holonomic constraints, which avoids the question of the validity of the equations of motion. In addition, vakonomics is developed in the context of ordinary differential equations, but the minimization procedure at the foundation of the impetus-striction formulation permits us to extend to quite general infinite dimensional systems involving partial differential equations as well. Maddocks & Pego [44] have already shown this formulation to be useful in modelling incompressible fluid flow.

Dirac [18] also developed a detailed theory concerning constrained dynamical systems. He considered unconstrained Lagrangian dynamics in which the Lagrangian is convex in the velocity, but not strictly convex, and obtained a constrained Hamiltonian formulation in which the Hamiltonian is strictly convex in the momentum variables. In the impetus-striction formulation we instead considered a constrained Lagrangian system in which the Lagrangian is strictly convex in the velocity, and obtain an unconstrained Hamiltonian formulation in which the Hamiltonian is convex, but not strictly convex, in the impetuses. In this sense, the impetus-striction formulation is dual to the Dirac theory.

Chapters 5 and 6 have focused on the analysis of solitary waves. There have been many prior discussions of travelling waves in rods. My interest in the stability of solitary waves in an infinite rod was largely sparked by the recent work of Coleman & Dill [13]. Subsequently, Coleman & Xu [16] performed numerical studies which show that the interaction of solitary waves differ qualitatively at subsonic and supersonic wave speeds, and Coleman et al. [14] presented explicit representations of spatial travelling waves by means of Euler angles. From their formulæ it is apparent that for a solitary wave which is not planar, the twist density

$$\mathbf{b}^3(\boldsymbol{\phi}) \cdot \boldsymbol{\phi}_s = \cos\theta\, \phi_s + \psi_s \tag{7.8}$$

approaches a nonzero constant, and so the term

$$\tfrac{1}{2} K_3 \int_{-\infty}^{\infty} \mathbf{b}^3(\boldsymbol{\phi}(s)) \cdot \boldsymbol{\phi}_s(s)\, ds \tag{7.9}$$

in the potential energy (2.6.1.29) is infinite. Hence the methods developed in this dissertation would need to be adapted in order to analyze the stability properties of nonplanar solitary waves.

Somewhat remarkably, the family of functions that are here studied as solitary wave profiles of our dynamic rod model have previously arisen in several other contexts. First, with the appropriate re-identification of parameters, the solitary wave profiles are also equilibria of an infinite elastica and as such were described by Euler [21]. See also Love [41], pp. 402-405. Second, the family of solitary waves for our dynamical system exactly coincides with that for the sine-Gordon equation (see, e.g. [20]), and stability of the solitary wave family within the context of the much simpler sine-Gordon dynamics has been examined previously [28]. It should be emphasized that there is some subtlety in the necessary re-identification of parameters in the two contexts. For example in the sine-Gordon equation the natural choice of physical parameters leads to a coupling between the amplitude and wave speed of the solitary wave, whereas the amplitude is naturally decoupled from the wave speed in the context of our rod model.

The same family of solution profiles was also found by Konno, Ichikawa & Wadati [37] to be solitary waves for an approximate model of a "stretched rope." Their model was first proposed in [31] as an approximation for the nonlinear transverse oscillation of an elastic beam under tension, and was further studied in [60], [38] and [39]. The basic system that arises is a single partial differential equation whose highest order derivatives are second order in time, and fourth order in space. A stretching is introduced which restricts attention to uni-directional propagation, and it is concluded that the amplitude of the solitary wave decreases with increasing wave speed. This stands in direct contradiction to the conclusions I draw, which discrepancy can be explained by the following crucial differences in modelling. The rod model I have used allows the tension, or force, acting across each cross-section of the rod to vary as a function of both space and time. Consequently the Lagrangian form of our dynamical system comprises a system of three partial differential equations which express balance of two components of linear acceleration with force, and angular acceleration with bending moment. In contrast, in [31] the force is assumed to be constant both in space and time, which seems an inappropriate approximation for large amplitude solitary waves. As a consequence of this last hypothesis their governing system is a single partial differential equation expressing moment balance.

The existence of solitary waves within the rod model studied in this dissertation follows immediately and trivially from a simple phase plane analysis. Thus the focus of my work falls upon the consideration of solitary waves as special solutions of the full dynamical system. This emphasis stands in contrast to that adopted by Antman & Liu [4] who considered the existence of solitary waves in more sophisticated rod models that encompass effects of shear, compression, and viscosity, but who did not discuss issues of stability in the full dynamics. Simo et al. [58] have formulated a Hamiltonian system describing the dynamics of such unconstrained rods. Their formulation of the dynamics for this more intricate constitutive model is in some ways simpler than the system derived here, because there are no constraints. But I find no straightforward generalization of their system to the dynamics of an infinite rod subject to the constraints of inextensibility and unshearability. In any case, a stability analysis of travelling waves in extensible and shearable rods remains an open problem.

Appendix A

THE SPATIAL ELASTICA: EULER ANGLES

A.1 Introduction

In Chapter 6 we formulated the dynamics of three-dimensional motion for an inextensible, unshearable rod in terms of Euler parameters. A parallel development in terms of Euler angles is presented here for comparison. The Hamiltonian system derived in section A.2 is a natural generalization of the planar dynamics involving a single angle obtained in Chapter 5. The invariants of motion and associated symmetries are delineated in section A.3. Finally, the variational analysis of a planar solitary wave profile appearing in sections A.4 and A.5 is directly analogous to the calculations appearing in sections 6.4 and 6.5.

A.2 Hamiltonian Formulation Using Euler Angles

For the three-dimensional dynamics the time derivative of the inextensibility constraint (2.5.6) is given by

$$\mathbf{h}_{st} = (\mathbf{d}_3)_t = \mathsf{D}_3[\boldsymbol{\phi}]^T \boldsymbol{\phi}_t \tag{A.2.1}$$

where the matrix $\mathsf{D}_3[\boldsymbol{\phi}]$ is given by equation (2.6.1.19). Using (A.2.1), the action principle can be expressed as

$$\mathcal{L} \equiv T - V + \int_{-L}^{L} \boldsymbol{\Lambda} \cdot \left\{ \mathbf{h}_{st} - \mathsf{D}_3[\boldsymbol{\phi}]^T \boldsymbol{\phi}_t \right\} \, ds, \tag{A.2.2}$$

where the kinetic energy T is given by equation (2.6.1.27) and the potential energy V is given by equation (2.6.1.29). Once again we define the impetuses

$$\mathbf{g} \equiv \frac{\delta \mathcal{L}}{\delta \mathbf{h}_t} = \rho \mathbf{h}_t - \boldsymbol{\Lambda}_s \tag{A.2.3}$$

and

$$\boldsymbol{\chi} \equiv \frac{\delta \mathcal{L}}{\delta \boldsymbol{\phi}_t} = \mathsf{I}_B[\boldsymbol{\phi}] \cdot \boldsymbol{\phi}_t - \mathsf{D}_3[\boldsymbol{\phi}] \boldsymbol{\Lambda}. \tag{A.2.4}$$

We next invert the relations (A.2.3) and (A.2.4) to obtain $\mathbf{h}_t$ and $\boldsymbol{\phi}_t$ as function of $(\mathbf{g}, \boldsymbol{\phi}, \boldsymbol{\chi}, \boldsymbol{\Lambda})$:

$$\mathbf{h}_t = \rho^{-1}(\mathbf{g} + \boldsymbol{\Lambda}_s) \tag{A.2.5}$$

and

$$\boldsymbol{\phi}_t = \mathsf{I}_B[\boldsymbol{\phi}]^{-1}\left(\boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\boldsymbol{\Lambda}\right), \tag{A.2.6}$$

where

$$\mathsf{I}_B[\boldsymbol{\phi}]^{-1} = \frac{1}{s^2\theta}\begin{pmatrix} I_1^{-1}c^2\psi + I_2^{-1}s^2\psi & (I_2^{-1} - I_1^{-1})s\psi c\psi s\theta & -(I_1^{-1}c^2\psi + I_2^{-1}s^2\psi)c\theta \\ \cdot & (I_1^{-1}s^2\psi + I_2^{-1}c^2\psi)s^2\theta & (I_2^{-1} - I_1^{-1})s\psi c\psi c\theta \\ \cdot & \cdot & \begin{array}{r}(I_1^{-1}c^2\psi + I_2^{-1}s^2\psi)c^2\theta \\ +I_3^{-1}s^2\theta\end{array} \end{pmatrix}. \tag{A.2.7}$$

From equation (A.2.7) we observe that it must be assumed $\sin\theta \neq 0$ for the inverse (A.2.6) of (A.2.4) to be well-defined. This condition is certainly satisfied for planar motion where $\theta = \pi/2$.

Equation (A.2.6) implies that the angular momentum is given by

$$\boldsymbol{\pi} = \mathsf{I}\boldsymbol{\omega} = \mathsf{I}\,\mathsf{B}[\boldsymbol{\phi}]\boldsymbol{\phi}_t = \mathsf{B}[\boldsymbol{\phi}]^{-T}\left[\boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\boldsymbol{\Lambda}\right] = \mathsf{B}[\boldsymbol{\phi}]^{-T}\boldsymbol{\chi} + \mathsf{B}[\boldsymbol{\phi}]^{-T}\mathsf{D}_3[\boldsymbol{\phi}]\boldsymbol{\Lambda}, \tag{A.2.8}$$

in which $\mathsf{B}^{-T} \equiv (\mathsf{B}^T)^{-1}$. Using the definition (2.6.1.19) of $\mathsf{D}_3[\boldsymbol{\phi}]$ one can show that the product $\mathsf{B}[\boldsymbol{\phi}]^{-T}\mathsf{D}_3[\boldsymbol{\phi}]$ has an especially simple form, namely

$$\mathsf{B}[\boldsymbol{\phi}]^{-T}\mathsf{D}_3[\boldsymbol{\phi}] = \begin{pmatrix} -\mathbf{d}_2(\boldsymbol{\phi})^T \\ \mathbf{d}_1(\boldsymbol{\phi})^T \\ \mathbf{0}^T \end{pmatrix}. \tag{A.2.9}$$

We next define the pre-Hamiltonian to be the Legendre transform of $\mathcal{L}$. Due to the specific form of $\mathcal{L}$, the pre-Hamiltonian is the sum of the kinetic and potential energies, expressed in terms of Hamiltonian variables using equations (A.2.5) and (A.2.6). Thus the pre-Hamiltonian is

$$\begin{aligned} M(\mathbf{g}, \boldsymbol{\phi}, \boldsymbol{\chi}, \boldsymbol{\Lambda}) \;&=\; \tfrac{1}{2}\textstyle\int_{-L}^{L} \left\{\rho^{-1}|\mathbf{g} + \boldsymbol{\Lambda}_s|^2 + (\boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\boldsymbol{\Lambda}) \cdot \mathsf{I}_B[\boldsymbol{\phi}]^{-1}\,(\boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\boldsymbol{\Lambda})\right\} ds \\ &\quad + \tfrac{1}{2}\textstyle\int_{-L}^{L} \boldsymbol{\phi}_s \cdot \mathsf{K}_B[\boldsymbol{\phi}]\boldsymbol{\phi}_s \, ds. \end{aligned} \tag{A.2.10}$$

However, in light of equation (A.2.9) it might be simpler to express M as

$$\begin{aligned} M(\mathbf{g}, \boldsymbol{\phi}, \boldsymbol{\chi}, \boldsymbol{\Lambda}) &= \tfrac{1}{2}\textstyle\int_{-L}^{L} \left\{\rho^{-1}|\mathbf{g} + \boldsymbol{\Lambda}_s|^2\right\} ds \\ &\quad + \tfrac{1}{2}\textstyle\int_{-L}^{L} \left\{I_1^{-1}(\mathbf{b}_1(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} - \mathbf{d}_2 \cdot \boldsymbol{\Lambda})^2 + I_2^{-1}(\mathbf{b}_2(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} + \mathbf{d}_1 \cdot \boldsymbol{\Lambda})^2 + I_3^{-1}(\mathbf{b}_3(\boldsymbol{\phi}) \cdot \boldsymbol{\chi})^2\right\} ds \\ &\quad + \tfrac{1}{2}\textstyle\int_{-L}^{L} K_j(\mathbf{b}^j[\boldsymbol{\phi}] \cdot \boldsymbol{\phi}_s)^2, \end{aligned} \tag{A.2.11}$$

where the vectors $\{\mathbf{b}_j(\boldsymbol{\phi})\}$ and $\{\mathbf{b}^j(\boldsymbol{\phi})\}$ are defined in equations (2.6.1.22) and (2.6.1.12) - (2.6.1.14).

The striction $\mathbf{\Lambda}$ is defined to be the minimizer of the pre-Hamiltonian M, and is again denoted by $\tilde{\mathbf{\Lambda}}$. Hence the striction must satisfy the first variation condition

$$\frac{\delta M}{\delta \mathbf{\Lambda}} = -\rho^{-1}(\mathbf{g} + \mathbf{\Lambda}_s)_s + \mathsf{D}_3[\boldsymbol{\phi}]^T \, \mathsf{I}_B[\boldsymbol{\phi}]^{-1} \left(\boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\mathbf{\Lambda}\right) = \mathbf{0} \tag{A.2.12}$$

with the boundary conditions

$$\rho^{-1}(\mathbf{g} + \mathbf{\Lambda}_s)_s = \mathbf{0} \qquad \text{at} \quad s = \pm L. \tag{A.2.13}$$

Here

$$\rho^{-1}(\mathbf{g} + \mathbf{\Lambda}_s)_s = \mathbf{h}_{st} \tag{A.2.14}$$

and

$$\mathsf{D}_3[\boldsymbol{\phi}]^T \, \mathsf{I}_B[\boldsymbol{\phi}]^{-1}(\boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\mathbf{\Lambda}) = \mathsf{D}_3[\boldsymbol{\phi}]^T \, \boldsymbol{\phi}_t = (\mathbf{d}_3)_t, \tag{A.2.15}$$

so the condition (A.2.12) on the striction is simply an enforcement of the inextensibility constraint, as in the planar formulation. The proof of the existence and uniqueness of the striction is virtually the same as that given in section 6.2.

We now define the Hamiltonian

$$\begin{aligned}
H(\mathbf{g}, \boldsymbol{\phi}, \boldsymbol{\chi}) &\equiv M(\mathbf{g}, \boldsymbol{\phi}, \boldsymbol{\chi}, \tilde{\mathbf{\Lambda}}) \\
&= \tfrac{1}{2} \textstyle\int_{-L}^{L} \{\rho^{-1}|\mathbf{g} + \tilde{\mathbf{\Lambda}}_s|^2 + (\boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\mathbf{\Lambda}) \cdot \mathsf{I}_B[\boldsymbol{\phi}]^{-1} \, (\boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\mathbf{\Lambda})\} \, ds \\
&\quad + \tfrac{1}{2} \textstyle\int_{-L}^{L} \boldsymbol{\phi}_s \cdot \mathsf{K}_B[\boldsymbol{\phi}]\boldsymbol{\phi}_s \, ds \\
&= \tfrac{1}{2} \textstyle\int_{-L}^{L} \rho^{-1} \left|\mathbf{g} + \tilde{\mathbf{\Lambda}}_s\right|^2 \, ds \\
&\quad + \tfrac{1}{2} \textstyle\int_{-L}^{L} \left\{ I_1^{-1} \left(\mathbf{b}_1(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} - \mathbf{d}_2 \cdot \tilde{\mathbf{\Lambda}}\right)^2 + I_2^{-1} \left(\mathbf{b}_2(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} + \mathbf{d}_1 \cdot \tilde{\mathbf{\Lambda}}\right)^2 \right\} \, ds \\
&\quad + \tfrac{1}{2} \textstyle\int_{-L}^{L} I_3^{-1} \left(\mathbf{b}_3(\boldsymbol{\phi}) \cdot \boldsymbol{\chi}\right)^2 \, ds \\
&\quad + \tfrac{1}{2} \textstyle\int_{-L}^{L} K_j (\mathbf{b}^j[\boldsymbol{\phi}] \cdot \boldsymbol{\phi}_s)^2,
\end{aligned} \tag{A.2.16}$$

It remains to determine the Hamiltonian dynamical system. It is a straightforward calculation to show

$$\mathbf{g}_t = -\frac{\delta H}{\delta \mathbf{h}} = \mathbf{0}, \tag{A.2.17}$$

$$\mathbf{h}_t = \frac{\delta H}{\delta \mathbf{g}} = \rho^{-1}(\mathbf{g} + \tilde{\mathbf{\Lambda}}_s), \tag{A.2.18}$$

and

$$\begin{aligned}
\phi_t = \frac{\delta H}{\delta \boldsymbol{\chi}} &= \mathsf{I}_B^{-1}[\boldsymbol{\phi}] \left(\boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\tilde{\mathbf{\Lambda}}\right). \\
&= I_1^{-1}\mathbf{b}_1(\boldsymbol{\phi}) \left(\mathbf{b}_1(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} - \mathbf{d}_2 \cdot \tilde{\mathbf{\Lambda}}\right) \\
&\quad + I_2^{-1}\mathbf{b}_2(\boldsymbol{\phi}) \left(\mathbf{b}_2(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} + \mathbf{d}_1 \cdot \tilde{\mathbf{\Lambda}}\right) \\
&\quad + I_3^{-1}\mathbf{b}_3(\boldsymbol{\phi}) \left(\mathbf{b}_3(\boldsymbol{\phi}) \cdot \boldsymbol{\chi}\right)
\end{aligned} \tag{A.2.19}$$

Through an integration by parts we also find

$$\begin{aligned}
\boldsymbol{\chi}_t = -\frac{\delta H}{\delta \boldsymbol{\phi}} &= I_1^{-1}\left(\mathbf{b}_1(\boldsymbol{\phi})\cdot\boldsymbol{\chi} - \mathbf{d}_2\cdot\tilde{\boldsymbol{\Lambda}}\right)\left[\left(\frac{\partial \mathbf{b}_1}{\partial \boldsymbol{\phi}}\right)^T \boldsymbol{\chi} - \mathsf{D}_2(\boldsymbol{\phi})\,\tilde{\boldsymbol{\Lambda}}\right] \\
&\quad + I_2^{-1}\left(\mathbf{b}_2(\boldsymbol{\phi})\cdot\boldsymbol{\chi} + \mathbf{d}_1\cdot\tilde{\boldsymbol{\Lambda}}\right)\left[\left(\frac{\partial \mathbf{b}_2}{\partial \boldsymbol{\phi}}\right)^T \boldsymbol{\chi} + \mathsf{D}_1(\boldsymbol{\phi})\,\tilde{\boldsymbol{\Lambda}}\right] \\
&\quad + K_j\left(\mathbf{b}^j(\boldsymbol{\phi})\cdot\boldsymbol{\phi}_s\right)\left[\left(\frac{\partial \mathbf{b}^j(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}\right)^T - \left(\frac{\partial \mathbf{b}^j(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}\right)\right]\boldsymbol{\phi}_s \\
&\quad - K_j\left[\mathbf{b}^j(\boldsymbol{\phi})\cdot\boldsymbol{\phi}_s\right]_s \mathbf{b}^j(\boldsymbol{\phi}),
\end{aligned} \tag{A.2.20}$$

where

$$\frac{\partial \mathbf{b}_1}{\partial \boldsymbol{\phi}} = \begin{pmatrix} 0 & c\psi c\theta/s^2\theta & s\psi/s\theta \\ 0 & 0 & c\psi \\ 0 & -c\psi/s^2\theta & -s\psi c\theta/s\theta \end{pmatrix}, \tag{A.2.21}$$

$$\frac{\partial \mathbf{b}_2}{\partial \boldsymbol{\phi}} = \begin{pmatrix} 0 & -s\psi c\theta/s^2\theta & c\psi/s\theta \\ 0 & 0 & -s\psi \\ 0 & s\psi/s^2\theta & -c\psi c\theta/s\theta \end{pmatrix}, \tag{A.2.22}$$

$$\frac{\partial \mathbf{b}_3}{\partial \boldsymbol{\phi}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.2.23}$$

$$\frac{\partial \mathbf{b}^1}{\partial \boldsymbol{\phi}} = \begin{pmatrix} 0 & -c\psi c\theta & s\psi s\theta \\ 0 & 0 & c\psi \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.2.24}$$

$$\frac{\partial \mathbf{b}^2}{\partial \boldsymbol{\phi}} = \begin{pmatrix} 0 & s\psi c\theta & c\psi s\theta \\ 0 & 0 & -s\psi \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.2.25}$$

and

$$\frac{\partial \mathbf{b}^1}{\partial \boldsymbol{\phi}} = \begin{pmatrix} 0 & -c\psi c\theta & s\psi s\theta \\ 0 & 0 & c\psi \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.2.26}$$

so

$$\left(\frac{\partial \mathbf{b}^1}{\partial \boldsymbol{\phi}}\right) - \left(\frac{\partial \mathbf{b}^1}{\partial \boldsymbol{\phi}}\right)^T = \mathbf{b}^2(\boldsymbol{\phi})\otimes\mathbf{b}^3(\boldsymbol{\phi}) - \mathbf{b}^3(\boldsymbol{\phi})\otimes\mathbf{b}^2(\boldsymbol{\phi}), \tag{A.2.27}$$

$$\left(\frac{\partial \mathbf{b}^2}{\partial \boldsymbol{\phi}}\right) - \left(\frac{\partial \mathbf{b}^2}{\partial \boldsymbol{\phi}}\right)^T = \mathbf{b}^3(\boldsymbol{\phi})\otimes\mathbf{b}^1(\boldsymbol{\phi}) - \mathbf{b}^1(\boldsymbol{\phi})\otimes\mathbf{b}^3(\boldsymbol{\phi}), \tag{A.2.28}$$

and

$$\left(\frac{\partial \mathbf{b}^3}{\partial \boldsymbol{\phi}}\right) - \left(\frac{\partial \mathbf{b}^3}{\partial \boldsymbol{\phi}}\right)^T = \mathbf{b}^1(\boldsymbol{\phi})\otimes\mathbf{b}^2(\boldsymbol{\phi}) - \mathbf{b}^2(\boldsymbol{\phi})\otimes\mathbf{b}^1(\boldsymbol{\phi}). \tag{A.2.29}$$

Notice from equation (2.6.1.22) that $\mathbf{b}_3(\boldsymbol{\phi}) = \mathbf{e}_3$, so equations (A.2.21) and (A.2.22) imply

$$\frac{\partial \mathbf{b}_1}{\partial \boldsymbol{\phi}} \mathbf{b}_3(\boldsymbol{\phi}) = \quad \mathbf{b}_2(\boldsymbol{\phi}) \tag{A.2.30}$$

and

$$\frac{\partial \mathbf{b}_2}{\partial \boldsymbol{\phi}} \mathbf{b}_3(\boldsymbol{\phi}) = -\mathbf{b}_1(\boldsymbol{\phi}). \tag{A.2.31}$$

There does not seem to be a simple representation of the vectors

$$\mathbf{c}_{jk}(\boldsymbol{\phi}) \equiv \frac{\partial \mathbf{b}_j}{\partial \boldsymbol{\phi}} \mathbf{b}_k(\boldsymbol{\phi}) \tag{A.2.32}$$

in terms of the basis $\{\mathbf{b}^j(\boldsymbol{\phi})\}$ when neither j nor k is 3, although the relationship

$$\mathbf{c}_{21}(\boldsymbol{\phi}) = \mathbf{c}_{12}(\boldsymbol{\phi}) + \mathbf{b}_3(\boldsymbol{\phi}) \tag{A.2.33}$$

is useful in our variational analysis. By contrast, in Euler parameters the basis vectors $\mathsf{B}_j\boldsymbol{\beta}$ are linear in $\boldsymbol{\beta}$, so the direct analog of equations (A.2.30) - (A.2.32) is the relatively simple commutation relation (2.6.2.41).

It is convenient for our variational analysis to decompose $\delta H/\delta\boldsymbol{\chi}$ into its components with respect to the basis $\mathbf{b}_j(\boldsymbol{\phi})$. By taking inner products of $(A.2.19)$ with the vectors $\mathbf{b}^j(\boldsymbol{\phi})$ and utilizing the identity (2.6.1.20) for dual bases, we find

$$\frac{\delta H}{\delta \boldsymbol{\chi}} \cdot \mathbf{b}^1(\boldsymbol{\phi}) = I_1^{-1}\left[\mathbf{b}_1(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} - \mathbf{d}_2 \cdot \tilde{\boldsymbol{\Lambda}}\right], \tag{A.2.34}$$

$$\frac{\delta H}{\delta \boldsymbol{\chi}} \cdot \mathbf{b}^2(\boldsymbol{\phi}) = I_2^{-1}\left[\mathbf{b}_2(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} + \mathbf{d}_1 \cdot \tilde{\boldsymbol{\Lambda}}\right], \tag{A.2.35}$$

$$\frac{\delta H}{\delta \boldsymbol{\chi}} \cdot \mathbf{b}^3(\boldsymbol{\phi}) = I_3^{-1}\,\mathbf{b}_3(\boldsymbol{\phi}) \cdot \boldsymbol{\chi}. \tag{A.2.36}$$

In the same way we can express $\delta H/\delta\boldsymbol{\phi}$ with respect to the basis $\mathbf{b}^j(\boldsymbol{\phi})$ by taking inner products of $(A.2.20)$ with $\mathbf{b}_j(\boldsymbol{\phi})$. Exploiting equations (A.2.27) – (A.2.29) and (A.2.30) – (A.2.32) together with the definitions (2.6.1.17) and (2.6.1.18) of $\mathsf{D}_1[\boldsymbol{\phi}]$ and $\mathsf{D}_2[\boldsymbol{\phi}]$ and the identity (2.6.1.20), we find

$$\begin{aligned}\frac{\delta H}{\delta \boldsymbol{\phi}} \cdot \mathbf{b}_1(\boldsymbol{\phi}) = {} & I_1^{-1}\left[\mathbf{b}_1(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} - \mathbf{d}_2(\boldsymbol{\phi}) \cdot \tilde{\boldsymbol{\Lambda}}\right]\left[\mathbf{c}_{11}(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} - \mathbf{d}_3(\boldsymbol{\phi}) \cdot \tilde{\boldsymbol{\Lambda}}\right] \\ & + I_2^{-1}\left[\mathbf{b}_2(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} + \mathbf{d}_1(\boldsymbol{\phi}) \cdot \tilde{\boldsymbol{\Lambda}}\right]\left[\mathbf{c}_{21}(\boldsymbol{\phi}) \cdot \boldsymbol{\chi}\right] \\ & + (K_2 - K_3)\left[\mathbf{b}^2(\boldsymbol{\phi}) \cdot \boldsymbol{\phi}_s\right]\left[\mathbf{b}^3(\boldsymbol{\phi}) \cdot \boldsymbol{\phi}_s\right] \\ & - K_1\left[\mathbf{b}^1(\boldsymbol{\phi}) \cdot \boldsymbol{\phi}_s\right]_s,\end{aligned} \tag{A.2.37}$$

$$\begin{aligned}\frac{\delta H}{\delta \boldsymbol{\phi}} \cdot \mathbf{b}_2(\boldsymbol{\phi}) = {} & I_1^{-1}\left[\mathbf{b}_1(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} - \mathbf{d}_2(\boldsymbol{\phi}) \cdot \tilde{\boldsymbol{\Lambda}}\right]\left[\mathbf{c}_{12}(\boldsymbol{\phi}) \cdot \boldsymbol{\chi}\right] \\ & + I_2^{-1}\left[\mathbf{b}_2(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} + \mathbf{d}_1(\boldsymbol{\phi}) \cdot \tilde{\boldsymbol{\Lambda}}\right]\left[\mathbf{c}_{22}(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} + \mathbf{d}_3(\boldsymbol{\phi}) \cdot \tilde{\boldsymbol{\Lambda}}\right] \\ & + (K_3 - K_1)\left[\mathbf{b}^3(\boldsymbol{\phi}) \cdot \boldsymbol{\phi}_s\right]\left[\mathbf{b}^1(\boldsymbol{\phi}) \cdot \boldsymbol{\phi}_s\right] \\ & - K_2\left[\mathbf{b}^2(\boldsymbol{\phi}) \cdot \boldsymbol{\phi}_s\right]_s,\end{aligned} \tag{A.2.38}$$

and

$$\begin{aligned}
\frac{\delta H}{\delta \boldsymbol{\phi}} \cdot \mathbf{b}_3(\boldsymbol{\phi}) &= I_1^{-1}\left[\mathbf{b}_1(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} - \mathbf{d}_2(\boldsymbol{\phi}) \cdot \tilde{\boldsymbol{\Lambda}}\right]\left[\mathbf{b}^2(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} + \mathbf{d}_1(\boldsymbol{\phi}) \cdot \tilde{\boldsymbol{\Lambda}}\right] \\
&\quad + I_2^{-1}\left(\mathbf{b}_2(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} + \mathbf{d}_1 \cdot \tilde{\boldsymbol{\Lambda}}\right)\left[-\mathbf{b}_1(\boldsymbol{\phi}) \cdot \boldsymbol{\chi} - \mathbf{d}_2(\boldsymbol{\phi}) \cdot \tilde{\boldsymbol{\Lambda}}\right] \\
&\quad + (K_1 - K_2)\left[\mathbf{b}^1(\boldsymbol{\phi}) \cdot \boldsymbol{\phi}_s\right]\left[\mathbf{b}^2(\boldsymbol{\phi}) \cdot \boldsymbol{\phi}_s\right] \\
&\quad - K_3\left[\mathbf{b}^3(\boldsymbol{\phi}) \cdot \boldsymbol{\phi}_s\right]_s . \qquad \text{(A.2.39)}
\end{aligned}$$

One advantage of formulating the dynamics using Euler angles is that the Hamiltonian system takes a particularly simple form in the isotropic case, where $I_1 = I_2$ and $K_1 = K_2$. The matrix D_3 (2.6.1.19) is always independent of the angle ψ, but in the isotropic case the matrices I_B and K_B are also independent of ψ. This is attributable to the fact that the angle ψ describes the rotational symmetry about the axis $\mathbf{d}_3(\boldsymbol{\phi})$. Specifically,

$$\mathsf{I}_B = \begin{pmatrix} I_1 s^2\theta + I_3 c^2\theta & 0 & I_3 c\theta \\ \cdot & I_1 & 0 \\ \cdot & \cdot & I_3 \end{pmatrix}, \qquad \text{(A.2.40)}$$

and

$$\mathsf{K}_B = \begin{pmatrix} K_1 s^2\theta + K_3 c^2\theta & 0 & K_3 c\theta \\ \cdot & K_1 & 0 \\ \cdot & \cdot & K_3 \end{pmatrix}. \qquad \text{(A.2.41)}$$

The kinetic and potential energies in this case are

$$T = \tfrac{1}{2}\textstyle\int_{-L}^{L} I_1(\sin^2\theta\,\phi_t^2 + \theta_t^2) + I_3(\cos\theta\,\phi_t + \psi_t)^2 \qquad \text{(A.2.42)}$$

and

$$V = \tfrac{1}{2}\textstyle\int_{-L}^{L} I_1(\sin^2\theta\,\phi_s^2 + \theta_s^2) + I_3(\cos\theta\,\phi_s + \psi_s)^2, \qquad \text{(A.2.43)}$$

so

$$\begin{aligned}
\frac{\delta V}{\delta \phi} &= -K_1\left(\sin^2\theta\,\phi_s\right)_s - K_3\left\{\cos\theta(\cos\theta\,\phi_s + \psi_s)\right\}_s, \\
\frac{\delta V}{\delta \theta} &= K_1\left(\sin\theta\cos\theta\,\phi_s^2 - \theta_{ss}\right) - K_3\left\{\sin\theta(\cos\theta\,\phi_s + \psi_s)\phi_s\right\}, \\
\frac{\delta V}{\delta \psi} &= -K_3(\cos\theta\,\phi_s + \psi_s)_s. \qquad \text{(A.2.44)}
\end{aligned}$$

The equation for $(\chi_3)_t$ is therefore especially simple, namely

$$(\chi_3)_t = K_3(c\theta\,\phi_s + \psi_s)_s, \qquad \text{(A.2.45)}$$

which is in the form of a conservation law.

In the reduced system, the angles ϕ, θ, ψ are subject to the constraints

$$\int_{-\infty}^{\infty}(1 - x_s)\,ds = \int_{-\infty}^{\infty}(1 - \sin\theta\cos\phi)\,ds = 4a, \qquad \text{(A.2.46)}$$

$$\int_{-\infty}^{\infty} y_s\, ds = \int_{-\infty}^{\infty} \sin\theta \sin\phi\, ds = 0, \tag{A.2.47}$$

and

$$\int_{-\infty}^{\infty} z_s\, ds = \cos\theta\, ds = 0, \tag{A.2.48}$$

A.3 Hamiltonian Form of Conservation Laws

Henceforth in this appendix we assume the rod is infinite and transversely isotropic.

In analogy with the invariant (5.4.12), we again have an infinite set of invariants

$$C(\mathbf{z}) \equiv \int_{-\infty}^{\infty} \mathbf{\Lambda}^*(s) \cdot [\mathbf{h}_s(s,t) - \mathbf{d}_3(s,t) + \mathbf{e}_1] \tag{A.3.1}$$

associated with the inextensibility condition. The first variation of C is

$$\frac{\delta C}{\delta \mathbf{z}} = (-(\mathbf{\Lambda}^*)_s, -\left(\frac{\partial \mathbf{d}_3}{\partial \boldsymbol{\phi}}\right)^T \mathbf{\Lambda}^*, \mathbf{0}, \mathbf{0}) = (-(\mathbf{\Lambda}^*)_s, -\mathsf{D}_3[\boldsymbol{\phi}]\,\mathbf{\Lambda}^*, \mathbf{0}, \mathbf{0}), \tag{A.3.2}$$

and the generator of the associated gauge symmetry is

$$\mathbf{J}\frac{\delta C}{\delta \mathbf{z}} = (\mathbf{0}, \mathbf{0}, (\mathbf{\Lambda}^*)_s, \mathsf{D}_3[\boldsymbol{\phi}]\,\mathbf{\Lambda}^*). \tag{A.3.3}$$

As in the planar case, the translational symmetry of the Hamiltonian with respect to arc-length implies the invariance of the linear impulse, given by

$$I \equiv \int_{-\infty}^{\infty} \{\mathbf{g} \cdot \mathbf{h}_s + \boldsymbol{\chi} \cdot \boldsymbol{\phi}_s\}\, ds. \tag{A.3.4}$$

Using the inextensibility condition (2.5.6) and the definitions of $\mathbf{g}$ (A.2.3) and $\boldsymbol{\chi}$ (A.2.4) we can rewrite the impulse given by (A.3.4) in the form given in Chapter 3:

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \left\{\left(\mathbf{g} + \tilde{\mathbf{\Lambda}}_s\right) \cdot \mathbf{h}_s + \left(\boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\,\tilde{\mathbf{\Lambda}}\right) \cdot \boldsymbol{\phi}_s\right\} ds \\ &= \int_{-\infty}^{\infty} \left\{\rho \mathbf{h}_t \cdot \mathbf{h}_s + \mathsf{B}[\boldsymbol{\phi}]^T\, \mathsf{IB}[\boldsymbol{\phi}]\,\boldsymbol{\phi}_t \cdot \boldsymbol{\phi}_s\right\} ds \\ &= \int_{-\infty}^{\infty} \{\mathbf{p} \cdot \mathbf{h}_s + \boldsymbol{\pi} \cdot \mathbf{u}\}\, ds \end{aligned} \tag{A.3.5}$$

The first variation of I is

$$\frac{\delta I}{\delta \mathbf{z}} = (-\mathbf{g}_s, -\chi_s, \mathbf{h}_s, \boldsymbol{\phi}_s). \tag{A.3.6}$$

The invariance of the linear impulse is associated with the translational symmetry of the rod with respect to the arc-length s. The infinitesimal generator of this symmetry is

$$\mathbf{J}\frac{\delta I}{\delta \mathbf{z}} = (\mathbf{h}_s, \boldsymbol{\phi}_s, \mathbf{g}_s, \boldsymbol{\chi}_s). \tag{A.3.7}$$

Using the expression (A.2.8) for the angular momentum together with the fact that

$$\mathsf{B}[\boldsymbol{\phi}]^{-1}\,\mathbf{e}_3 = \mathbf{b}_3(\boldsymbol{\phi}) = \mathbf{e}_3 \tag{A.3.8}$$

as a consequence of equations (2.6.1.21) and (2.6.1.22), the isotropic angular momentum A takes an extremely simple form in terms of the Hamiltonian state variables, namely

$$\begin{aligned} A \equiv \int_{-\infty}^{\infty} \mathsf{I}\boldsymbol{\omega}\cdot\mathbf{e}_3 &= \int_{-\infty}^{\infty} \mathbf{e}_3\cdot \mathsf{B}[\boldsymbol{\phi}]^{-T}\left(\boldsymbol{\chi}+\mathsf{D}_3[\boldsymbol{\phi}]\,\tilde{\boldsymbol{\Lambda}}\right)\,ds \\ &= \int_{-\infty}^{\infty} \mathbf{b}_3(\boldsymbol{\phi})\cdot\left(\boldsymbol{\chi}+\mathsf{D}_3[\boldsymbol{\phi}]\,\tilde{\boldsymbol{\Lambda}}\right)\,ds \\ &= \int_{-\infty}^{\infty} \mathbf{e}_3\cdot(\boldsymbol{\chi}+\mathsf{D}_3[\boldsymbol{\phi}]\,\tilde{\boldsymbol{\Lambda}})\,ds \\ &= \int_{-\infty}^{\infty} \mathbf{e}_3\cdot\boldsymbol{\chi}\,ds, \end{aligned} \tag{A.3.9}$$

where the last equality follows from equation (2.6.1.23). The first variation of A is

$$\frac{\delta A}{\delta \mathbf{z}} = (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{e}_3). \tag{A.3.10}$$

That A in its Hamiltonian form is conserved can now be seen readily from the conservation law (A.2.45) found in the last section. We reach the same conclusion through Noether's Theorem by noting that the Hamiltonian system for the isotropic rod is invariant under the transformation $\psi \to \psi + \epsilon$, whose infinitesimal generator is

$$\mathbf{J}\frac{\delta A}{\delta \mathbf{z}} = (\mathbf{0},\mathbf{e}_3,\mathbf{0},\mathbf{0}). \tag{A.3.11}$$

We denote the total angular momentum about the origin by

$$\boldsymbol{\Omega} \equiv \int_{-\infty}^{\infty} \{\boldsymbol{\pi}+\mathbf{r}\times\mathbf{p}\}\,ds \tag{A.3.12}$$

and let $\Omega_k^o \equiv \boldsymbol{\Omega}\cdot\mathbf{e}_k$. In terms of the Hamiltonian variables

$$\begin{aligned} \boldsymbol{\Omega}^o &= \int_{-\infty}^{\infty} \left\{(\mathsf{B}_o)^{-T}[\boldsymbol{\phi}]\left(\boldsymbol{\chi}+\mathsf{D}_3[\boldsymbol{\phi}]\tilde{\boldsymbol{\Lambda}}\right)+(\mathbf{h}+s\mathbf{e}_1)\times(\mathbf{g}+\tilde{\boldsymbol{\Lambda}}_s)\right\}\,ds \\ &= \int_{-\infty}^{\infty} \left\{(\mathsf{B}_o)^{-T}[\boldsymbol{\phi}]\boldsymbol{\chi}+\mathbf{d}_3\times\tilde{\boldsymbol{\Lambda}}+(\mathbf{h}+s\mathbf{e}_1)\times(\mathbf{g}+\tilde{\boldsymbol{\Lambda}})\right\}\,ds \end{aligned} \tag{A.3.13}$$

which may be rewritten through an integration by parts and the inextensibility condition as

$$\begin{aligned} \boldsymbol{\Omega}^o &= \int_{-\infty}^{\infty} \left\{(\mathsf{B}_o[\boldsymbol{\phi}])^{-T}\boldsymbol{\chi}+(\mathbf{h}+s\mathbf{e}_1)\times\mathbf{g}+(\mathbf{d}_3-\mathbf{e}_1-\mathbf{h}_s)\times\tilde{\boldsymbol{\Lambda}}\right\}\,ds \\ &= \int_{-\infty}^{\infty} \left\{(\mathsf{B}_o[\boldsymbol{\phi}])^{-T}\boldsymbol{\chi}+(\mathbf{h}+s\mathbf{e}_1)\times\mathbf{g}\right\}\,ds. \end{aligned} \tag{A.3.14}$$

We define the vectors

$$\mathbf{b}_j^o(\boldsymbol{\phi}) \equiv \mathsf{B}^o[\boldsymbol{\phi}]^{-T}\,\mathbf{e}_j \tag{A.3.15}$$

in analogy with the vectors $\mathsf{B}_j^o\boldsymbol{\beta}$ used for Euler parameters, so that the components of angular momentum can be written as

$$\Omega_k^o = \int_{-\infty}^{\infty} \left\{\mathbf{b}_j^o(\boldsymbol{\phi})\cdot\boldsymbol{\gamma}+[(\mathbf{h}+s\mathbf{e}_1)\times\mathbf{g}]\cdot\mathbf{e}_k\right\}\,ds. \tag{A.3.16}$$

The first variation of Ω_k^o is therefore

$$\frac{\delta \Omega_k^o}{\delta \mathbf{z}} = \left(-\mathbf{e}_k \times \mathbf{g},\, -\left(\frac{\partial \mathbf{b}_j^o}{\partial \boldsymbol{\phi}}\right)^T \boldsymbol{\gamma},\, \mathbf{e}_k \times (\mathbf{h} + s\mathbf{e}_1),\, \mathbf{b}_j^o(\boldsymbol{\phi}) \right). \tag{A.3.17}$$

The associated infinitesimal generator is

$$\mathbf{J}\frac{\delta \Omega_k^o}{\delta \mathbf{z}} = \left(\mathbf{e}_k \times (\mathbf{h} + s\mathbf{e}_1),\, \mathbf{b}_j^o(\boldsymbol{\phi}),\, \mathbf{e}_k \times \mathbf{g},\, \left(\frac{\partial \mathbf{b}_j^o}{\partial \boldsymbol{\phi}}\right)^T \boldsymbol{\gamma} \right), \tag{A.3.18}$$

which generates the rotation about the $\mathbf{e}_k$ axis.

The transverse components of linear momentum

$$P_2^o \equiv \int_{-\infty}^{\infty} \rho \mathbf{h}_t \cdot \mathbf{e}_2 \, ds \tag{A.3.19}$$

and

$$P_3^o \equiv \int_{-\infty}^{\infty} \rho \mathbf{h}_t \cdot \mathbf{e}_3 \, ds \tag{A.3.20}$$

are also conserved. In terms of the Hamiltonian variables, these can be written as

$$\begin{aligned} P_2^o &= \int_{-\infty}^{\infty} (\mathbf{g} + \tilde{\mathbf{\Lambda}}_s) \cdot \mathbf{e}_2 \, ds \\ &= \int_{-\infty}^{\infty} \mathbf{g} \cdot \mathbf{e}_2 \, ds + \tilde{\mathbf{\Lambda}} \cdot \mathbf{e}_2 |_{-\infty}^{\infty} \end{aligned} \tag{A.3.21}$$

and

$$P_3^o = \int_{-\infty}^{\infty} \mathbf{g} \cdot \mathbf{e}_3 \, ds + \tilde{\mathbf{\Lambda}} \cdot \mathbf{e}_3 |_{-\infty}^{\infty}. \tag{A.3.22}$$

The first variations therefore are

$$\frac{\delta P_k^o}{\delta \mathbf{z}} = (\mathbf{0}, \mathbf{0}, \mathbf{e}_k, \mathbf{0}) \qquad \text{for } k = 2, 3, \tag{A.3.23}$$

which correspond to the generators

$$\mathbf{J}\frac{\delta P_k^o}{\delta \mathbf{z}} = (\mathbf{e}_k, \mathbf{0}, \mathbf{0}, \mathbf{0}) \tag{A.3.24}$$

of the translational symmetry groups $(\mathbf{h} + \epsilon \mathbf{e}_k, 0, 0, 0)$. The striction is unchanged under these symmetries.

The commutation relations among these invariants are the same as those described in section 6.3. For the reduced system, provided we choose the linear impulse $\mathbf{g}$ to be constant in s as we as t, the associated linear impulse

$$I_{\text{red}} \equiv \int_{-\infty}^{\infty} \boldsymbol{\chi} \cdot \boldsymbol{\phi}_s \, ds \tag{A.3.25}$$

remains a conserved quantity.

A.4 A Variational Characterization of Solitary Waves

Once again, we determine the extremals of the invariant $F = H + cI$ and show that they are the travelling waves solutions. The development is essentially the same as in section 5.5, so we only outline the argument in this section. The equations for an extremal of F are

$$\mathbf{0} = \frac{\delta F}{\delta \mathbf{h}} = -c\bar{\mathbf{g}}', \tag{A.4.1}$$

$$\mathbf{0} = \frac{\delta F}{\delta \mathbf{g}} = \rho^{-1}(\bar{\mathbf{g}} + \bar{\mathbf{\Lambda}}') + c\bar{\mathbf{h}}', \tag{A.4.2}$$

$$0 = \frac{\delta F}{\delta \boldsymbol{\chi}} \cdot \bar{\mathbf{b}}^1 = I_1^{-1}\left(\bar{\mathbf{b}}_1 \cdot \boldsymbol{\chi} - \bar{\mathbf{d}}_2 \cdot \bar{\mathbf{\Lambda}}\right) + c\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}', \tag{A.4.3}$$

$$0 = \frac{\delta F}{\delta \boldsymbol{\chi}} \cdot \bar{\mathbf{b}}^2 = I_2^{-1}\left(\bar{\mathbf{b}}_2 \cdot \boldsymbol{\chi} + \bar{\mathbf{d}}_1 \cdot \bar{\mathbf{\Lambda}}\right) + c\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}', \tag{A.4.4}$$

$$0 = \frac{\delta F}{\delta \boldsymbol{\chi}} \cdot \bar{\mathbf{b}}^3 = I_3^{-1}\,\bar{\mathbf{b}}_3 \cdot \boldsymbol{\chi} + c\bar{\mathbf{b}}^3 \cdot \bar{\boldsymbol{\phi}}', \tag{A.4.5}$$

$$\begin{aligned}
0 = \frac{\delta F}{\delta \boldsymbol{\phi}} \cdot \bar{\mathbf{b}}_1 \quad = \quad & I_1^{-1}\left(\bar{\mathbf{b}}_1 \cdot \boldsymbol{\chi} - \bar{\mathbf{d}}_2 \cdot \bar{\mathbf{\Lambda}}\right)\left(\bar{\mathbf{c}}_{11} \cdot \boldsymbol{\chi} - \bar{\mathbf{d}}_3\,\bar{\mathbf{\Lambda}}\right) \\
& + I_2^{-1}\left(\bar{\mathbf{b}}_2 \cdot \boldsymbol{\chi} + \bar{\mathbf{d}}_1 \cdot \bar{\mathbf{\Lambda}}\right)\left(\bar{\mathbf{c}}_{21} \cdot \boldsymbol{\chi}\right) \\
& + (K_2 - K_3)\left(\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{b}}^3 \cdot \bar{\boldsymbol{\phi}}'\right) \\
& - K_1\left(\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}'\right)' \\
& - c\bar{\mathbf{b}}_1 \cdot \boldsymbol{\chi}',
\end{aligned} \tag{A.4.6}$$

$$\begin{aligned}
0 = \frac{\delta F}{\delta \boldsymbol{\phi}} \cdot \bar{\mathbf{b}}_2 \quad = \quad & I_1^{-1}\left(\bar{\mathbf{b}}_1 \cdot \boldsymbol{\chi} - \bar{\mathbf{d}}_2 \cdot \bar{\mathbf{\Lambda}}\right)\left(\bar{\mathbf{c}}_{12} \cdot \boldsymbol{\chi}\right) \\
& + I_2^{-1}\left(\bar{\mathbf{b}}_2 \cdot \boldsymbol{\chi} + \bar{\mathbf{d}}_1 \cdot \bar{\mathbf{\Lambda}}\right)\left(\bar{\mathbf{c}}_{22} \cdot \boldsymbol{\chi} + \bar{\mathbf{d}}_3\,\bar{\mathbf{\Lambda}}\right) \\
& + (K_3 - K_1)\left(\bar{\mathbf{b}}^3 \cdot \bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}'\right) \\
& - K_2\left(\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}'\right)' \\
& - c\bar{\mathbf{b}}_2 \cdot \boldsymbol{\chi}',
\end{aligned} \tag{A.4.7}$$

and

$$\begin{aligned}
0 = \frac{\delta F}{\delta \boldsymbol{\phi}} \cdot \bar{\mathbf{b}}_3 \quad = \quad & I_1^{-1}\left(\bar{\mathbf{b}}_1 \cdot \boldsymbol{\chi} - \bar{\mathbf{d}}_2 \cdot \bar{\mathbf{\Lambda}}\right)\left(\boldsymbol{\chi} \cdot \bar{\mathbf{b}}^2 + \bar{\mathbf{d}}_1 \cdot \bar{\mathbf{\Lambda}}\right) \\
& + I_2^{-1}\left(\bar{\mathbf{b}}_2 \cdot \boldsymbol{\chi} + \bar{\mathbf{d}}_1 \cdot \bar{\mathbf{\Lambda}}\right)\left(-\boldsymbol{\chi} \cdot \bar{\mathbf{b}}_1 - \bar{\mathbf{d}}_2 \cdot \bar{\mathbf{\Lambda}}\right) \\
& + (K_1 - K_2)\left(\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}'\right) \\
& - K_3\left(\bar{\mathbf{b}}^3 \cdot \bar{\boldsymbol{\phi}}'\right)' \\
& - c\bar{\mathbf{b}}_3 \cdot \boldsymbol{\chi}',
\end{aligned} \tag{A.4.8}$$

In addition we have the equation governing $\bar{\mathbf{\Lambda}}$,

$$\mathbf{0} = -\rho^{-1}\left(\bar{\mathbf{g}} + \bar{\mathbf{\Lambda}}'\right)' + \mathsf{D}_3[\bar{\boldsymbol{\phi}}]^T\,\mathsf{B}[\bar{\boldsymbol{\phi}}]^{-1}\mathsf{I}^{-1}\mathsf{B}[\bar{\boldsymbol{\phi}}]^{-T}\left(\bar{\boldsymbol{\chi}} + \mathsf{D}_3[\bar{\boldsymbol{\phi}}]\bar{\mathbf{\Lambda}}\right). \tag{A.4.9}$$

From equation (A.4.1) we see that $\bar{\mathbf{g}}$ is a constant. By combining equations (A.4.2), (A.4.3) - (A.4.5) and (A.4.9) we obtain

$$c\bar{\mathbf{h}}'' - c\mathsf{D}_3^T\bar{\boldsymbol{\phi}}' = c(\bar{\mathbf{h}}' - \bar{\mathbf{d}}_3)' = 0, \tag{A.4.10}$$

which together with boundary conditions implies

$$\bar{\mathbf{h}}' = \bar{\mathbf{d}}_3 - \mathbf{e}_1, \tag{A.4.11}$$

We next solve equation (A.4.2) for $\bar{\mathbf{\Lambda}}'$ and exploit (A.4.11) to obtain

$$\bar{\mathbf{\Lambda}}' = -\bar{\mathbf{g}} - c\rho(\bar{\mathbf{d}}_3 - \mathbf{e}_1). \tag{A.4.12}$$

Thus both $\bar{\mathbf{h}}$ and $\bar{\mathbf{\Lambda}}$ are determined when we find $\bar{\mathbf{g}}$ and $\bar{\boldsymbol{\phi}}$.

Equations (A.4.3) - (A.4.5) can be solved to obtain

$$\begin{aligned}
\bar{\mathbf{b}}_1\cdot\bar{\boldsymbol{\chi}} &= -cI_1\bar{\mathbf{b}}^1\cdot\bar{\boldsymbol{\phi}}' + \bar{\mathbf{d}}_2\cdot\bar{\mathbf{\Lambda}}, && \text{(A.4.13)}\\
\bar{\mathbf{b}}_2\cdot\bar{\boldsymbol{\chi}} &= -cI_2\bar{\mathbf{b}}^2\cdot\bar{\boldsymbol{\phi}}' - \bar{\mathbf{d}}_1\cdot\bar{\mathbf{\Lambda}}, && \text{(A.4.14)}\\
\bar{\mathbf{b}}_3\cdot\bar{\boldsymbol{\chi}} &= -cI_3\bar{\mathbf{b}}^3\cdot\bar{\boldsymbol{\phi}}'. && \text{(A.4.15)}
\end{aligned}$$

Equations (A.4.13) – (A.4.15) can next be exploited to rewrite equations (A.4.6) – (A.4.8) more simply as

$$\begin{aligned}
c\bar{\mathbf{b}}_1\cdot\bar{\boldsymbol{\chi}}' = \; & -cI_1\left(\bar{\mathbf{b}}^1\cdot\bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{c}}_{11}\cdot\bar{\boldsymbol{\chi}} - \bar{\mathbf{d}}_3\,\bar{\mathbf{\Lambda}}\right)\\
& -cI_2\left(\bar{\mathbf{b}}^2\cdot\bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{c}}_{21}\cdot\bar{\boldsymbol{\chi}}\right)\\
& +(K_2 - K_3)\left(\bar{\mathbf{b}}^2\cdot\bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{b}}^3\cdot\bar{\boldsymbol{\phi}}'\right)\\
& -K_1\left(\bar{\mathbf{b}}^1\cdot\bar{\boldsymbol{\phi}}'\right)', && \text{(A.4.16)}
\end{aligned}$$

$$\begin{aligned}
c\bar{\mathbf{b}}_2\cdot\bar{\boldsymbol{\chi}}' = \; & -cI_1\left(\bar{\mathbf{b}}^1\cdot\bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{c}}_{12}\cdot\bar{\boldsymbol{\chi}}\right)\\
& -cI_2\left(\bar{\mathbf{b}}^2\cdot\bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{c}}_{22}\cdot\bar{\boldsymbol{\chi}} + \bar{\mathbf{d}}_3\,\bar{\mathbf{\Lambda}}\right)\\
& +(K_3 - K_1)\left(\bar{\mathbf{b}}^3\cdot\bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{b}}^1\cdot\bar{\boldsymbol{\phi}}'\right)\\
& -K_2\left(\bar{\mathbf{b}}^2\cdot\bar{\boldsymbol{\phi}}'\right)', && \text{(A.4.17)}
\end{aligned}$$

and

$$\begin{aligned}
c\bar{\mathbf{b}}_3\cdot\bar{\boldsymbol{\chi}}' = \; & -cI_1\left(\bar{\mathbf{b}}^1\cdot\bar{\boldsymbol{\phi}}'\right)\left(\bar{\boldsymbol{\chi}}\cdot\bar{\mathbf{b}}^2 + \bar{\mathbf{d}}_1\cdot\bar{\mathbf{\Lambda}}\right)\\
& -cI_2\left(\bar{\mathbf{b}}^2\cdot\bar{\boldsymbol{\phi}}'\right)\left(-\bar{\boldsymbol{\chi}}\cdot\bar{\mathbf{b}}_1 - \bar{\mathbf{d}}_2\cdot\bar{\mathbf{\Lambda}}\right)\\
& +(K_1 - K_2)\left(\bar{\mathbf{b}}^1\cdot\bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{b}}^2\cdot\bar{\boldsymbol{\phi}}'\right)\\
& -K_3\left(\bar{\mathbf{b}}^3\cdot\bar{\boldsymbol{\phi}}'\right)'. && \text{(A.4.18)}
\end{aligned}$$

On the other hand one can differentiate equations (A.4.13) - (A.4.15) to obtain

$$
\begin{aligned}
c\bar{\mathbf{b}}_1 \cdot \boldsymbol{\chi}' &= -c\left(\bar{\mathbf{b}}_1\right)' \cdot \bar{\boldsymbol{\chi}}' - c^2 I_1 \left(\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}'\right)' \\
&\quad + \bar{\boldsymbol{\phi}}' \cdot \mathsf{D}_2[\bar{\boldsymbol{\phi}}]\bar{\boldsymbol{\Lambda}} + \bar{\mathbf{d}}_2 \cdot \bar{\boldsymbol{\Lambda}}', && \text{(A.4.19)} \\
c\bar{\mathbf{b}}_2 \cdot \boldsymbol{\chi}' &= -c\left(\bar{\mathbf{b}}_2\right)' \cdot \bar{\boldsymbol{\chi}}' - c^2 I_2 \left(\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}'\right)' \\
&\quad - \bar{\boldsymbol{\phi}}' \cdot \mathsf{D}_1[\bar{\boldsymbol{\phi}}]\bar{\boldsymbol{\Lambda}} - \bar{\mathbf{d}}_1 \cdot \bar{\boldsymbol{\Lambda}}', && \text{(A.4.20)} \\
c\bar{\mathbf{b}}_3 \cdot \boldsymbol{\chi}' &= -c\left(\bar{\mathbf{b}}_3\right)' \cdot \bar{\boldsymbol{\chi}}' - c^2 I_3 \left(\bar{\mathbf{b}}^3 \cdot \bar{\boldsymbol{\phi}}'\right)'. && \text{(A.4.21)}
\end{aligned}
$$

In order to produce a system of equations governing $\bar{\boldsymbol{\phi}}$, one can first equate the pairs of equations (A.4.16) and (A.4.13), (A.4.17) and (A.4.14) and (A.4.18) and (A.4.15). If one then utilizes equations (A.2.30) – (A.2.33), (A.4.12) and the identity

$$
\bar{\boldsymbol{\phi}}' = \bar{\mathbf{b}}_1 \left(\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}'\right) + \bar{\mathbf{b}}_2 \left(\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}'\right) + \bar{\mathbf{b}}_3 \left(\bar{\mathbf{b}}^3 \cdot \bar{\boldsymbol{\phi}}'\right). \tag{A.4.22}
$$

Ultimately one obtains the system of differential equations

$$
\begin{aligned}
\left(K_1 - c^2 I_1\right)\left(\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}'\right)' &= \left[\left(K_2 - c^2 I_2\right) - \left(K_3 - c^2 I_3\right)\right]\left(\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{b}}^3 \cdot \bar{\boldsymbol{\phi}}'\right) \\
&\quad + c\bar{\mathbf{d}}_2 \cdot \left(\bar{\mathbf{g}} - c\rho\mathbf{e}_1\right), && \text{(A.4.23)}
\end{aligned}
$$

$$
\begin{aligned}
\left(K_2 - c^2 I_2\right)\left(\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}'\right)' &= \left[\left(K_3 - c^2 I_3\right) - \left(K_1 - c^2 I_1\right)\right]\left(\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{b}}^3 \cdot \bar{\boldsymbol{\phi}}'\right) \\
&\quad - c\bar{\mathbf{d}}_1 \cdot \left(\bar{\mathbf{g}} - c\rho\mathbf{e}_1\right), && \text{(A.4.24)}
\end{aligned}
$$

and

$$
\left(K_3 - c^2 I_3\right)\left(\bar{\mathbf{b}}^3 \cdot \bar{\boldsymbol{\phi}}'\right)' = \left[\left(K_1 - c^2 I_1\right) - \left(K_2 - c^2 I_2\right)\right]\left(\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}'\right). \tag{A.4.25}
$$

For an isotropic rod equation (A.4.25) can be rewritten simply as

$$
\left(K_3 - c^2 I_3\right)\left(\bar{\psi}' + \cos\bar{\theta}\,\bar{\phi}'\right)' = 0 \tag{A.4.26}
$$

Thus if $K_3 - c^2 I_3 \neq 0$ we conclude from the boundary conditions that

$$
\bar{\psi}' + \cos\bar{\theta}\,\bar{\phi}' = \psi'(\pm\infty), \tag{A.4.27}
$$

which shows that the twist rate must be constant. If the twist rate is nonzero, it follows that the potential energy (2.6.1.29) is infinite.

If equations (A.4.23) – (A.4.25) are multiplied by $\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}'$, $\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}'$ and $\bar{\mathbf{b}}^3 \cdot \bar{\boldsymbol{\phi}}'$ respectively and summed, the cross terms cancel and one obtains

$$
\begin{aligned}
\left(K_j - c^2 I_j\right)\left(\bar{\mathbf{b}}^j \cdot \bar{\boldsymbol{\phi}}'\right)\left(\bar{\mathbf{b}}^j \cdot \bar{\boldsymbol{\phi}}'\right)' &= -c\left(\bar{\mathbf{g}} - c\rho\mathbf{e}_1\right) \cdot \left[\left(\bar{\mathbf{b}}^1 \cdot \bar{\boldsymbol{\phi}}'\right)\bar{\mathbf{d}}_2 - \left(\bar{\mathbf{b}}^2 \cdot \bar{\boldsymbol{\phi}}'\right)\bar{\mathbf{d}}_1\right] \\
&= -c\left(\bar{\mathbf{g}} - c\rho\mathbf{e}_1\right)\mathsf{D}_3[\bar{\boldsymbol{\phi}}]^T\bar{\boldsymbol{\phi}}' \\
&= -c\left(\bar{\mathbf{g}} - c\rho\mathbf{e}_1\right)\left(\bar{\mathbf{d}}_3\right)', && \text{(A.4.28)}
\end{aligned}
$$

where we have exploited the expression (2.6.1.19) for $\mathsf{D}_3[\boldsymbol{\phi}]$ to simplify the right hand side. The boundary conditions imply that

$$\bar{\mathbf{g}} = \bar{g}_1 \mathbf{e}_1 \tag{A.4.29}$$

for some $\bar{g}_1$. Therefore equation (A.4.28) takes the form of an exact derivative with first integral

$$\tfrac{1}{2}\left(K_j - c^2 I_j\right)\left(\bar{\mathbf{b}}^j \cdot \bar{\boldsymbol{\phi}}'\right)^2 = c\left(\bar{g}_1 - c\rho\right)\left(1 - \mathbf{e}_1 \cdot \bar{\mathbf{d}}_3\right). \tag{A.4.30}$$

In the case of planar dynamics we fix $\bar{\theta} = \pi/2$ and $\bar{\psi} = 0$. Under these assumptions $\bar{\boldsymbol{\phi}} = \bar{\phi}\,\mathbf{e}_1$, $\bar{\mathbf{b}}^1 = -\mathbf{e}_1$, $\bar{\mathbf{b}}^2 = \mathbf{e}_2$ and $\bar{\mathbf{b}}^3 = \mathbf{e}_3$. Therefore the system of equations (A.4.23) - (A.4.25) reduces to a single equation, which can be written in the form (5.5.19). Thus the planar solitary wave is a critical point of F with respect to spatial variations.

A.5 Analysis of the Second Variation for Planar Solitary Waves

In this section we compute the second variation of the functional F_{red}, for the purpose of comparison with the calculation in section 6.5. The variational analysis is based upon the function

$$F_{\text{red}} \equiv H_{\text{red}} + cI_{\text{red}}, \tag{A.5.1}$$

where H_{red} is the reduced Hamiltonian defined in section 5.3, I is the linear impulse as in Chapter 3,

$$I \equiv \int_{-\infty}^{\infty} \left\{\mathbf{g}\cdot\mathbf{h}' + \boldsymbol{\chi}\cdot\boldsymbol{\phi}'\right\}\, ds \tag{A.5.2}$$

and I_{red} is its reduced form

$$I_{\text{red}} \equiv \int_{-\infty}^{\infty} \boldsymbol{\chi}\cdot\boldsymbol{\phi}'\, ds. \tag{A.5.3}$$

We reiterate that the rod is henceforth assumed to be isotropic.

As before we rewrite F_{red} by completing squares as

$$F_{\text{red}} \equiv F_1 + F_2. \tag{A.5.4}$$

Here

$$\begin{aligned}
F_1(\boldsymbol{\phi},\boldsymbol{\chi}) &\equiv \tfrac{1}{2}\textstyle\int_{-\infty}^{\infty}\left\{\rho^{-1}|\mathbf{g}_c(s)|^2 + \chi_c\cdot\mathsf{B}[\boldsymbol{\phi}]^{-1}\,\mathsf{I}^{-1}\mathsf{B}[\boldsymbol{\phi}]^{-T}\chi_c\right\} ds, &\text{(A.5.5)}\\
\mathbf{g}_c(\boldsymbol{\phi},\boldsymbol{\chi}) &\equiv \bar{\mathbf{g}} + \tilde{\boldsymbol{\Lambda}}'(\boldsymbol{\phi},\boldsymbol{\chi}) + c\rho(\mathbf{d}_3 - \mathbf{e}_1), &\text{(A.5.6)}\\
\chi_c(\boldsymbol{\phi},\boldsymbol{\chi}) &\equiv \boldsymbol{\chi} + \mathsf{D}_3[\boldsymbol{\phi}]\tilde{\boldsymbol{\Lambda}} + c\mathsf{B}[\boldsymbol{\phi}]^T\,\mathsf{I}\mathsf{B}[\boldsymbol{\phi}]\boldsymbol{\phi}', &\text{(A.5.7)}
\end{aligned}$$

and

$$F_2(\boldsymbol{\phi}) = \int_{-\infty}^{\infty}\left\{\tfrac{1}{2}\boldsymbol{\phi}'\cdot\mathsf{B}[\boldsymbol{\phi}]^T(\mathsf{K} - c^2\mathsf{I})\mathsf{B}[\boldsymbol{\phi}]\boldsymbol{\phi}' - \tfrac{1}{2}c^2\rho|\mathbf{d}_3 - \mathbf{e}_1|^2 - c\bar{\mathbf{g}}\cdot(\mathbf{d}_3 - \mathbf{e}_1)\right\} ds. \tag{A.5.8}$$

By inserting the expression (5.5.23) for $\bar{\mathbf{g}}$ into equation (A.5.8) and using the fact that $|\mathbf{d}_3| = 1 = |\mathbf{e}_1|$ we can rewrite F_2 more simply as

$$F_2(\boldsymbol{\phi}) = \int_{-\infty}^{\infty} \left\{ \tfrac{1}{2}\boldsymbol{\phi}' \cdot \mathsf{B}[\boldsymbol{\phi}]^T(\mathsf{K} - c^2\mathsf{I})\mathsf{B}[\boldsymbol{\phi}]\boldsymbol{\phi}' + \tfrac{(K_1 - c^2 I_1)}{a^2}\mathbf{e}_1 \cdot (\mathbf{d}_3 - \mathbf{e}_1) \right\} ds \tag{A.5.9}$$

It remains to calculate the second variation of F_2. The second variation of the term

$$\tfrac{1}{2}\boldsymbol{\phi}' \cdot \mathsf{B}[\boldsymbol{\phi}]^T(\mathsf{K} - c^2\mathsf{I})\mathsf{B}[\boldsymbol{\phi}]\boldsymbol{\phi}' \tag{A.5.10}$$

is

$$\begin{aligned}(K_1 - c^2 I_1) &\left\{ \delta\bar{\phi}'^2 - 2\bar{\phi}'\delta\psi\delta\theta' - \bar{\phi}'^2(\delta\psi^2 + \delta\theta^2) \right\} \\ &+(K_2 - c^2 I_2)(\delta\theta'^2 + \bar{\phi}'\delta\psi)^2 \\ &\quad+(K_3 - c^2 I_3)(\delta\psi' - \bar{\phi}'\delta\theta)^2,\end{aligned} \tag{A.5.11}$$

which can be rewritten as

$$\begin{aligned}(K_1 - c^2 I_1) &\left\{ \delta\bar{\phi}'^2 - (\delta\theta' + \bar{\phi}'\delta\psi)^2 + \delta\theta'^2 - \bar{\phi}'^2\delta\theta^2 \right\} \\ &+(K_2 - c^2 I_2)(\delta\theta'^2 + \bar{\phi}'\delta\psi)^2 \\ &\quad+(K_3 - c^2 I_3)(\delta\psi' - \bar{\phi}'\delta\theta)^2.\end{aligned} \tag{A.5.12}$$

The second variation of the term

$$\frac{(K_1 - c^2 I_1)}{a^2}\mathbf{e}_1 \cdot (\mathbf{d}_3 - \mathbf{e}_1) \tag{A.5.13}$$

is

$$\frac{(K_1 - c^2 I_1)}{a^2}\cos\bar{\phi}(\delta\phi^2 + \delta\theta^2). \tag{A.5.14}$$

Thus the second variation of F_2 is

$$\begin{aligned}\delta^2 F_2 \quad = \quad & (K_1 - c^2 I_1)(\delta\bar{\phi}'^2 + \frac{1}{a^2}\cos\bar{\phi}\delta\phi^2) \\ &+(K_1 - c^2 I_1)\left\{ \delta\theta'^2 + (\frac{1}{a^2}\cos\bar{\phi} - \bar{\phi}'^2)\delta\theta^2 \right\} \\ &+\left\{ (K_2 - c^2 I_2) - (K_1 - c^2 I_1) \right\} (\delta\theta' + \bar{\phi}'\delta\psi)^2 \\ &+(K_3 - c^2 I_3)(\delta\psi' - \bar{\phi}'\delta\theta)^2,\end{aligned} \tag{A.5.15}$$

which can be rewritten as

$$\delta^2 F_2 \quad = \quad (K_1 - c^2 I_1)\langle \delta\phi, P(\delta\phi) \rangle \tag{A.5.16}$$

$$+(K_1 - c^2 I_1)\langle \delta\theta, S(\delta\theta) \rangle \tag{A.5.17}$$

$$+\left\{ (K_2 - c^2 I_2) - (K_1 - c^2 I_1) \right\} \int_{-\infty}^{\infty} (\delta\theta' + \bar{\phi}'\delta\psi)^2\, ds \tag{A.5.18}$$

$$+(K_3 - c^2 I_3) \int_{-\infty}^{\infty} (\delta\psi' - \bar{\phi}'\delta\theta)^2\, ds \tag{A.5.19}$$

Here P is the operator define in equation (5.6.12) and S is given by

$$S(f) \equiv -f'' + (\frac{1}{a^2}\cos\bar{\phi} - \bar{\phi}'^2)f. \tag{A.5.20}$$

Using the equation (5.5.25) for $\bar{\phi}'$, S may also be written as

$$S(f) \equiv -f'' + \frac{1}{a^2}(3\cos\bar{\phi} - 2)f. \tag{A.5.21}$$

Before closing, the problem of performing a variational analysis in terms of Euler angles for a more general configuration deserves some discussion. It was pointed out in section 6.5 that a variational analysis in terms of $\delta\phi$, $\delta\theta$ and $\delta\psi$ would be awkward for a configuration which is nonplanar or has twist because equations (6.5.1.8) - (6.5.1.10) would not apply. However one could imitate equation (6.5.1.2) and parameterize $\delta\boldsymbol{\phi}$ as

$$\delta\boldsymbol{\phi} = \zeta_1\mathbf{b}_1(\boldsymbol{\phi}) + \zeta_2\mathbf{b}_2(\boldsymbol{\phi}) + \zeta_3\mathbf{b}_3(\boldsymbol{\phi}), \tag{A.5.22}$$

where

$$\zeta_j = \mathbf{b}^j(\boldsymbol{\phi}) \cdot \delta\boldsymbol{\phi}. \tag{A.5.23}$$

Of course the primary motivation for a three-dimensional parameterization of $\delta\boldsymbol{\beta}$ was to enforce the linearized constraint (6.5.1.1). But the particular choice of parameterizations in equation (A.5.22) implies

$$\begin{aligned} \delta\mathbf{d}_1 &= -\zeta_2\mathbf{d}_3 + \zeta_3\mathbf{d}_2, && \text{(A.5.24)} \\ \delta\mathbf{d}_2 &= \zeta_1\mathbf{d}_3 - \zeta_3\mathbf{d}_1, && \text{(A.5.25)} \\ \delta\mathbf{d}_3 &= -\zeta_1\mathbf{d}_2 + \zeta_2\mathbf{d}_1, && \text{(A.5.26)} \end{aligned}$$

so ζ_j again has the geometric interpretation as an infinitesimal rotation about $\mathbf{d}_j$. But as always, the configuration under examination must satisfy $\sin\theta \neq 0$ for the singularity to be avoided and the vectors $\mathbf{b}_j(\boldsymbol{\phi})$ (2.6.1.22) to be defined.

Appendix B

A NONLOCAL HAMILTONIAN FORMULATION FOR THE FINITE PLANAR ELASTICA

In this appendix we present an alternative Hamiltonian formulation for the dynamics of a planar elastica of finite length and without twist. This formulation predates the impetus-striction formulation. Instead it is built upon the analysis in [12] where a Lagrangian formulation was used to prove that the buckled equilibria of a finite, planar, inextensible rod that minimize the potential energy are in fact dynamically stable in the sense of Lyapunov. However because this model does not appear to carry over to the case of an infinite rod, we were motivated to develop the impetus-striction formulation.

This Hamiltonian evolutionary system has several distinctive features. First the Hamiltonian density is nonlocal in the sense that the integrand appearing in the Hamiltonian is itself an integral over the real line. Second there are constraints on the phase space in the sense that the functions of arc-length s that determine the configuration of the rod at any time t must satisfy certain integral side constraints. Such constrained Hamiltonian systems appear frequently and are well understood in finite-dimensional phase space associated with time evolution arising from ordinary differential equations, but we are unaware of many other physical examples involving partial differential equations.

The motion of the rod is governed by the following system of partial differential equations for the unknowns $\lambda(s,t), \nu(s,t)$ and $\phi(s,t)$

$$\lambda_s = -\rho\partial_{tt}\{\int_{-L}^{s} \cos\phi(u,t)\,du\} \quad (= -\rho\partial_{tt}x), \tag{B.1}$$

$$\nu_s = \rho\partial_{tt}\{\int_{-L}^{s} \sin\phi(u,t)\,du\} \quad (= \rho\partial_{tt}y), \tag{B.2}$$

and

$$(K_1\phi_s)_s + \lambda\sin\phi + \nu\cos\phi = I_1\phi_{tt}. \tag{B.3}$$

These equations express the balance between the two components of linear acceleration and force, and angular acceleration and moment, respectively. As before ρ is the mass per unit length of the rod, I_1 is the axial moment of inertia of the cross-section, and $(\lambda(s,t), -\nu(s,t))$ are the components of force exerted across the cross-section of the rod at

s. It will be assumed throughout that the moment $\mathcal{M}(s)$ across the cross-section at s is related to the deformation at s through the linear constitutive law $\mathcal{M}(s) = K_1\phi_s(s)$, where K_1 is the bending stiffness. In order to later obtain solitary waves in an infinite rod, we restrict our attention to the uniform case in which ρ, I_1 and K_1 are constants.

In addition to (2.5.26) and (2.5.26), which imply $y(-L,t) = 0$, and $x(-L,t) = -L+2a$, we impose the following additional boundary conditions: First, the moment vanishes at either end of the rod, and second, the end at $s = L$ remains on the y-axis, with fixed displacement R from its reference location at L, so

$$\begin{aligned} \phi_s(-L,t) &= 0, & \text{(B.4)}\\ \phi_s(L,t) &= 0, & \text{(B.5)}\\ y(L,t) &= 0, & \text{(B.6)}\\ x(L,t) &= L+2a. & \text{(B.7)} \end{aligned}$$

Equations (2.5.26) and (2.5.26) for $x(s,t)$ and $y(s,t)$ taken together with boundary conditions (B.6), (B.7) imply that

$$\int_{-L}^{L} \sin\phi(u,t)\, du = 0, \tag{B.8}$$

and

$$\int_{-L}^{L} [1-\cos\phi(u,t)]\, du = 4a. \tag{B.9}$$

We also have the constraints on ϕ_t arising from the time derivatives of (B.8) and (B.9), namely

$$\int_{-L}^{L} \cos\phi(u,t)\, \phi_t(u,t)\, du = 0, \tag{B.10}$$

$$\int_{-L}^{L} \sin\phi(u,t)\, \phi_t(u,t)\, du = 0. \tag{B.11}$$

The kinetic energy is

$$T(\phi) \quad = \quad \tfrac{1}{2}\textstyle\int_{-L}^{L} \left\{ I_1\phi_t(s)^2 + \rho|\mathbf{r}_t(s)|^2 \right\} ds \tag{B.12}$$

where the dependence of ϕ upon t has been suppressed for brevity. In our model, the rod is fixed at both ends, and we choose a nonlocal representation of the kinetic energy in terms of the variable ϕ which reflects this symmetry. From equations (2.5.26) and (B.6), we can express y_t in two ways,

$$y_t = \int_{-L}^{s} \partial_t \sin\phi(u) du = -\int_{s}^{L} \partial_t \sin\phi(u) du, \tag{B.13}$$

with similar expressions for x_t, so that

$$\begin{aligned} |\mathbf{r}_t(s)|^2 \quad = \quad & -\int_{-L}^{s} \partial_t \cos\phi(u) du \int_{s}^{L} \partial_t \cos\phi(u) du \\ & -\int_{-L}^{s} \partial_t \sin\phi(u) du \int_{s}^{L} \partial_t \sin\phi(u) du. \end{aligned} \tag{B.14}$$

After substitution of (B.14) for $|\mathbf{r}_t(s)|^2$ and a change in order of integration, (B.12) can be rewritten in the form

$$T(\phi) = \tfrac{1}{2}\int_{-L}^{L}\left\{I_1\phi_t(s)^2 + \int_{-L}^{L} k(s,u)\cos\left[\phi(s)-\phi(u)\right]\phi_t(u)\phi_t(s)\,du\right\}\,ds, \qquad \text{(B.15)}$$

where the kernel $k(s,u)$ is defined by

$$k(s,u) = -\tfrac{1}{2}\int_{\min(s,u)}^{\max(s,u)}\rho\,dv = -\tfrac{1}{2}\rho|s-u|. \qquad \text{(B.16)}$$

The potential energy corresponding to a linear bending law is

$$V(\phi) = \tfrac{1}{2}K_1\int_{-L}^{L}\phi_s(s)^2\,ds. \qquad \text{(B.17)}$$

The equations of motion can be obtained from Hamilton's principle, which states that the action

$$\int_{t_0}^{t_1}\{T(\phi)-V(\phi)\}\;dt \qquad \text{(B.18)}$$

is stationary for all t_0 and t_1, among admissible variations of ϕ. The admissible variations are those satisfying the constraints, (B.8) and (B.9). By the Lagrange multiplier rule, Hamilton's principle can be rewritten as the requirement that the modified action

$$\int_{t_0}^{t_1}\left\{T(\phi)-V(\phi)+\hat{\nu}(t)y(L,t)+\hat{\lambda}(t)x(L,t)\right\}\;dt \qquad \text{(B.19)}$$

is stationary for all t_0 and t_1. (It is not necessary to include the constraints (B.10) and (B.11) in Hamilton's principle (B.19), since these constraints are null Lagrangians.) Now the admissible variations need only satisfy the boundary conditions (B.4) and (B.5), which are natural and so need not be imposed in (B.19). With T given by (B.12), Hamilton's principle leads to equations (B.1) – (B.3) (cf. [12], p. 4). However, using the nonlocal form (B.15) for the kinetic energy yields a more useful form of the dynamical system in which the variables $\lambda(s,t)$ and $\nu(s,t)$ do not appear, namely the integro-partial differential equation for $\hat{\lambda}(t) = \frac{1}{2}[\lambda(-L,t)+\lambda(L,t)]$, $\hat{\nu}(t) = \frac{1}{2}[\nu(-L,t)+\nu(L,t)]$, and $\phi(s,t)$

$$\begin{aligned} K_1\phi_{ss}+\hat{\lambda}(t)\sin\phi+\hat{\nu}(t)\cos\phi \;&=\; I_1\phi_{tt}+\int_{-L}^{L}k(s,u)\cos\left[\phi(s)-\phi(u)\right]\phi_{tt}(u)\,du \\ &\quad +\int_{-L}^{L}k(s,u)\sin\left[\phi(s)-\phi(u)\right]\phi_t(u)^2\;du. \qquad \text{(B.20)} \end{aligned}$$

In (B.20), $\hat{\lambda}(t)$ and $\hat{\nu}(t)$ are time-dependent Lagrange multipliers representing unknown reactions of the end supports. They are determined (in principle) by the requirement that constraints (B.8) and (B.9) hold for all times $t > 0$. At the end of the next section, we show how to obtain explicit closed form expressions for $\hat{\lambda}(t)$ and $\hat{\nu}(t)$ in terms of $\phi(s,t)$ and $\phi_t(s,t)$.

It is worth noting that the form of the kernel k and the interpretation of the Lagrange multipliers depend heavily upon our formulation of the kinetic energy in (B.12), where we have invoked constraints at both ends of the rod. There are two other natural formulations. We could instead formulate the kinetic energy using the constraint that the end at $s = -L$ is fixed, through the substitution of

$$|\mathbf{r}_t(s)|^2 = \left[\int_{-L}^{s} \partial_t \cos\phi(u)du\right]^2 + \left[\int_{-L}^{s} \partial_t \sin\phi(u)du\right]^2 \tag{B.21}$$

into (B.12) (Cf. [12]). The associated kernel in the nonlocal formulation analogous to (B.15) would be $k(s,u) = -\rho\max(s,u)$, and the Lagrange multipliers in (B.19) are given by $\hat{\lambda}(t) = \lambda(L,t)$ and $\hat{\nu}(t) = \nu(L,t)$. Alternatively, we could use the constraint that the end at $s = L$ is fixed, through the substitution of

$$|\mathbf{r}_t(s)|^2 = \left[\int_{s}^{L} \partial_t \cos\phi(u)du\right]^2 + \left[\int_{s}^{L} \partial_t \sin\phi(u)du\right]^2 \tag{B.22}$$

into (B.12). Then the associated kernel in the nonlocal formulation (B.15) would be $k(s,u) = \rho\min(s,u)$, and the Lagrange multipliers in (B.19) are given by $\hat{\lambda}(t) = \lambda(-L,t)$ and $\hat{\nu}(t) = \nu(-L,t)$.

We remark that the above dynamical system, albeit for slightly different boundary and side conditions, was first examined in [12]. Closely related systems have also been studied by Coleman and Dill [13] and by Xu [62].

Define the operator L_ϕ by

$$L_\phi f(s) \equiv f(s) + {I_1}^{-1} \int_{-L}^{L} k(s,u)\cos\left[\phi(s) - \phi(u)\right] f(u)\, du. \tag{B.23}$$

By an argument in [12] (p. 16), with appropriate modifications for the new kernel defined by (B.16), one can show that the operator L_ϕ is (formally) self-adjoint and invertible, so that we may define

$$K_\phi \equiv L_\phi^{-1}. \tag{B.24}$$

The kinetic energy in (B.15) can now be rewritten as

$$T(\phi) = \tfrac{1}{2} I_1 \textstyle\int_{-L}^{L} \phi_t(s)\left[L_\phi \phi_t\right](s)\, ds, \tag{B.25}$$

and the equation of motion (B.20) can then be rewritten in the form

$$\begin{aligned} K_1\phi_{ss} + \hat{\lambda}\sin\phi + \hat{\nu}\cos\phi \quad &= \quad I_1 L_\phi \phi_{tt} \\ &\quad + \int_{-L}^{L} k(s,u)\sin\left[\phi(s) - \phi(u)\right]\phi_t(u)^2\, du. \end{aligned} \tag{B.26}$$

Accordingly we define a variable $\chi(s,t)$ conjugate to $\phi(s,t)$ by

$$\chi \equiv I_1 L_\phi \phi_t. \tag{B.27}$$

Notice that, like the kinetic energy T, the conjugate variable χ is defined nonlocally. Inversion of (B.27) using (B.24) yields

$$\phi_t = {I_1}^{-1} K_\phi \chi. \tag{B.28}$$

After inserting (B.28) into the constraints (B.10) and (B.11) on ϕ_t, we find the following constraints on p

$$\int_{-L}^{L} \cos\phi(u,t)\, K_\phi \chi(u,t)\, du = 0, \tag{B.29}$$

$$\int_{-L}^{L} \sin\phi(u,t)\, K_\phi \chi(u,t)\, du = 0. \tag{B.30}$$

Now from (B.27) and (B.23),

$$\chi_t = I_1 L_\phi \phi_{tt}(s) - \int_{-L}^{L} k(s,u) \sin\left[\phi(s) - \phi(u)\right]\left[\phi_t(s) - \phi_t(u)\right] \phi_t(u)\, du, \tag{B.31}$$

so

$$\begin{aligned} I_1 L_\phi \phi_{tt} + \int_{-L}^{L} k(s,u) \sin\left[\phi(s) - \phi(u)\right] \phi_t(u)^2\, du = \\ \chi_t + \int_{-L}^{L} k(s,u) \sin\left[\phi(s) - \phi(u)\right] \phi_t(u)\phi_t(s)\, du. \end{aligned} \tag{B.32}$$

Substitution of (B.32) into the equation of motion (B.26) yields

$$\begin{aligned} \chi_t &= K_1 \partial_{ss}\phi + \hat{\lambda}\sin\phi + \hat{\nu}\cos\phi \\ &\quad - \int_{-L}^{L} k(s,u) \sin\left[\phi(s) - \phi(u)\right] \phi_t(u)\phi_t(s)\, du \\ &= K_1 \partial_{ss}\phi + \hat{\lambda}\sin\phi + \hat{\nu}\cos\phi \\ &\quad - {I_1}^{-2} K_\phi\chi(s) \int_{-L}^{L} k(s,u) \sin\left[\phi(s) - \phi(u)\right] K_\phi \chi(u)\, du, \end{aligned} \tag{B.33}$$

where the second identity follows after two applications of (B.28).

In terms of ϕ and χ, the kinetic energy given in (B.25) takes the form

$$T(\phi,\chi) = \tfrac{1}{2}{I_1}^{-1} \textstyle\int_{-L}^{L} \chi(s) K_\phi \chi(s)\, ds, \tag{B.34}$$

so the total energy is

$$H(\phi,\chi) = \tfrac{1}{2} \textstyle\int_{-L}^{L} \left[{I_1}^{-1}\chi(s) K_\phi \chi(s) + K_1 \phi_s(s)^2\right] ds. \tag{B.35}$$

We next compute the first variations of the energy H. First note that from the expression for the kinetic energy (B.34),

$$\begin{aligned} T(\phi+\epsilon\zeta, p+\epsilon\eta) &= \tfrac{1}{2}{I_1}^{-1} \textstyle\int_{-L}^{L} (\chi+\epsilon\eta)(s)\left[K_{\phi+\epsilon\zeta})(\chi+\epsilon\eta)\right](s)\, ds \\ &= \tfrac{1}{2}{I_1}^{-1} \textstyle\int_{-L}^{L} \chi(s)\left[K_\phi \chi\right](s)\, ds \\ &\quad + \tfrac{1}{2}{I_1}^{-1} \textstyle\int_{-L}^{L} \chi(s)\left[(K_{\phi+\epsilon\zeta} - K_\phi)\chi\right](s)\, ds \\ &\quad + \epsilon {I_1}^{-1} \int_{-L}^{L} \chi(s)\left[K_\phi \eta\right](s)\, ds \\ &\quad + O(\epsilon^2). \end{aligned} \tag{B.36}$$

From the definition of K_ϕ in (B.24), the identity $K_\phi L_\phi = 1 = L_\phi K_\phi$ and so

$$K_\psi - K_\phi = -K_\phi(L_\psi - L_\phi)K_\psi. \tag{B.37}$$

By a Taylor series expansion,

$$\left[L_{\phi+\epsilon\zeta}f\right](s) = \left[L_\phi f\right](s) + \epsilon\left[L_\phi^{(1)}(\zeta)f\right](s) + O(\epsilon^2), \tag{B.38}$$

where

$$\left[L_\phi^{(1)}(\zeta)f\right](s) \equiv -{I_1}^{-1}\int_{-L}^{L} k(s,u)\sin\left[\phi(s)-\phi(u)\right]\left[\zeta(s)-\zeta(u)\right]f(u)\,du. \tag{B.39}$$

Combining (B.36)-(B.39), we can expand T in powers of ϵ as

$$\begin{aligned} T(\phi+\epsilon\zeta,\chi+\epsilon\eta) &= T(\phi,\chi) \\ &\quad -\tfrac{1}{2}\epsilon{I_1}^{-1}\textstyle\int_{-L}^{L}\chi(s)\left[K_\phi L_\phi^{(1)}(\zeta)K_\phi\chi\right](s)\,ds \\ &\quad +\epsilon{I_1}^{-1}\displaystyle\int_{-L}^{L}\chi(s)\left[K_\phi\eta\right](s) + O(\epsilon^2). \end{aligned} \tag{B.40}$$

From the order ϵ terms in (B.40), we find that the variation of the kinetic energy with respect to p satisfies

$$\delta_\chi T(\phi,\chi) = {I_1}^{-1}K_\phi\chi(s). \tag{B.41}$$

The calculation of the ϕ variation is more complicated. Using (B.39) in (B.40), we find that

$$\begin{aligned} &2I_1^2\langle\delta_\phi T(\phi,p),\zeta\rangle \\ &= \int_{-L}^{L}\int_{-L}^{L} k(s,u)\sin\left[\phi(s)-\phi(u)\right]\left(\zeta(s)-\zeta(u)\right)K_\phi\chi(u)K_\phi\chi(s)\,du\,ds \\ &= \int_{-L}^{L}\int_{-L}^{L} k(s,u)\sin\left[\phi(s)-\phi(u)\right]\zeta(s)K_\phi\chi(u)K_\phi\chi(s)\,du\,ds \\ &\quad -\int_{-L}^{L}\int_{-L}^{L} k(s,u)\sin\left[\phi(s)-\phi(u)\right]\zeta(u)K_\phi\chi(u)K_\phi\chi(s)\,du\,ds, \end{aligned} \tag{B.42}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual L^2 inner product. Reversing the roles of s and u in the second integral, and using the symmetry of $k(s,u)$ shows that the two integrals on the right hand side of (B.42) are equal. Consequently it may be concluded that

$$\delta_\phi T(\phi,\chi) = {I_1}^{-2}K_\phi\chi(s)\int_{-L}^{L} k(s,u)\sin\left[\phi(s)-\phi(u)\right]K_\phi\chi(u)\,du. \tag{B.43}$$

A similar, but much simpler, expansion of the potential energy produces

$$V(\phi+\epsilon\zeta) = V(\phi) + \epsilon K_1\int_{-L}^{L}\phi_s(s)\zeta_s(s)\,ds + O(\epsilon^2), \tag{B.44}$$

so

$$\begin{aligned} \delta_\phi V(\phi) &= -K_1\phi_{ss}, &\text{(B.45)}\\ \delta_\chi V(\phi) &= 0. &\text{(B.46)}\end{aligned}$$

Then, combining equations (B.41), (B.43), (B.45) and (B.46) for the first variations in T and V, we obtain

$$\begin{aligned} \delta_\phi H &= {I_1}^{-2} K_\phi\chi(s)\int_{-L}^{L} k(s,u)\sin\left[\phi(s)-\phi(u)\right] K_\phi\chi(u)\,du, \\ &\quad -K_1\phi_{ss} &\text{(B.47)}\\ \delta_\chi H &= {I_1}^{-1}K_\phi\chi(s). &\text{(B.48)}\end{aligned}$$

Comparing equation (B.28) for ϕ_t with equation (B.48) for $\delta_\chi H(\phi,\chi)$, and equation (B.33) for χ_t with equation (B.47) for $\delta_\chi H(\phi,\chi)$, we find that

$$\begin{aligned} \phi_t &= \delta_\chi H(\phi,\chi), &\text{(B.49)}\\ \chi_t &= -\delta_\phi H(\phi,\chi)+\hat\lambda\sin\phi+\hat\nu\cos\phi. &\text{(B.50)}\end{aligned}$$

We now turn to an explanation of the sense in which equations (B.49) and (B.50) can be interpreted as a constrained Hamiltonian system. Using (B.48), the constraints (B.29) and (B.30) can be expressed as

$$\int_{-L}^{L}\sin\phi(s)H_\chi(s)\,ds = 0, \tag{B.51}$$

$$\int_{-L}^{L}\cos\phi(s)H_\chi(s)\,ds = 0, \tag{B.52}$$

where H_χ denotes $\delta_\chi H(\phi,\chi)$. Then the the action

$$\int_0^T \left[\langle\chi,\phi_t\rangle - H(\phi,\chi)\right]\,dt \tag{B.53}$$

can be made stationary amongst all pairs (ϕ,χ) in the submanifold of phase space defined by the four constraints (B.8), (B.9), (B.51) and (B.52) by making

$$\int_0^T\left\{\langle p,\phi_t\rangle - H(\phi,\chi)+\int_{-L}^{L}\left[\mu_1\sin\phi+\mu_2(1-\cos\phi)+\mu_3\cos\phi H_\chi+\mu_4\sin\phi H_\chi\right]\,ds\right\}\,dt \tag{B.54}$$

stationary amongst all pairs (ϕ,χ). The associated first order conditions are

$$\begin{aligned} \phi_t &= H_p-\mu_3\cos\phi H_{\chi\chi}-\mu_4\sin\phi H_{\chi\chi}, &\text{(B.55)}\\ \chi_t &= -H_\phi+\mu_1\cos\phi+\mu_2\sin\phi+\mu_3(\cos\phi H_{\chi\phi}+\sin\phi H_\chi)\\ &\quad +\mu_4(\sin\phi H_{\chi\phi}+\cos\phi H_\chi), &\text{(B.56)}\end{aligned}$$

where μ_1, μ_2, μ_3 and μ_4 are time dependent Lagrange multipliers. But in light of the constraints (B.51) and (B.52), the time derivative of the constraints (B.8) and (B.9), and the fact that the operator

$$H_{\chi\chi} \equiv K_\phi \tag{B.57}$$

is positive definite, the L^2 inner-products of (B.55) with $\sin\phi$ and $\cos\phi$ provide a system of two scalar equations which imply that $\mu_3(t) \equiv \mu_4(t) \equiv 0$. This result is the Hamiltonian manifestation of an observation that was previously made in the context of the Lagrangian form of the stationary action principle, namely that the time derivatives of the constraints (B.8) and (B.9) are null Lagrangians.

Equations (B.49) and (B.50) can now be recognized as being of the form (B.55) and (B.56) with $\mu_1 = \hat{\nu}$, $\mu_2 = \hat{\lambda}$, and $\mu_3 = \mu_4 = 0$. However equations (B.49) and (B.50) do not form a well-posed evolutionary system in phase-space until the multipliers $\hat{\lambda}$ and $\hat{\nu}$ are determined in terms of the phase space variables (ϕ, χ). The conditions to obtain such a relationship are that the dynamics must remain on the manifold defined by the four constraints (B.8), (B.9), (B.51) and (B.52). The explicit calculations are as follows.

Assume that the initial data for ϕ and χ lie on the constraint manifold. We shall obtain expressions for $\hat{\lambda}(t)$ and $\hat{\nu}(t)$ such that the time derivatives of (B.51) and (B.52) are identically zero. It then follows that (B.51) and (B.52) hold for all $t > 0$. Since (B.29) and (B.30) are the statements that the time derivatives of (B.8) and (B.9) are themselves zero, (B.8) and (B.9) must then hold for all $t > 0$ also. Setting the time derivative of (B.51) to zero implies

$$\int_{-L}^{L} \cos\phi(s)\phi_t(s)H_\chi(s)\,ds + \int_{-L}^{L} \sin\phi(s)\left(H_{\chi\chi}(s)\,\chi_t(s) + H_{\chi\phi}(s)\,\phi_t(s)\right)\,ds = 0. \tag{B.58}$$

Inserting the time evolution (B.49) and (B.50) into (B.58) and rearranging terms, we obtain

$$\begin{aligned} \hat{\lambda}\int_{-L}^{L} \sin\phi\, H_{\chi\chi}\sin\phi\,ds + \hat{\nu}\int_{-L}^{L}\cos\phi\, H_{\chi\chi}\sin\phi\,ds &= \\ \int_{-L}^{L}\sin\phi\,\left(H_{\chi\chi}H_\phi - H_{\chi\phi}H_\chi\right)\,ds - \int_{-L}^{L} H_\chi\cos\phi H_\chi\,ds.& \end{aligned} \tag{B.59}$$

Operating similarly on (B.52), and collecting the resulting equation with (B.59) produces the system of equations

$$M\cdot\begin{pmatrix}\hat{\lambda}(t)\\ \hat{\nu}(t)\end{pmatrix} = b \tag{B.60}$$

where

$$M = \begin{pmatrix} \int_{-L}^{L}\sin\phi\, H_{\chi\chi}\sin\phi\,ds & \int_{-L}^{L}\cos\phi\, H_{\chi\chi}\sin\phi\,ds \\ \int_{-L}^{L}\sin\phi\, H_{\chi\chi}\cos\phi\,ds & \int_{-L}^{L}\cos\phi\, H_{\chi\chi}\cos\phi\,ds \end{pmatrix} \tag{B.61}$$

and

$$b = \begin{pmatrix} \int_{-L}^{L}\sin\phi\,(H_{\chi\chi}H_\phi - H_{\chi\phi}H_\chi)\,ds - \int_{-L}^{L} H_\chi\cos\phi H_\chi\,ds. \\ \int_{-L}^{L}\cos\phi\,(H_{\chi\chi}H_\phi - H_{\chi\phi}H_\chi)\,ds + \int_{-L}^{L} H_\chi\sin\phi H_\chi\,ds \end{pmatrix}. \tag{B.62}$$

Since $H_{\chi\chi} = K_\phi$ is positive and self-adjoint, we see by the Cauchy-Schwartz inequality that the matrix M in (B.61) is invertible, and so equation (B.60) can be solved to obtain formulæ for $\hat{\lambda}(t)$ and $\hat{\nu}(t)$ in terms of ϕ and χ as desired.

Appendix C

DETAILED CALCULATIONS IN THE SPATIAL VARIATIONAL ANALYSIS

C.1 Introduction

This appendix contains some of the lengthy calculations required for the spatial variational analysis in Chapter 6. In section C.2 we discuss the motivation for the parametrizations of the variations use in Section 6.4. In sections C.3 and C.4 calculate of the linearized constraints and the second variation, respectively.

C.2 Motivation of Parametrization

The parametrization we used in Section 6.4 was motivated by the expression of the linearized inextensibility constraint in terms of Lagrangian variables. Taking the time derivative of (6.5.1.16), the perturbation must therefore satisfy

$$\delta\mathbf{h}_{st} = \delta\mathsf{D}_3[\boldsymbol{\beta}]^T\bar{\boldsymbol{\beta}}_t + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]^T\delta\boldsymbol{\beta}_t. \tag{C.2.1}$$

From equations (6.2.6) and (6.2.9) the perturbations in the Hamiltonian variables are related to the perturbations in the Lagrangian variables by

$$\delta\mathbf{h}_t = \rho^{-1}(\delta\mathbf{g} + \delta\tilde{\boldsymbol{\Lambda}}_s) \tag{C.2.2}$$

and

$$\begin{aligned}
\delta\boldsymbol{\beta}_t \quad = \quad & \tfrac{1}{4}\mathsf{B}[\delta\boldsymbol{\beta}]^T\,\mathsf{I}^{-1}\,\mathsf{B}[\bar{\boldsymbol{\beta}}]\,(\bar{\boldsymbol{\gamma}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\bar{\boldsymbol{\Lambda}}) \\
+ \quad & \tfrac{1}{4}\mathsf{B}[\bar{\boldsymbol{\beta}}]^T\,\mathsf{I}^{-1}\,\mathsf{B}[\delta\boldsymbol{\beta}]\,(\bar{\boldsymbol{\gamma}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\bar{\boldsymbol{\Lambda}}) \\
& \tfrac{1}{4}\mathsf{B}[\bar{\boldsymbol{\beta}}]^T\,\mathsf{I}^{-1}\,\mathsf{B}[\bar{\boldsymbol{\beta}}]\,(\delta\boldsymbol{\gamma} + \delta\mathsf{D}_3[\boldsymbol{\beta}]\bar{\boldsymbol{\Lambda}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\delta\tilde{\boldsymbol{\Lambda}}),
\end{aligned} \tag{C.2.3}$$

or equivalently

$$\begin{aligned}
\delta\boldsymbol{\gamma} + \delta\mathsf{D}_3[\boldsymbol{\beta}]\bar{\boldsymbol{\Lambda}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\delta\tilde{\boldsymbol{\Lambda}} \quad = \quad & 4\mathsf{B}[\delta\boldsymbol{\beta}]^T\,\mathsf{I}\,\mathsf{B}[\bar{\boldsymbol{\beta}}]\,(\bar{\boldsymbol{\gamma}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\bar{\boldsymbol{\Lambda}}) \\
& 4\mathsf{B}[\bar{\boldsymbol{\beta}}]^T\,\mathsf{I}\,\mathsf{B}[\delta\boldsymbol{\beta}]\,(\bar{\boldsymbol{\gamma}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\bar{\boldsymbol{\Lambda}}) \\
& 4\mathsf{B}[\bar{\boldsymbol{\beta}}]^T\,\mathsf{I}\,\mathsf{B}[\bar{\boldsymbol{\beta}}]\,\delta\boldsymbol{\beta}_t.
\end{aligned} \tag{C.2.4}$$

We differentiate $\delta\boldsymbol{\beta}$ (6.5.1.2) and use the relation

$$\boldsymbol{\beta}_t = -c\boldsymbol{\beta}_s \tag{C.2.5}$$

to find

$$\begin{aligned}
\delta\boldsymbol{\beta}_t &= \tfrac{1}{2}(\zeta_1)_t \mathsf{B}_1\bar{\boldsymbol{\beta}} + \tfrac{1}{2}(\zeta_2)_t \mathsf{B}_2\bar{\boldsymbol{\beta}} + \tfrac{1}{2}(\zeta_3)_t \mathsf{B}_3\bar{\boldsymbol{\beta}} \\
&\quad -\tfrac{1}{2}c\zeta_1\mathsf{B}_1\bar{\boldsymbol{\beta}}_s - \tfrac{1}{2}c\zeta_2\mathsf{B}_2\bar{\boldsymbol{\beta}}_s - \tfrac{1}{2}c\zeta_3\mathsf{B}_3\bar{\boldsymbol{\beta}}_s \\
&= \tfrac{1}{2}(\zeta_1)_t \mathsf{B}_1\bar{\boldsymbol{\beta}} + \tfrac{1}{2}\left[(\zeta_2)_t - \tfrac{1}{2}c\bar{\phi}'\zeta_3\right]\mathsf{B}_2\bar{\boldsymbol{\beta}} \\
&\quad +\tfrac{1}{2}\left[(\zeta_3)_t + \tfrac{1}{2}c\bar{\phi}'\zeta_2\right]\mathsf{B}_3\bar{\boldsymbol{\beta}} - \tfrac{1}{4}c\bar{\phi}_s\zeta_1\bar{\boldsymbol{\beta}}
\end{aligned} \tag{C.2.6}$$

Inserting equation (C.2.6) into (C.2.4) and simplifying we obtain

$$\begin{aligned}
\delta\boldsymbol{\gamma} + \delta\mathsf{D}_3[\boldsymbol{\beta}]\boldsymbol{\Lambda} + \mathsf{D}_3[\boldsymbol{\beta}]\delta\boldsymbol{\Lambda} &= 2I_1\left[(\zeta_1)_t - c(\zeta_1)_s\right]\mathsf{B}_1\bar{\boldsymbol{\beta}} \\
&\quad +2I_1\left[(\zeta_2)_t - c\bar{\phi}_s\zeta_3\right]\mathsf{B}_2\bar{\boldsymbol{\beta}} - cI_1\bar{\phi}'\zeta_3\mathsf{B}_2\bar{\boldsymbol{\beta}} \\
&\quad +2I_3\left[(\zeta_3)_t + c\bar{\phi}_s\zeta_2\right]\mathsf{B}_3\bar{\boldsymbol{\beta}} + cI_1\bar{\phi}'\zeta_2\mathsf{B}_3\bar{\boldsymbol{\beta}} \\
&\quad -\zeta_1 cI_1\bar{\phi}'\bar{\boldsymbol{\beta}},
\end{aligned} \tag{C.2.7}$$

which also satisfies the linearized constraint namely

$$\delta\boldsymbol{\gamma}\cdot\boldsymbol{\beta} + \boldsymbol{\gamma}\cdot\delta\boldsymbol{\beta} = 0. \tag{C.2.8}$$

We next differentiate equation (6.5.1.18) with respect to time we obtain

$$\delta\mathbf{h}_t = \int_{-\infty}^{s}[-\zeta_1(u)\bar{\mathbf{d}}_2(u)]_t\,du - \mathbf{e}_3\int_{-\infty}^{s}\eta_2(u)\,du. \tag{C.2.9}$$

Because

$$(\bar{\mathbf{d}}_2)_t = -c(\bar{\mathbf{d}}_2)_s, \tag{C.2.10}$$

equation (C.2.9) can be rewritten through an integration by parts as

$$\begin{aligned}
\delta\mathbf{h}_t(s) &= c\zeta_1(s)\bar{\mathbf{d}}_2(s) \\
&\quad -\int_{-\infty}^{s}[(\zeta_1)_t(u) + c(\zeta_1)_s(u)]\bar{\mathbf{d}}_2(u)\,du, \\
&\quad -\mathbf{e}_3\int_{-\infty}^{s}(\zeta_2)_t(u)\,du.
\end{aligned} \tag{C.2.11}$$

It follows from equations (C.2.2) and (C.2.11) that

$$\begin{aligned}
\delta\mathbf{g} + \delta\boldsymbol{\Lambda}_s &= -\rho\int_{-\infty}^{s}(\zeta_1)_t(u)\bar{\mathbf{d}}_2(u)\,du + \rho\int_{-\infty}^{s}(\zeta_2)_t(u)\bar{\mathbf{d}}_1(u)\,du \\
&\quad -\rho c\int_{-\infty}^{s}\bar{\phi}_s\zeta_1(u)\bar{\mathbf{d}}_3(u)\,du.
\end{aligned} \tag{C.2.12}$$

Finally by identifying $(\zeta_k)_t$ as η_k in equations (C.2.7) and (C.2.12) we obtain the desired equations (6.5.1.27) and (6.5.1.26).

C.3 Linearized Constraints

We now compute the linearized constraints involving $\mathbf{g}$ and $\boldsymbol{\gamma}$ in terms of the parametrizations (6.5.1.27) and (6.5.1.26). To simplify the presentation we will assume that the linearized constraints (6.5.1.19) – (6.5.1.17) on the configuration are satisfied.

Because the linearizations are about a travelling wave solution, we will repeatedly use the identities from equations (6.4.3) and (6.4.9) – (6.4.11) which imply

$$\bar{\mathbf{g}} + \bar{\boldsymbol{\Lambda}}' = -\rho c \bar{\mathbf{h}}' \tag{C.3.1}$$

and

$$\bar{\boldsymbol{\gamma}} + \mathsf{D}_3[\bar{\boldsymbol{\beta}}]\tilde{\boldsymbol{\Lambda}} = -cI_1\bar{\boldsymbol{\beta}}_s = \tfrac{1}{2}cI_1\bar{\phi}_s\mathsf{B}_1\bar{\boldsymbol{\beta}}. \tag{C.3.2}$$

We recall from equation (6.3.5) that the linear impulse is

$$I \;=\; \int_{-\infty}^{\infty}\left\{\left(\mathbf{g}+\tilde{\boldsymbol{\Lambda}}'\right)\cdot\mathbf{h}' + \left(\boldsymbol{\gamma}+\mathsf{D}_3\tilde{\boldsymbol{\Lambda}}\right)\cdot\boldsymbol{\beta}'\right\}\,ds \tag{C.3.3}$$

so the linearized constraint on the impulse is

$$\begin{aligned} 0 \;=\; & \int_{-\infty}^{\infty}\left(\delta\mathbf{g}+\delta\tilde{\boldsymbol{\Lambda}}'\right)\cdot\mathbf{h}' + \left(\mathbf{g}+\tilde{\boldsymbol{\Lambda}}'\right)\cdot\delta\mathbf{h}' \\ & + \int_{-\infty}^{\infty}\left(\delta\boldsymbol{\gamma}+\delta\mathsf{D}_3\tilde{\boldsymbol{\Lambda}}+\mathsf{D}_3\delta\tilde{\boldsymbol{\Lambda}}\right)\cdot\boldsymbol{\beta}'\,ds + \left(\boldsymbol{\gamma}+\mathsf{D}_3\tilde{\boldsymbol{\Lambda}}\right)\cdot\delta\boldsymbol{\beta}'\,ds \end{aligned} \tag{C.3.4}$$

Using the equations (6.4.40), (6.4.50), (C.4.9), and (6.5.1.27) for $\bar{\boldsymbol{\beta}}'$, $\bar{\boldsymbol{\gamma}}$, $\delta\boldsymbol{\beta}$ and $\delta\boldsymbol{\gamma}+\delta\mathsf{D}_3\tilde{\boldsymbol{\Lambda}}+\mathsf{D}_3\delta\tilde{\boldsymbol{\Lambda}}$ we find

$$\left(\delta\boldsymbol{\gamma}+\delta\mathsf{D}_3\tilde{\boldsymbol{\Lambda}}+\mathsf{D}_3\delta\tilde{\boldsymbol{\Lambda}}\right)\cdot\boldsymbol{\beta}' = -I_1\bar{\phi}'\left(\eta_1 - c\zeta_1'\right) \tag{C.3.5}$$

and

$$\left(\delta\boldsymbol{\gamma}+\delta\mathsf{D}_3\tilde{\boldsymbol{\Lambda}}+\mathsf{D}_3\delta\tilde{\boldsymbol{\Lambda}}\right)\cdot\delta\boldsymbol{\beta}' = cI_1\bar{\phi}'\zeta_1'. \tag{C.3.6}$$

Using an integration by parts, the differential equation (5.5.24) for $\bar{\phi}$ and the linearized constraint (6.5.1.19) we can rewrite

$$\int_{-\infty}^{\infty}\bar{\phi}'\zeta_1'\,ds = -\frac{1}{a^2}\int_{-\infty}^{\infty}\sin\bar{\phi}\zeta_1\,ds = 0, \tag{C.3.7}$$

so equations (C.3.5) - (C.3.7) yield

$$\int_{-\infty}^{\infty}\left(\delta\boldsymbol{\gamma}+\delta\mathsf{D}_3\tilde{\boldsymbol{\Lambda}}+\mathsf{D}_3\delta\tilde{\boldsymbol{\Lambda}}\right)\cdot\boldsymbol{\beta}'\,ds + \left(\boldsymbol{\gamma}+\mathsf{D}_3\tilde{\boldsymbol{\Lambda}}\right)\cdot\delta\boldsymbol{\beta}'\,ds = cI_1\int_{-\infty}^{\infty}\bar{\phi}'\zeta_1'\,ds. \tag{C.3.8}$$

In order to simplify the translational part of equation (C.3.4) we first use equation (6.4.3) to obtain

$$\int_{-\infty}^{\infty}\left(\delta\mathbf{g}+\delta\tilde{\boldsymbol{\Lambda}}'\right)\cdot\mathbf{h}' + \left(\mathbf{g}+\tilde{\boldsymbol{\Lambda}}'\right)\cdot\delta\mathbf{h}' = \int_{-\infty}^{\infty}\bar{\mathbf{h}}'\cdot\left(\mathbf{g}+\tilde{\boldsymbol{\Lambda}}'-c\rho\delta\mathbf{h}'\right)ds. \tag{C.3.9}$$

The right hand side of equation (C.3.9) may be simplified by using the constraint (2.5.6) and the linearized constraint (6.5.1.16) to rewrite

$$\int_{-\infty}^{\infty} \bar{\mathbf{h}}' \cdot \delta\mathbf{h}'\, ds = \int_{-\infty}^{\infty} \bar{\mathbf{d}}_3 \cdot \delta\mathbf{d}_3 - \mathbf{e}_1 \cdot \delta\mathbf{h}'\, ds = -\mathbf{e}_1 \cdot \delta\mathbf{h}|_{-\infty}^{\infty} = 0, \tag{C.3.10}$$

where the final equality is a consequence of equations (6.5.1.7) and (6.5.1.14). The substitution of equations (C.3.10) and (6.5.1.26) into equation (C.3.9) produces

$$\begin{aligned}
&\int_{-\infty}^{\infty} \{\bar{\mathbf{h}}' \cdot (\delta\mathbf{g} + \delta\boldsymbol{\Lambda}') + (\mathbf{g} + \boldsymbol{\Lambda}') \cdot \delta\mathbf{h}'\}\, ds = \\
&\quad c\int_{-\infty}^{\infty} [\bar{\mathbf{d}}_3(s) - \mathbf{e}_1]\, ds \cdot \zeta_1(s)\bar{\mathbf{d}}_2(s) \\
&\quad\quad - \int_{-\infty}^{\infty} [\bar{\mathbf{d}}_3(s) - \mathbf{e}_1] \cdot \int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)]\bar{\mathbf{d}}_2(u)\, du\, ds \\
&\quad\quad - \int_{-\infty}^{\infty} [\bar{\mathbf{d}}_3(s) - \mathbf{e}_1]\, ds \cdot \mathbf{e}_3 \int_{-\infty}^{s} \eta_2(u)\, du \\
&= c\int_{-\infty}^{\infty} \sin\bar{\phi}(s)\zeta_1(s)\, ds \\
&\quad\quad - \int_{-\infty}^{\infty} [\bar{\mathbf{d}}_3(s) - \mathbf{e}_1] \cdot \int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)]\bar{\mathbf{d}}_2(u)\, du\, ds \\
&= +\int_{-\infty}^{\infty} [\cos\bar{\phi}(s) - 1] \int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)] \sin\bar{\phi}(u)\, du\, ds \\
&\quad\quad - \int_{-\infty}^{\infty} \sin\bar{\phi}(s) \int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)] \cos\bar{\phi}(u)\, du\, ds
\end{aligned} \tag{C.3.11}$$

where we have used the identity $(\bar{\mathbf{d}}_3 - \mathbf{e}_1) \cdot \mathbf{e}_3 = 0$ and the linearized constraint (6.5.1.19). Combining equations (C.3.8) and (C.3.11), the linearized constraint on the impulse becomes

$$\begin{aligned}
0 &= \rho\int_{-\infty}^{\infty} [\cos\bar{\phi}(s) - 1] \int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)] \sin\bar{\phi}(u)\, du\, ds \\
&\quad -\rho\int_{-\infty}^{\infty} \sin\bar{\phi}(s) \int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)] \cos\bar{\phi}(u)\, du\, ds \\
&\quad -I_1 \int_{-\infty}^{\infty} \bar{\phi}' \eta_1\, ds.
\end{aligned} \tag{C.3.12}$$

We recall from equation (6.3.8) that the isotropic angular momentum can be expressed as

$$A = \tfrac{1}{2} \textstyle\int_{-\infty}^{\infty} \mathsf{B}_3\boldsymbol{\beta} \cdot \left(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\, ds \tag{C.3.13}$$

so its linearization is

$$\begin{aligned}
0 &= \tfrac{1}{2} \textstyle\int_{-\infty}^{\infty} \mathsf{B}_3\delta\boldsymbol{\beta} \cdot \left(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\, ds \\
&\quad + \tfrac{1}{2} \textstyle\int_{-\infty}^{\infty} \mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \left(\delta\boldsymbol{\gamma} + \delta\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}} + \mathsf{D}_3[\boldsymbol{\beta}]\delta\tilde{\boldsymbol{\Lambda}}\right)\, ds
\end{aligned} \tag{C.3.14}$$

Using the expressions (C.4.8), (6.4.50), (6.5.1.27) and (6.4.40), equation (C.3.14) we find

$$\tfrac{1}{2} \textstyle\int_{-\infty}^{\infty} \mathsf{B}_3\delta\boldsymbol{\beta} \cdot \left(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\, ds = \textstyle\int_{-\infty}^{\infty} \tfrac{1}{2} c I_1 \bar{\phi}' \zeta_2\, ds \tag{C.3.15}$$

and

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \mathsf{B}_3\bar{\boldsymbol{\beta}}\cdot\left(\delta\boldsymbol{\gamma}+\delta\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}+\mathsf{D}_3[\boldsymbol{\beta}]\delta\tilde{\boldsymbol{\Lambda}}\right)\,ds=\int_{-\infty}^{\infty}\left\{-\tfrac{1}{2}cI_1\bar{\phi}'\zeta_2+I_3\left[\eta_3+c\bar{\phi}'\zeta_2\right]\right\}\,ds, \tag{C.3.16}$$

so equation (C.3.14) may be rewritten as

$$0=I_3\int_{-\infty}^{\infty}\left[\eta_3+c\bar{\phi}'\zeta_2\right]\,ds. \tag{C.3.17}$$

Using equations (6.3.14) and (2.6.2.27), the total angular momentum is given by

$$\boldsymbol{\Omega}\;\equiv\;\int_{-\infty}^{\infty}\left\{\tfrac{1}{2}\mathbf{e}_k\,\mathsf{B}_k^o\boldsymbol{\beta}\cdot\left(\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)+\rho(\mathbf{h}+s\mathbf{e}_1)\times\left(\mathbf{g}+\tilde{\boldsymbol{\Lambda}}'\right)\right\}\,ds, \tag{C.3.18}$$

so the linearized constraint is

$$\begin{aligned}\mathbf{0}\;=\;&\int_{-\infty}^{\infty}\left\{\tfrac{1}{2}\mathbf{e}_k\,\mathsf{B}_k^o\delta\boldsymbol{\beta}\cdot\left(\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)+\tfrac{1}{2}\mathbf{e}_k\,\mathsf{B}_k^o\boldsymbol{\beta}\cdot\left(\delta\boldsymbol{\gamma}+\mathsf{D}_3[\delta\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}+\mathsf{D}_3[\delta\boldsymbol{\beta}]\delta\tilde{\boldsymbol{\Lambda}}\right)\right\}\,ds\\&+\int_{-\infty}^{\infty}\left\{\bar{\mathbf{r}}\times\left(\delta\mathbf{g}+\delta\tilde{\boldsymbol{\Lambda}}'\right)+\delta\mathbf{h}\times\left(\mathbf{g}+\tilde{\boldsymbol{\Lambda}}'\right)\right\}\,ds,\end{aligned} \tag{C.3.19}$$

where $\bar{\mathbf{r}}\equiv\bar{\mathbf{h}}+s\mathbf{e}_1=\bar{x}\mathbf{e}_1+\bar{y}\mathbf{e}_2$.

To simplify the angular terms in equation (C.3.19) we exploit the multiplicative properties of the skew matrices B_k^o and B_k and the symmetric matrices D_j^k. Specifically, using equations (2.6.2.57), (2.6.2.68), (2.6.2.69) and (6.5.1.2) we can rewrite

$$\begin{aligned}\tfrac{1}{2}\mathbf{e}_k\,\mathsf{B}_k^o\delta\boldsymbol{\beta}\cdot\left(\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\;&=\;\tfrac{1}{2}\zeta_j\left[\mathsf{B}_l\bar{\boldsymbol{\beta}}\cdot\left(\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\right]\left(\mathsf{B}_k^o\mathsf{B}_j\bar{\boldsymbol{\beta}}\cdot\mathsf{B}_l\bar{\boldsymbol{\beta}}\right)\,\mathbf{e}_k\\&=\;-\tfrac{1}{2}\zeta_j\left[\mathsf{B}_l\bar{\boldsymbol{\beta}}\cdot\left(\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\right]\left(\mathsf{D}_j^k\bar{\boldsymbol{\beta}}\cdot\mathsf{B}_l\bar{\boldsymbol{\beta}}\right)\,\mathbf{e}_k\\&=\;-\tfrac{1}{2}\zeta_j\left[\mathsf{B}_l\bar{\boldsymbol{\beta}}\cdot\left(\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\right]\left(\bar{\boldsymbol{\beta}}\cdot\mathsf{D}_j^k\mathsf{B}_l\bar{\boldsymbol{\beta}}\right)\,\mathbf{e}_k\\&=\;-\tfrac{1}{2}\zeta_j\left[\mathsf{B}_l\bar{\boldsymbol{\beta}}\cdot\left(\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\right]\left(\delta_{k1}\bar{\boldsymbol{\beta}}\cdot\mathsf{B}_k^o\bar{\boldsymbol{\beta}}-\epsilon_{jli}\,\bar{\boldsymbol{\beta}}\cdot\mathsf{D}_i^k\bar{\boldsymbol{\beta}}\right)\,\mathbf{e}_k\\&=\;\tfrac{1}{2}\epsilon_{jli}\,\zeta_j\left[\mathsf{B}_l\bar{\boldsymbol{\beta}}\cdot\left(\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\right]\left(\bar{\boldsymbol{\beta}}\cdot\mathsf{D}_i^k\bar{\boldsymbol{\beta}}\right)\,\mathbf{e}_k\\&=\;\tfrac{1}{2}\epsilon_{jli}\,\zeta_j\left[\mathsf{B}_l\bar{\boldsymbol{\beta}}\cdot\left(\boldsymbol{\gamma}+\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right)\right](\mathbf{e}_k\cdot\mathbf{d}_i)\,\mathbf{e}_k\\&=\;cI_1\bar{\phi}'\left(\zeta_2\mathbf{d}_3-\zeta_3\mathbf{d}_2\right).\end{aligned} \tag{C.3.20}$$

Similarly we can rewrite

$$\left(\mathsf{B}_j\bar{\boldsymbol{\beta}}\cdot\mathsf{B}_k^o\bar{\boldsymbol{\beta}}\right)\mathbf{e}_k=-\left(\bar{\boldsymbol{\beta}}\cdot\mathsf{B}_j\mathsf{B}_k^o\bar{\boldsymbol{\beta}}\right)\mathbf{e}_k=\left(\bar{\boldsymbol{\beta}}\cdot\mathsf{D}_j^k\bar{\boldsymbol{\beta}}\right)\mathbf{e}_k=(\mathbf{d}_j\cdot\mathbf{e}_j)\,\mathbf{e}_k=\mathbf{d}_j, \tag{C.3.21}$$

from which we conclude

$$\begin{aligned}&\tfrac{1}{2}\mathbf{e}_k\,\mathsf{B}_k^o\boldsymbol{\beta}\cdot\left(\delta\boldsymbol{\gamma}+\delta\mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}+\mathsf{D}_3[\boldsymbol{\beta}]\delta\tilde{\boldsymbol{\Lambda}}\right)\\&\quad=\;I_1\left[\eta_1-c\zeta_1'\right]\mathbf{d}_1+\left[I_1\left(\eta_2-c\bar{\phi}'\zeta_3\right)-\tfrac{1}{2}cI_1\bar{\phi}'\zeta_3\right]\mathbf{d}_2\\&\qquad+\left[I_3\left(\eta_3+c\bar{\phi}'\zeta_2\right)+\tfrac{1}{2}cI_1\bar{\phi}'\zeta_2\right]\mathbf{d}_3.\end{aligned} \tag{C.3.22}$$

Adding equations (C.3.20) and (C.3.22) we obtain

$$\begin{aligned}\tfrac{1}{2}\mathbf{e}_k \mathsf{B}_k^o \boldsymbol{\beta} \cdot \left(\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]\tilde{\boldsymbol{\Lambda}}\right) + \tfrac{1}{2}\mathbf{e}_k \mathsf{B}_k^o \boldsymbol{\beta} \cdot (\boldsymbol{\gamma} + \mathsf{D}_3[\boldsymbol{\beta}]) &= I_1\left(\eta_1 - c\zeta_1'\right)\mathbf{d}_1 + I_1\left(\eta_2 - c\bar{\phi}'\zeta_3\right)\mathbf{d}_2 \\ &\quad + I_3\left(\eta_3 + c\bar{\phi}'\zeta_2\right)\mathbf{d}_3. \qquad \text{(C.3.23)}\end{aligned}$$

To simplify the translational terms in (C.3.19), we employ equation (C.3.1) and integrate by parts to obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\left\{\bar{\mathbf{r}}(s)\times\left(\delta\mathbf{g}(s) + \delta\tilde{\boldsymbol{\Lambda}}'(s)\right) + \delta\mathbf{h}(s)\times\left(\mathbf{g}(s) + \tilde{\boldsymbol{\Lambda}}'(s)\right)\right\} ds \\ &= \int_{-\infty}^{\infty}\left\{\bar{\mathbf{r}}(s)\times\left(\delta\mathbf{g}(s) + \delta\tilde{\boldsymbol{\Lambda}}'(s)\right) - \rho c\delta\mathbf{h}(s)\times\bar{\mathbf{h}}'(s)\right\} ds \\ &= -\rho c\int_{-\infty}^{\infty}\mathbf{e}_1\times\delta\mathbf{h}(s)\, ds \\ &\quad + \int_{-\infty}^{\infty}\left\{\bar{\mathbf{r}}(s)\times\left(\delta\mathbf{g}(s) + \delta\tilde{\boldsymbol{\Lambda}}'(s)\right) + \rho c(\bar{\mathbf{h}}' + \mathbf{e}_1)\times\delta\mathbf{h}\right\} ds \\ &= -\rho c\int_{-\infty}^{\infty}\mathbf{e}_1\times\delta\mathbf{h}(s)\, ds \\ &\quad + \int_{-\infty}^{\infty}\bar{\mathbf{r}}(s)\times(\delta\mathbf{g}(s) + \delta\tilde{\boldsymbol{\Lambda}}'(s) - \rho c\delta\mathbf{h}'(s))\, ds. \qquad \text{(C.3.24)}\end{aligned}$$

Using equations (6.5.1.16), (6.5.1.18) and (C.2.11) we can write

$$\begin{aligned}\delta\mathbf{g}(s) + \delta\tilde{\boldsymbol{\Lambda}}'(s) - \rho c\delta\mathbf{h}'(s) &= 2\rho c\zeta_1(s)\bar{\mathbf{d}}_2(s) - \rho\int_{-\infty}^{s}[\eta_1(u) + c\zeta_1'(u)]\bar{\mathbf{d}}_2(u)\, du \\ &\quad -\rho\mathbf{e}_3\int_{-\infty}^{s}[\eta_2(u) - c\zeta_2'(u)]\, du, \qquad \text{(C.3.25)}\end{aligned}$$

$$\bar{\mathbf{r}}(s)\times\bar{\mathbf{d}}_2(u) = [\bar{x}(s)\cos\bar{\phi}(u) + \bar{y}(s)\sin\bar{\phi}(u)]\mathbf{e}_3, \qquad \text{(C.3.26)}$$

$$\bar{\mathbf{r}}(s)\times\mathbf{e}_3 = \bar{y}(s)\mathbf{e}_1 - \bar{x}(s)\mathbf{e}_2, \qquad \text{(C.3.27)}$$

and

$$\mathbf{e}_1\times\delta\mathbf{h}(s) = \int_{-\infty}^{s}\zeta_2(u)\, du\mathbf{e}_2 - \int_{-\infty}^{s}\cos\bar{\phi}(u)\,\zeta_1(u)\, du\mathbf{e}_3. \qquad \text{(C.3.28)}$$

Combining equation (C.3.24) - (C.3.27) and rearranging terms we obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty} &\left\{ \bar{\mathbf{r}}(s) \times \left(\delta\mathbf{g}(s) + \delta\tilde{\boldsymbol{\Lambda}}'(s) \right) + \delta\mathbf{h}(s) \times \left(\mathbf{g}(s) + \tilde{\boldsymbol{\Lambda}}'(s) \right) \right\} \, ds \\
= \quad & -\rho \int_{-\infty}^{\infty} \bar{y}(s) \int_{-\infty}^{s} [\eta_2(u) - c\zeta_2'(u)] \, du \, ds \mathbf{e}_1 \\
& -\rho c \int_{-\infty}^{\infty} \int_{-\infty}^{s} \zeta_2(u) \, du \, ds \mathbf{e}_2 \\
& +\rho \int_{-\infty}^{\infty} \bar{x}(s) \int_{-\infty}^{s} [\eta_2(u) - c\zeta_2'(u)] \, du \, ds \mathbf{e}_2 \\
& +2\rho c \int_{-\infty}^{\infty} [\bar{x}(s) \cos\bar{\phi}(s) + \bar{y}(s) \sin\bar{\phi}(s)] \zeta_1(s) \, ds \mathbf{e}_3 \\
& -\rho \int_{-\infty}^{\infty} \bar{x}(s) \int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)] \cos\bar{\phi}(u) \, du \, ds \mathbf{e}_3 \\
& -\rho \int_{-\infty}^{\infty} \bar{y}(s) \int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)] \sin\bar{\phi}(u) \, du \, ds \mathbf{e}_3 \\
& +\rho c \int_{-\infty}^{\infty} \int_{-\infty}^{s} \cos\bar{\phi}(u) \zeta_1(u) \, du \, ds \mathbf{e}_3.
\end{aligned} \tag{C.3.29}
$$

By further combining equations (C.3.23) and (C.3.29), the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ components of the linearized constraint on the total angular momentum are respectively

$$
\begin{aligned}
0 \quad = \quad & -I_1 \int_{-\infty}^{\infty} [\eta_2(s) - c\zeta_2'(s)] \sin\bar{\phi}(s) \, ds - I_3 \int_{-\infty}^{\infty} [\eta_3(s) + c\bar{\phi}'(s)\zeta_2(s)] \cos\bar{\phi}(s) \, ds \\
& -\rho \int_{-\infty}^{\infty} \bar{y}(s) \int_{-\infty}^{s} [\eta_2(u) - c\zeta_2'(u)] \, du \, ds,
\end{aligned} \tag{C.3.30}
$$

$$
\begin{aligned}
0 \quad = \quad & I_1 \int_{-\infty}^{\infty} [\eta_2(s) - c\zeta_2'(s)] \cos\bar{\phi}(s) \, ds + I_3 \int_{-\infty}^{\infty} [\eta_3(s) + c\bar{\phi}'(s)\zeta_2(s)] \sin\bar{\phi}(s) \, ds \\
& -c\rho \int_{-\infty}^{\infty} \int_{-\infty}^{s} \zeta_2(u) \, du \, ds + \rho \int_{-\infty}^{\infty} \bar{x}(s) \int_{-\infty}^{s} [\eta_2(u) - c\zeta_2'(u)] \, du \, ds
\end{aligned} \tag{C.3.31}
$$

and

$$
\begin{aligned}
0 \quad = \quad & -I_1 \int_{-\infty}^{\infty} \eta_1(s) \, ds + c\rho \int_{-\infty}^{\infty} \int_{-\infty}^{s} \cos\bar{\phi}(u) \zeta_1(u) \, du \, ds \\
& +2c\rho \int_{-\infty}^{\infty} [\bar{x}(s) \cos\bar{\phi}(s) + \bar{y}(s) \sin\bar{\phi}(s)] \zeta_1(s) \, ds \\
& -\rho \int_{-\infty}^{\infty} \bar{x}(s) \int_{-\infty}^{s} [\eta_1(u) + c\zeta_2'(u)] \cos\bar{\phi}(u) \, du \, ds \\
& -\rho \int_{-\infty}^{\infty} \bar{y}(s) \int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)] \sin\bar{\phi}(u) \, du \, ds,
\end{aligned} \tag{C.3.32}
$$

Finally, the linearized constraints on two transverse components of linear momentum are

$$
\int_{-\infty}^{\infty} \mathbf{e}_2 \cdot \left(\mathbf{g}(s) + \tilde{\boldsymbol{\Lambda}}'(s) \right) \, ds = 0, \tag{C.3.33}
$$

and

$$
\int_{-\infty}^{\infty} \mathbf{e}_3 \cdot \left(\mathbf{g}(s) + \tilde{\boldsymbol{\Lambda}}'(s) \right) \, ds = 0. \tag{C.3.34}
$$

Using equation (C.2.12) we can rewrite (C.3.33) as

$$\begin{aligned} 0 &= c\int_{-\infty}^{\infty} \cos\bar{\phi}(s)\,\zeta_1(s)\,ds - \int_{-\infty}^{\infty}\int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)]\cos\bar{\phi}(u)\,du\,ds \\ &= -\int_{-\infty}^{\infty}\int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)]\cos\bar{\phi}(u)\,du\,ds, \end{aligned} \tag{C.3.35}$$

where the second equality follows from the linearized constraint (6.5.1.20). Similarly equation (C.3.34) can be written as

$$0 = \int_{-\infty}^{\infty}\int_{-\infty}^{s} \eta_2(u)\,du\,ds. \tag{C.3.36}$$

C.4 Second Variation

Finally we compute the second variations of the functionals F_1 and F_2.

To compute $\delta^2 F_1$, we first combine equations (6.5.2.6), (6.5.1.26) and (6.5.1.17) to obtain

$$\delta\mathbf{g}_c = -\rho\int_{-\infty}^{s} \left(\eta_1(u) + c\zeta_1'(u)\right)\bar{\mathbf{d}}_2(u)\,du + \rho\int_{-\infty}^{s} \left(\eta_2(u) + c\zeta_2'(u)\right)\bar{\mathbf{d}}_1\,du. \tag{C.4.1}$$

Similarly we can combine equations (6.5.2.7) and (6.5.1.27) to obtain

$$\delta\boldsymbol{\gamma}_c = 2I_j\mathsf{B}_j\bar{\boldsymbol{\beta}}\left(\eta_j + c\zeta_j'\right). \tag{C.4.2}$$

Differentiating equation (6.5.2.5) and inserting expressions (C.4.1) and (C.4.2), the second variation of F_1 can be written as

$$\begin{aligned} \delta^2 F_1 &= \rho\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{s} [\eta_1(u) + c\zeta_1'(u)]\bar{\mathbf{d}}_2(u)\,du\right\}^2 ds \\ &\quad +\rho\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{s} [\eta_2(u) + c\zeta_2'(u)]\,du\right\}^2 ds \\ &\quad +I_j\int_{-\infty}^{\infty}\left\{\eta_j(s) + c\zeta_j'(s)\right\}^2 ds. \end{aligned} \tag{C.4.3}$$

From equation (6.5.2.14), the second variation of F_2 is given by

$$\begin{aligned}
\delta^2 F_2(\delta\boldsymbol{\beta}) \;=\;& 4\int_{-\infty}^{\infty}(K_j - c^2 I_j)(\mathsf{B}_j\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\beta}' + \mathsf{B}_j\delta\boldsymbol{\beta}\cdot\bar{\boldsymbol{\beta}}')^2\,ds \\
&+8\int_{-\infty}^{\infty}(K_1 - c^2 I_1)(\mathsf{B}_1\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}')(\mathsf{B}_1\delta\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}')\,ds \\
&-32\int_{-\infty}^{\infty}(K_1 - c^2 I_1)(\mathsf{B}_1\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}')(\mathsf{B}_j\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\beta}' + \mathsf{B}_j\delta\boldsymbol{\beta}\cdot\bar{\boldsymbol{\beta}}')(\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\beta})\,ds \\
&-8\int_{-\infty}^{\infty}(K_1 - c^2 I_1)(\mathsf{B}_1\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}')^2\left[|\delta\boldsymbol{\beta}|^2 - 6(\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\beta})^2\right]\,ds \\
&+2\int_{-\infty}^{\infty}\left(K_1 - c^2 I_1\right)\frac{1}{a^2}\left[(\delta\beta_1-\delta\beta_3)^2+(\delta\beta_2-\delta\beta_4)^2\right]\,ds \\
&-16\int_{-\infty}^{\infty}(K_1 - c^2 I_1)(\bar{\beta}_2\delta\boldsymbol{\beta}_2+\bar{\beta}_3\delta\boldsymbol{\beta}_3)(\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\beta})\,ds \\
&-2\int_{-\infty}^{\infty}\left(K_1 - c^2 I_1\right)\frac{1}{a^2}\left[(\beta_1-\beta_3)^2+(\beta_2-\beta_4)^2\right]\left[|\delta\boldsymbol{\beta}|^2 - 4(\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\beta})^2\right]\,ds \\
&+4\int_{-\infty}^{\infty}\left(K_1 - c^2 I_1\right)(\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\gamma}+\bar{\boldsymbol{\gamma}}\cdot\delta\boldsymbol{\beta})(\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\beta})'\,ds,
\end{aligned} \tag{C.4.4}$$

where we have taken advantage of the fact that $\mathsf{B}_2\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}' = 0 = \mathsf{B}_3\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}}'$. Notice that the second variation of F_2 is neutral in the direction $\bar{\boldsymbol{\beta}}$, in accordance with the homogeneity of F_2 with respect to $\boldsymbol{\beta}$.

From the parametrization (6.5.1.2) of $\delta\boldsymbol{\beta}$ we obtain the following list of identities which we use to simplify the second variation of F_2:

$$4|\delta\boldsymbol{\beta}|^2 = \left(\zeta_1^2+\zeta_2^2+\zeta_3^2\right), \tag{C.4.5}$$

$$2\mathsf{B}_1\delta\boldsymbol{\beta} = -\zeta_1\bar{\boldsymbol{\beta}} - \zeta_2\mathsf{B}_3\bar{\boldsymbol{\beta}} + \zeta_3\mathsf{B}_2\bar{\boldsymbol{\beta}}, \tag{C.4.6}$$

$$2\mathsf{B}_2\delta\boldsymbol{\beta} = \zeta_1\mathsf{B}_3\bar{\boldsymbol{\beta}} - \zeta_2\bar{\boldsymbol{\beta}} - \zeta_3\mathsf{B}_1\bar{\boldsymbol{\beta}}, \tag{C.4.7}$$

$$2\mathsf{B}_3\delta\boldsymbol{\beta} = -\zeta_1\mathsf{B}_2\bar{\boldsymbol{\beta}} + \zeta_2\mathsf{B}_1\bar{\boldsymbol{\beta}} - \zeta_3\bar{\boldsymbol{\beta}}, \tag{C.4.8}$$

$$2\delta\boldsymbol{\beta}' = \tfrac{1}{2}\bar{\phi}'\zeta_1\bar{\boldsymbol{\beta}} + \zeta_1'\mathsf{B}_1\bar{\boldsymbol{\beta}} + \left(\zeta_2' + \tfrac{1}{2}\bar{\phi}'\zeta_3\right)\mathsf{B}_2\bar{\boldsymbol{\beta}} + \left(\zeta_3' - \tfrac{1}{2}\bar{\phi}'\zeta_2\right)\mathsf{B}_3\bar{\boldsymbol{\beta}}, \tag{C.4.9}$$

and

$$2\left(\mathsf{B}_1\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\beta}' + \mathsf{B}_1\delta\boldsymbol{\beta}\cdot\bar{\boldsymbol{\beta}}'\right) = \zeta_1', \tag{C.4.10}$$

$$2\left(\mathsf{B}_2\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\beta}' + \mathsf{B}_2\delta\boldsymbol{\beta}\cdot\bar{\boldsymbol{\beta}}'\right) = \zeta_2' + \bar{\phi}'\zeta_3, \tag{C.4.11}$$

$$2\left(\mathsf{B}_3\bar{\boldsymbol{\beta}}\cdot\delta\boldsymbol{\beta}' + \mathsf{B}_3\delta\boldsymbol{\beta}\cdot\bar{\boldsymbol{\beta}}'\right) = \zeta_3' - \bar{\phi}'\zeta_2, \tag{C.4.12}$$

$$\begin{aligned}
&2\left[(\delta\boldsymbol{\beta}_1-\delta\boldsymbol{\beta}_3)^2+(\delta\boldsymbol{\beta}_2-\delta\boldsymbol{\beta}_4)^2\right] \\
&\quad= \tfrac{1}{2}(1+\cos\bar{\phi})\zeta_1^2 + \tfrac{1}{2}(1+\cos\bar{\phi})\zeta_2^2 + \tfrac{1}{2}(1-\cos\bar{\phi})\zeta_3^2 + \sin\bar{\phi}\zeta_2\zeta_3.
\end{aligned} \tag{C.4.13}$$

and

$$4\mathsf{B}_1\delta\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}_s = -\tfrac{1}{2}\bar{\phi}'\zeta_1^2 + \tfrac{1}{2}\bar{\phi}'\zeta_2^2 + \tfrac{1}{2}\bar{\phi}'\zeta_3^2 - \zeta_2\zeta_3' + \zeta_3\zeta_2'. \tag{C.4.14}$$

The terms in the integrand of $\delta^2 F_2$ (C.4.4) can therefore be rewritten as

$$\begin{aligned}
4(K_1 - c^2 I_1)(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \delta\boldsymbol{\beta}' + \mathsf{B}_1\delta\boldsymbol{\beta} \cdot \bar{\boldsymbol{\beta}}')^2 &= (K_1 - c^2 I_1)\zeta_1'^2, &\text{(C.4.15)}\\
4(K_2 - c^2 I_2)(\mathsf{B}_2\bar{\boldsymbol{\beta}} \cdot \delta\boldsymbol{\beta}' + \mathsf{B}_2\delta\boldsymbol{\beta} \cdot \bar{\boldsymbol{\beta}}')^2 &= (K_2 - c^2 I_2)\left(\zeta_2' + \bar{\phi}'\zeta_3\right)^2, &\text{(C.4.16)}\\
4(K_3 - c^2 I_3)(\mathsf{B}_3\bar{\boldsymbol{\beta}} \cdot \delta\boldsymbol{\beta}' + \mathsf{B}_3\delta\boldsymbol{\beta} \cdot \bar{\boldsymbol{\beta}}')^2 &= (K_3 - c^2 I_3)\left(\zeta_3' - \bar{\phi}'\zeta_2\right)^2, &\text{(C.4.17)}
\end{aligned}$$

$$8\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)\left(\mathsf{B}_1\delta\boldsymbol{\beta} \cdot \delta\boldsymbol{\beta}'\right) - 8\left(\mathsf{B}_1\bar{\boldsymbol{\beta}} \cdot \bar{\boldsymbol{\beta}}'\right)^2 |\delta\boldsymbol{\beta}|^2 = -\bar{\phi}'\left(\zeta_2^2 + \zeta_3^2\right) + \bar{\phi}'\left(\zeta_2\zeta_3' - \zeta_3\zeta_2'\right), \quad \text{(C.4.18)}$$

and

$$\begin{aligned}
&2\left[(\delta\beta_1 - \delta\beta_3)^2 + (\delta\beta_2 - \delta\beta_4)^2\right]\\
&\quad -2\left[(\beta_1 - \beta_3)^2 + (\beta_2 - \beta_4)^2\right]|\delta\boldsymbol{\beta}|^2 = \cos\bar{\phi}\zeta_1^2 + \cos\bar{\phi}\zeta_2^2 + \sin\bar{\phi}\zeta_2\zeta_3. \quad \text{(C.4.19)}
\end{aligned}$$

Combining equations (C.4.15) - (C.4.19) we find that the second variation of F_2 can be expressed as

$$\begin{aligned}
\delta^2 F_2 = & \int_{-\infty}^{\infty} \left(K_1 - c^2 I_1\right) \zeta_1'^2 \, ds\\
& + \int_{-\infty}^{\infty} \left(K_2 - c^2 I_2\right) \left(\zeta_2' + \bar{\phi}'\zeta_3\right)^2 \, ds\\
& + \int_{-\infty}^{\infty} \left(K_3 - c^2 I_3\right) \left(\zeta_3' - \bar{\phi}'\zeta_2\right)^2 \, ds\\
& - \int_{-\infty}^{\infty} \left(K_1 - c^2 I_1\right) \bar{\phi}'^2 \left(\zeta_2^2 + \zeta_3^2\right) \, ds\\
& + \int_{-\infty}^{\infty} \left(K_1 - c^2 I_1\right) \bar{\phi}' \left(\zeta_2\zeta_3' - \zeta_3\zeta_2'\right) \, ds\\
& + \int_{-\infty}^{\infty} \left(K_1 - c^2 I_1\right) \frac{1}{a^2} \left(\cos\bar{\phi}\zeta_1^2 + \cos\bar{\phi}\zeta_2^2 + \sin\bar{\phi}\zeta_2\zeta_3\right) \, ds. \quad \text{(C.4.20)}
\end{aligned}$$

We can then expand the second integrand on the right hand side and substitute equation (5.5.25) for $\bar{\phi}'$ to obtain

$$\begin{aligned}
\delta^2 F_2 = & \left(K_1 - c^2 I_1\right) \int_{-\infty}^{\infty} \left\{\zeta_1'^2 + \frac{1}{a^2}\cos\bar{\phi}\zeta_1^2\right\} ds\\
& + \left(K_1 - c^2 I_1\right) \int_{-\infty}^{\infty} \left\{\zeta_2'^2 + \frac{1}{a^2}\cos\bar{\phi}\zeta_2^2 - \frac{2}{a^2}(1 - \cos\bar{\phi})\zeta_2^2\right\} ds\\
& + \left(K_1 - c^2 I_1\right) \int_{-\infty}^{\infty} \left\{\bar{\phi}'\left(\zeta_2\zeta_3' + \zeta_3\zeta_2'\right) + \frac{1}{a^2}\sin\bar{\phi}\zeta_2\zeta_3\right\} ds\\
& + \left[\left(K_2 - c^2 I_2\right) - \left(K_1 - c^2 I_1\right)\right] \int_{-\infty}^{\infty} \left(\zeta_2' + \bar{\phi}'\zeta_3\right)^2 ds\\
& + \left(K_3 - c^2 I_3\right) \int_{-\infty}^{\infty} \left(\zeta_3' - \bar{\phi}'\zeta_2\right)^2 ds. \quad \text{(C.4.21)}
\end{aligned}$$

Using the differential equation (5.5.24) for the planar solitary wave and an integration by parts we can rewrite

$$
\begin{aligned}
\int_{-\infty}^{\infty} \left\{ \bar{\phi}' \left(\zeta_2 \zeta_3' + \zeta_3 \zeta_2' \right) + \frac{1}{a^2} \sin \bar{\phi} \zeta_2 \zeta_3 \right\} ds &= \int_{-\infty}^{\infty} \left\{ \bar{\phi}' \left(\zeta_2 \zeta_3' + \zeta_3 \zeta_2' \right) + \bar{\phi}'' \zeta_2 \zeta_3 \right\} ds \\
&= \int_{-\infty}^{\infty} \left\{ \bar{\phi}' \left(\zeta_2 \zeta_3' + \zeta_3 \zeta_2' \right) - \bar{\phi}' \left(\zeta_2 \zeta_3' + \zeta_2' \zeta_3 \right) \right\} ds \\
&= 0, \qquad \text{(C.4.22)}
\end{aligned}
$$

which allows us to express the second variation more simply as

$$
\begin{aligned}
\delta^2 F_2 &= \Big(K_1 - c^2 I_1 \Big) \int_{-\infty}^{\infty} \left\{ \zeta_1'^2 + \frac{1}{a^2} \cos \bar{\phi}\, \zeta_1^2 \right\} ds \\
&\quad + \Big(K_1 - c^2 I_1 \Big) \int_{-\infty}^{\infty} \left\{ \zeta_2'^2 + \frac{1}{a^2} (3 \cos \bar{\phi} - 2) \zeta_2^2 \right\} ds \\
&\quad + \Big[\Big(K_2 - c^2 I_2 \Big) - \Big(K_1 - c^2 I_1 \Big) \Big] \int_{-\infty}^{\infty} \Big(\zeta_2' + \bar{\phi}' \zeta_3 \Big)^2 ds \\
&\quad + \Big(K_3 - c^2 I_3 \Big) \int_{-\infty}^{\infty} \Big(\zeta_3' - \bar{\phi}' \zeta_2 \Big)^2 ds. \qquad \text{(C.4.23)}
\end{aligned}
$$

In view of the interpretations (6.5.1.11) – (6.5.1.13) of the parameters ζ_j in terms of the Euler angles, it is not surprising that the second variation of F_2 in (C.4.23) is the same as the one (A.5.15) which we obtain in Appendix A.

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