ANALYSE III **Corrigé examen** 2018

Question 1

(a) $\partial S = C_1 \cup C_2$ where C_1 is the elipse $x^2 + 4y^2 = 4$ in the 2^{nd}, 3^{rd} and 4^{th} quadrants of the x-y plane and C_2 is the line $x + 2y = 2$ in the first quadrant.

C_1 and C_2 can be parametrised as
$C_1 = \alpha_1(t) = (2\cos(t), \sin(t))$ where $\frac{\pi}{2} < t < 2\pi$ and
$C_2 = \alpha_2(t) = (2(1-t), t)$ where $0 < t < 1$.

Both C_1 and C_2 are smooth since $\alpha'_{1,2}(t)$ exist and are continous.

$\alpha'_{1,2}(t) \neq (0,0) \Rightarrow \alpha$ is a regular parametric representations.

(b) Let $\mathbf{f}(x, y) = (xy^2 - xy, x^2 y)$

Now, $I_1 = \int_{\partial S} \mathbf{f}(x, y) \cdot d\mathbf{s} = \int_{C_1} \mathbf{f}(x, y) \cdot d\mathbf{s} + \int_{C_2} \mathbf{f}(x, y) \cdot d\mathbf{s}$

We then write the integral along the curve in terms of the parametric representation as
$\int_{C_1} \mathbf{f}(x, y) \cdot d\mathbf{s} = \int_{\frac{\pi}{2}}^{2\pi} \langle \mathbf{f}(\alpha_1(t)), \alpha'_1(t) \rangle dt$ and
$\int_{C_2} \mathbf{f}(x, y) \cdot d\mathbf{s} = \int_0^1 \langle \mathbf{f}(\alpha_2(t)), \alpha'_2(t) \rangle dt$

Since the orientation of the tangent is not mentioned in the question, full points are awarded for either orientation.

$I_2 = \int_{\partial S} \mathbf{f}(x, y) \cdot d\mathbf{s} = \int_{C_1} \mathbf{f}(x, y) \cdot d\mathbf{s} + \int_{C_2} \mathbf{f}(x, y) \cdot d\mathbf{s}$

$$
\begin{aligned}
\int_{C_1} \mathbf{f}(x, y) \cdot d\mathbf{s} &= \int_{\frac{\pi}{2}}^{2\pi} \langle \mathbf{f}(\alpha_1(t)), \alpha'_1(t) \rangle dt \\
&= \int_{\frac{\pi}{2}}^{2\pi} \langle (2\cos(t)\sin^2(t) - 2\cos(t)\sin(t), 4\cos^2(t)\sin(t)), (-2\sin(t), \cos(t)) \rangle dt \\
&= \int_{\frac{\pi}{2}}^{2\pi} -4\cos(t)\sin^3(t) + 4\cos(t)\sin^2(t) + 4\cos^3(t)\sin(t) dt \\
&= \left[-\sin^4(t) + \frac{4}{3}\sin^3(t) - \cos^4(t) \right]_{\frac{\pi}{2}}^{2\pi} \\
&= -\frac{4}{3}
\end{aligned}
$$

$$
\begin{aligned}
\int_{C_1} \mathbf{f}(x,y)\cdot d\mathbf{s} &= \int_0^1 \langle \mathbf{f}(\alpha_1(t)), \alpha_1'(t)\rangle dt \\
&= \int_0^1 \langle (2(1-t)t^2 - 2(1-t)t), 4(1-t)^2 t), (-2,1)\rangle dt \\
&= \int_0^1 (-4(1-t)t^2 + 4(1-t)t + 4(1-t)^2 t) dt \\
&= \int_0^1 (4t^3 - 4t^2 - 4t^2 + 4t + 4t^3 - 8t^2 + 4t) dt \\
&= \int_0^1 (8t^3 - 16t^2 + 8t) dt \\
&= \left[2t^4 - \frac{16}{3}t^3 + 4t^2 \right]_0^1 \\
&= \frac{2}{3}
\end{aligned}
$$

Now we have $I_2 = -\frac{4}{3} + \frac{2}{3} = -\frac{2}{3}$

(c) Let C be a simple closed, piece-wise smooth curve C in a 2D x-y plane and S be the surface on the plane bounded by C. Let f(x,y) and g(x,y) be functions with continous partial derivatives, defined on the plane. Now,

$$
\int_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy = \int_C f dx + g dy
$$

where the integration along C is done along positive orientation.

(d) Given that $\rho(x,y) = 1$. Mass M is given by

$$
M = \int_S dxdy
$$

If one takes $f = y$ and $g = 3x$, $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2$ and applies Green's theorem one can write

$$
\begin{aligned}
M &= \frac{1}{2} \int_{\partial S} (y, 3x) \cdot d\mathbf{s} \\
&= \frac{I_1}{2}
\end{aligned}
$$

Centre of mass along x, x_c is given by

$$
x_c = \frac{1}{M} \int_S x dxdy
$$

Now, taking $f = xy^2 - xy$ and $g = x^2y$ and applying the Green's theorem, one gets

$$\begin{aligned} x_c =& \frac{1}{M} \int_{\partial S} (xy^2 - xy, x^2y) \cdot d\mathbf{s} \\ =& \frac{2I_2}{I_1} \end{aligned}$$

Similarly one finds

$$y_c = \frac{2I_3}{I_1}$$

Question 2

(a) S_1 is the face of the tetrahedron with vertices at (0,0,0), (1,0,0), (0,1,0) and (0,0,1), that is opposite to the origin and bounded by the planes x=0, y=0 and z=0. A simple regular parametric representation of S_1 is
$S_1 = \alpha(x, y) = (x, y, 1 - x - y)$ where $0 < x < 1$ and $0 < y < 1 - x$.

$\partial_1\alpha(x, y) = (1, 0, -1)$ and $\partial_2\alpha(x, y) = (0, 1, -1)$.

$$\begin{aligned}\mathbf{N} =& \frac{\partial_1\alpha \times \partial_2\alpha}{||\partial_1\alpha \times \partial_2\alpha||}\\ =& \frac{1}{\sqrt{3}}(1, 1, 1)\end{aligned}$$

(b) Flux of $\mathbf{f}$ across S_1 in the direction of $\mathbf{N_1} = \mathbf{N}$ is given by $\int_S \langle \mathbf{f}, \mathbf{N}\rangle d\sigma$.
This can be written in terms of the parametric representation as

$$\begin{aligned}\int_S \langle \mathbf{f}, \mathbf{N}\rangle d\sigma =& \int_0^1 \int_0^{1-x} \langle \mathbf{f}(\alpha), \mathbf{N}\rangle ||\partial_1\alpha \times \partial_2\alpha|| dydx\\ =& \int_0^1 \int_0^{1-x} \langle ((1 - x - y)x, (1 - x - y)y), 1 - (1 - x - y)^2), (1, 1, 1)\rangle dydx\\ =& \int_0^1 \int_0^{1-x} \left(-2y^2 + (3 - 4x)y - 2x^2 + 3x\right) dydx\\ =& \int_0^1 \left(\frac{2}{3}x^3 - \frac{3}{2}x^2 + \frac{5}{6}\right) dx\\ =& \frac{1}{2}\end{aligned}$$

(c) $\nabla \cdot \mathbf{f} = 0$

$\nabla \times \mathbf{f} = (-y, x, 0)$

(d) See lecture notes

(e) By (c), $\mathbf{f}$ is incompressible. Since $\mathbb{R}^3$ is star-shaped, there exists a vector field $\mathbf{g} \in \mathbf{C^1}(\mathbb{R}^\mathbf{3}, \mathbb{R}^\mathbf{3})$ such that $\mathbf{f} = \nabla \times \mathbf{g}$.

Observe that since $\partial S_1 = \partial S_2 =: \partial S$, we have that $S_1 \cup S_2 \cup \partial S$ forms the boundary of a volume $W \subset \mathbb{R}^3$. Let $\mathbf{N_2}$ be the vector field normal to S_2, pointing toward the interior of W. Let $\mathbf{N_1}$ be the vector field normal to S_1 defined in (b). Applying Stokes' theorem twice, we get :

$$\begin{aligned}\int_{S_2} \langle \mathbf{f}, \mathbf{N_2}\rangle d\sigma = \int_{S_2} \langle \nabla \times \mathbf{g}, \mathbf{N_2}\rangle d\sigma = \int_{\partial S_2} \mathbf{g} \cdot \mathbf{dl}\\ = \int_{\partial S_1} \mathbf{g} \cdot \mathbf{dl} = \int_{S_1} \langle \nabla \times \mathbf{g}, \mathbf{N_1}\rangle d\sigma = \int_{S_1} \langle \mathbf{f}, \mathbf{N_1}\rangle d\sigma = \frac{1}{2}\end{aligned}$$

by (b). Here ∂S_i is positively oriented with respect to $\mathbf{N_i}$, $i = 1, 2$. Similarly, for $\mathbf{N_2}$ pointing outward W, we have

$$\int_{S_2} \langle \mathbf{f}, \mathbf{N_2}\rangle d\sigma = -\frac{1}{2}$$

Question 3

I. (a) Firstly we apply the standard separation of variables assumption $u_n(x,y) = f(x)g(y)$ then

$$\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = \lambda = -\frac{1}{g}\frac{\partial^2 g}{\partial y^2}$$

Then $f'' = \lambda f$.
In order to satisfy the given boundary conditions for $x = 0$ and $x = 1$ we must have $\lambda < 0$.
So f is of the form $f(x) = C\sin(n\pi x) + D\cos(n\pi x)$.
Also $g'' = -\lambda g$.
Therefore g is of the form

$$\begin{aligned} g(y) &= A\sinh(n\pi(1-y)) + B\cosh(n\pi(1-y)) \\ &= P\mathrm{e}^{\sqrt{-\lambda}ny} + Q\mathrm{e}^{-\sqrt{-\lambda}ny} \end{aligned}$$

Using the boundary conditions we see that $B = 0$ and $D = 0$. Some equivalent forms for g are

$$\begin{aligned} g(y) &= A\left(\mathrm{e}^{n\pi(1-y)} - \mathrm{e}^{-n\pi(1-y)}\right) \\ &= A\left(\mathrm{e}^{n\pi-n\pi y} - \mathrm{e}^{-n\pi+n\pi y}\right) \\ &= A\left(\mathrm{e}^{2n\pi-n\pi y} - \mathrm{e}^{n\pi y}\right) \\ &= A\left(\mathrm{e}^{-n\pi y} - \mathrm{e}^{-2n\pi+n\pi y}\right) \end{aligned}$$

Therefore $u_n = c_n\sinh(n\pi(1-y))\sin(n\pi x)$.
Now we seek a series solution $u = \sum c_n u_n$. The coefficients c_n are given by

$$c_n = \frac{2}{\sinh(n\pi)}\int_0^1 \phi(x)\sin(n\pi x)\,\mathrm{d}x$$

(b) The corresponding solutions are $v_n = d_n\sinh(n\pi y)\sin(n\pi x)$.

II. (a) Firstly $\Delta u_n = 0$ therefore, clearly, $\Delta\Delta u_n = 0$.
The Dirichlet boundary conditions are satisfied by I.(a).
Now we have

$$\frac{\partial^2 u_n}{\partial x^2} = f''(x)g(y) = \lambda u_n(x,y) \quad \text{and} \quad \frac{\partial^2 u_n}{\partial y^2} = f(x)g''(y) = -\lambda u_n(x,y)$$

with $\lambda \neq 0$. Therefore, at any boundary with zero Dirichlet boundary condition, we also have zero second derivative.
Equivalently, at any boundary with non-zero Dirichlet boundary condition, we also have non-zero second derivative.

(b) We have

$$g_{xx}(x,y) = (ax+by+c)w_{xx}(x,y) + 2aw_x(x,y)$$

and

$$g_{yy}(x,y) = (ax+by+c)w_{yy}(x,y) + 2bw_y(x,y)$$

Therefore

$$\Delta g = (ax + by + c)\Delta w + 2aw_x(x,y) + 2bw_y(x,y) = 2aw_x(x,y) + 2bw_y(x,y)$$

Alternatively, let $f(x,y) = ax + by + c$ then

$$\begin{aligned}\Delta g &= \nabla \cdot (\nabla g) = \nabla \cdot (f\nabla w + w\nabla f)\\ &= (\nabla f)\cdot(\nabla w) + f\underbrace{(\Delta w)}_{=0} + (\nabla w)\cdot(\nabla f) + w\underbrace{(\Delta f)}_{=0}\\ &= 2(\nabla f)\cdot(\nabla w)\end{aligned}$$

which, in general, is not equivalent to 0.
Finally

$$\begin{aligned}\Delta\Delta g &= \Delta(2(\nabla f)\cdot(\nabla w)) = \nabla\cdot\nabla(2(\nabla f)\cdot(\nabla w))\\ &= 2\nabla\cdot(\underbrace{\mathcal{H}(f)}_{=0}\nabla w + \mathcal{H}(w)\nabla f)\\ &= 2\nabla\cdot\begin{pmatrix} aw_{xx} + bw_{xy}\\ aw_{yx} + bw_{yy}\end{pmatrix}\\ &= 2aw_{xxx}(x,y) + 2bw_{xxy}(x,y) + 2aw_{xyy}(x,y) + 2bw_{yyy}(x,y)\\ &= 2a\frac{\partial}{\partial x}(\Delta w) + 2b\frac{\partial}{\partial y}(\Delta w) = 0\end{aligned}$$

(c) We look for solutions of the form $h_n(x,y) = \alpha_n y u_n(x,y) + \beta_n(1-y)v_n(x,y)$ where u_n, v_n are the solutions found in parts I.(a) and I.(b) respectively.
By II.(b) and the principle of superposition we can see that $\Delta\Delta h_n = 0$.
By construction, h_n satisfies both the conditions on the boundaries $x = 0$ and $x = 1$. Also, by construction, h_n satisfies the Dirichlet conditions on the boundaries $y = 0$ and $y = 1$. All that remains is to satisfy the second order derivatives on the top and bottom.
We calculate the derivatives of h_n

$$\frac{\partial h_n}{\partial y} = \alpha_n\left(y\frac{\partial u_n}{\partial y} + u_n\right) + \beta_n\left((1-y)\frac{\partial v_n}{\partial y} + v_n\right)$$

$$\frac{\partial^2 h_n}{\partial y^2} = \alpha_n\left(y\frac{\partial^2 u_n}{\partial y^2} + 2\frac{\partial u_n}{\partial y}\right) + \beta_n\left((1-y)\frac{\partial^2 v_n}{\partial y^2} - 2\frac{\partial v_n}{\partial y}\right)$$

Substituting the forms from parts I.(a) and I.(b) and evaluating at the boundaries gives

$$\frac{\partial^2 h_n}{\partial y^2}(x,0) = (-2\alpha_n n\pi\cosh(n\pi) - 2\beta_n n\pi)\sin(n\pi x)$$

$$\frac{\partial^2 h_n}{\partial y^2}(x,1) = (-2\alpha_n n\pi - 2\beta_n n\pi\cosh(n\pi))\sin(n\pi x)$$

The second derivative at the boundary $y = 1$ is zero so $\alpha_n + \beta_n\cosh(n\pi) = 0$. Using this to substitute for α_n at the boundary $y = 0$ we get

$$\frac{\partial^2 h_n}{\partial y^2}(x,0) = 2\beta_n n\pi\left(\cosh^2(n\pi) - 1\right)\sin(n\pi x) = 2\beta_n n\pi\sinh^2(n\pi)\sin(n\pi x)$$

Then

$$h_n(x,y) = -\beta_n \cosh(n\pi) y \sinh(n\pi(1-y)) \sin(n\pi x) + \beta_n (1-y) \sinh(n\pi y) \sin(n\pi x)$$

For the series solution to given boundary value problem the coefficients are given by

$$\beta_n = \frac{1}{2n\pi \sinh^2(n\pi)} \int_0^1 \psi(x) \sin(n\pi x)\, \mathrm{d}x$$