Differential Geometry of Framed Curves
Prof. John Maddocks
Summary of exercises — Giulio Corazza

Exercises for the Exam 2020-2021

1

1. **1.** Frenet frame of a (circular) helix: A basic exercise to get familiar with the standard notions of differential geometry for curves, namely tangent, principal normal and binormal making up the Frenet frame, and the Serret-Frenet equations which are a system of ODE with arc length as independent variable and with curvature and torsion as coefficients.
2. **2.** Change of parametrisation: What becomes of the Frenet frame, curvature and torsion under a change of parametrisation?
3. **3.** Formula for torsion when curve not parametrised by arc length.
4. **4.** Indicatrix of tangents of a planar curve: This exercise shows that the tantrix is itself a parametrised curve. The 3D version will be important later.
5. **5.** Evolute of a planar curve: This exercise asks you to prove some properties of the evolute of a planar curve. The evolute is a first example of an off-set curve which will appear a lot later on.
6. **6.** Construction of a planar curve with given curvature: This exercise shows that a planar curve is uniquely determined (up to a global rotation and translation) by its curvature expressed as a function of its arc-length.
7. **7.** General helices: This exercise gives equivalent characterisations of noncircular helices, for which the ratio of curvature and torsion is constant. They re-appear briefly in week 14 when considering equilibria of tubes.
8. **8.** A necessary and sufficient condition for a curve to lie on a sphere. Tantrices are one example of spherical curves, which are of interest when considering writhe.
9. **9.** A non-planar curve with zero torsion: This an example (rather esoteric) just to show that a curve with zero torsion need not be planar. If the curvature is strictly positive then a (smooth) curve is planar if and only of the torsion vanishes identically.

2

1. **1.** 3×3 skew matrices correspond to the action of a cross product in $\mathbb{R}^3$: This exercise gives a formula for skew matrices corresponding to the action of a cross product, and how things change under a linear mapping. Technical results that we exploit later.
2. **2.** Rotations in three dimensions: This exercise relates the properties of the eigenvalues and eigenvectors of a proper rotation matrix in three dimensions to the geometrical interpretation of the rotation, and the resulting version of an Euler–Rodrigues formula (ie a formula for the rotation matirx in terms of a parametrisation).
3. **3.** Interpretation of rotations applied on the left and on the right.
4. **4.** General form of the Darboux vector of an adapted framing of a curve: This exercise gives a general method for constructing any adapted framing of a given curve. The freedom is a single arbitrary scalar function appearing in the tangential component of the Darboux vector of the framing. Will be used late for constructing various interesting framings.

5. Frenet–Serret equations in $\mathbb{R}^n$ (optional, not examinable). For general educational purposes.

3

1. Composition of Darboux vectors: This exercise gives a formula relating the Darboux vectors of two framings of the same curve, including simplifications for the case in which both framings are adapted, and the further case when one framing is the Frenet frame.

2. Factorisation of curves in $SE(3)$: A framed curve in $\mathbb{R}^3$ can be regarded as a curve of matrices in $SE(3)$. This and the next exercise gives formulas for offsets of framed curves $\mathbb{R}^3$ as matrix factorising curves in $SE(3)$.

3. Offset of a curve in $\mathbb{R}^3$: This exercise defines an offset of a curve in $\mathbb{R}^3$ and asks you to give an adapted framing of the offset curve.

4

1. Link as signed area: Link can be defined as the signed area of the projection of the zodiacus onto the unit sphere. Central material.

2. Homotopy invariance of Link: This exercise proves the homotopy invariance of Link when two curves are (smoothly) deformed without intersecting each other (although intersecting themselves changes nothing).

3. The Hopf link: This exercise uses homotopy invariance to give a construction for explicitly computing the Link of the Hopf link as a double integral that can be evaluated. Also example of another slightly more complicated link.

5

1. Unknotted Curves and the Whitehead link: This is an example of a (2 component) link with vanishing Link that cannot be separated by a physical homotopy that doesn't pass either curve through itself. Thus two physically linked closed tubes can have centrelines with mathematical Link zero.

2. Non-smoothness of the link surface: This exercise shows that, in general, the Link surface is not smooth and gives some characterisations of the non-smooth points clarifying what can happen.

6

1. Curves on a sphere: This exercise illustrates the surface framing in the special case of a curve lying on a sphere, and relates the surface framing to the Frenet frame of the curve. The tantrix for 3D curves is also revisited.

2. Properties of the tangent indicatrix.

3. Generalised helices (revisited): This shows that the tantrix of a generalised helix is an arc of a circle.

4. Tangent indicatrix of a closed curve: This exercise gives a condition for a closed curve lying on the unit sphere to be the tantrix of a closed curve (the origin must lie in the convex hull of the curve).

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1. **1.** Doubly critical self-distance: This shows that chords of closest approach are orthogonal to the tangents of the curve at both ends of the chord. Simple and standard but important to know later on.
2. **2.** Evenness of writhe.
3. **3.** Invariance of writhe under translations, rotation and dilations, change of sign under reflexions.
4. **4.** Writhe of a planar curve: The Writhe of a planar curve is shown to be 0 (the opposite is not true in general, ie there are curves with zero writhe that are not planar!).
5. **5.** The writhe integrand vanishing pointwise is sufficient for the associated curve to be planar.
6. **6.** Writhe in terms of point tangent circles: another geometrical interpretation of the Writhe integrand is in terms of circles that are tangent at one point of the curve and intersect at another. The minimal radius over all such circles gives the global radius of curvature of the (closed) curve, which is a rather specialised topic.
7. **7.** Writhe of a (circular) helix: This exercise is an example of a curve (one of rather few) for which the Writhe can be calculated directly at least asymptotically via its integral definition. All a bit technical.

8

1. **1.** Proof of the C–F–W theorem: This sheet works through the steps of the proof of the C–F–W theorem appearing in the polycopie.

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1. **1.** Link and writhe for curves on spheres: The Link for several framings of a general curve on a sphere is calculated as well as the Writhe, in particular the Writhe of a curve on the surface of a sphere is always zero yet the curve need not be planar.
2. **2.** Writhe of an offset curve: Calculate the writhe of an offset curve by using the adapted framing of the offset curve, the symmetry of Link and the C–F–W theorem when exchanging roles of curve and offset curve.
3. **3.** Some provocative questions about potentially applying C–F–W for open curves.

10

1. **1.** Framings of closed curves: This exercise gets you to work with the Writhe framing and register angles as well as giving conditions for framings to be closed using the C–F–W theorem.
2. **2.** Writhe of a figure of 8 curve: This is another example of a curve for which the writhe can be approximated directly, this curve also demonstrates the jump in writhe as a curve passes through itself.
3. **3.** Writhe framing of a trefoil — just an example.

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1. Revisiting Euler-Rodrigues formula (proof).

2. Cayley transforms: Properties of the Cayley transform between matrices, particularly in three dimensions, and particularly between skew matrices and proper rotation matrices, and finally Cayley transforms in $SE(3)$.

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1. Euler–Rodrigues parameters: Properties of rotations expressed using quaternions aka Euler or Euler-Rodrigues parameters (NOT to be confused with Euler angles) Includes versions of the Euler–Rodrigues formula appropriate for Euler–Rodrigues parameters (ie how to get the rotation matrix from the parametrisation). Factorisation of a rotation matrix using Euler–Rodrigues parameters.

2. Composition rule for Euler-Rodrigues parameters: a purely algebraic proof based on factorisation of rotation matrices.

3. Composition rule for Cayley vectors based on the one for Euler-Rodrigues parameters through stereographic projection.

4. Darboux vector in terms of Euler-Rodrigues parameters.

5. Euler–Rodrigues parameters of particular curves in $SO(3)$: Euler–Rodrigues parameters for the Frenet frame of a helix and a multiply covered circle.

List of Exercises 2019-2020-2021 (students do not need to read the following)

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2. Composition rule for Cayley vectors based on the one for Euler-Rodrigues parameters through stereographic projection. Moved to Session 12 in academic year 2020-2021

3. Darboux vector in terms of Euler-Rodrigues parameters. Moved to Session 12 in academic year 2020-2021

4. Bertrand mates. Not done in academic year 2020-2021

5. Points of closest approach for double helices. The important observation is that if the pitch is short enough compared to the radius of the helical structure, then the chord of closest approach is NOT along a diameter. Not done in academic year 2020-2021

14

1. Ideal strings under frictionless contact: immediate results from the force balance equation. Not done in academic year 2020-2021

2. Equilibria of two strings with a single contact line: force balance for two strings in single contact. The total force acting across the two cross-sections is constant. There exist equilibria pairs of ideal strings with circular helical centrelines. The centrelines of a pair of ideal strings in equilibria form generalised helices. The centrelines of a pair of ideal strings in equilibrium are Bertrand mates. All equilibria are circular helices as final result. Not done in academic year 2020-2021