Differential Geometry of Framed Curves
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Session 2: Solutions T. Lessinnes

1. Given any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, we can compute the following:

$$\begin{aligned}
\boldsymbol{v}^T\left(|M|M^{-T}\mathrm{Sk}(\boldsymbol{u})M^{-1}\right)\boldsymbol{w} &= |M|\boldsymbol{v}^T\left(M^{-T}\mathrm{Sk}(\boldsymbol{u})M^{-1}\right)\boldsymbol{w}\\
&= |M|(M^{-1}\boldsymbol{v})^T\mathrm{Sk}(\boldsymbol{u})M^{-1}\boldsymbol{w}\\
&= |M|(M^{-1}\boldsymbol{v})\cdot(u\wedge M^{-1}\boldsymbol{w})\\
&= |M||M^{-1}\boldsymbol{v}\ \boldsymbol{u}\ M^{-1}\boldsymbol{w}|\\
&= |M||M^{-1}||\boldsymbol{v}M\boldsymbol{u}\boldsymbol{w}|\\
&= \boldsymbol{v}\cdot(M\boldsymbol{u}\wedge\boldsymbol{w})\\
&= \boldsymbol{v}^T\mathrm{Sk}(M\boldsymbol{u})\boldsymbol{w}
\end{aligned}$$

As this is true for any $\boldsymbol{v}$ and $\boldsymbol{w}$, we can conclude that $\mathrm{Sk}(M\boldsymbol{u}) = |M|M^{-T}\mathrm{Sk}(\boldsymbol{u})M^{-1}$.

When $M \in SO(3)$, we have $\mathrm{Sk}(M\boldsymbol{u}) = M\mathrm{Sk}(\boldsymbol{u})M^{-1}$ since $|M| = 1$ and $M^{-T} = M$.

The first equality $R' = \mathrm{Sk}(\tilde{\mathbf{u}})R$ was shown in lectures. We can substitute the Darboux vector $\tilde{\mathbf{u}}$ for $M\boldsymbol{u}$ in this formula and $\mathbf{v}$ for $\boldsymbol{u}$ and rearrange to obtain the second equality.

2. Rotations in three dimensions.

a) From the properties of the scalar product, $\forall \mathbf{w} \in \mathbb{R}^3$

$$\|Q\mathbf{w}\|^2 = Q\mathbf{w}\cdot Q\mathbf{w} = \mathbf{w}\cdot Q^TQ\mathbf{w} = \|\mathbf{w}\|^2 \tag{1}$$

where we have used the fact that Q is a rotation matrix, i.e. $Q^TQ = I$. If now λ is an eigenvalue for Q, let $\mathbf{w}$ be the corresponding eigenvector

$$\|Q\mathbf{w}\| = \|\lambda\mathbf{w}\| = |\lambda|\|\mathbf{w}\| \tag{2}$$

but then from equation (1) we obtain

$$|\lambda| = 1 \tag{3}$$

and therefore λ lies on the unit circle in the complex plane.

b) Note that the complex conjugate $\overline{\lambda}$ is an eigenvalue of Q (with corresponding eigenvector $\overline{\mathbf{w}}$) whenever λ is an eigenvalue of Q (with corresponding eigenvector $\mathbf{w}$). This follows from $\overline{Q} = Q$. Explicitly,

$$Q\mathbf{w} = \lambda\mathbf{w} \ \Rightarrow\ Q\overline{\mathbf{w}} = \overline{Q\mathbf{w}} = \overline{\lambda\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}}.$$

As Q has exactly 3 eigenvalues (counted according to multiplicity), we obtain from (3) that the set of eigenvalues of Q is

$$\mathrm{eig}(Q) = \{e^{ix}, e^{-ix}, 1\} \quad \text{or} \quad \mathrm{eig}(Q) = \{e^{ix}, e^{-ix}, -1\}$$

for some $x \in [0, \pi]$. But for $\mathrm{eig}(Q) = \{e^{ix}, e^{-ix}, -1\}$ we get[1] $\det(Q) = -1$, so that we must have

$$\mathrm{eig}(Q) = \{e^{ix}, e^{-ix}, 1\} \quad \text{for } x \in [0, \pi]. \tag{4}$$

[1] Here, we make use of

$$\det(A) = \prod_{i=1}^{n}\lambda_i,$$

for any matrix $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ repeated according to (algebraic) multiplicity.

This also shows that $\lambda = 1$ cannot have multiplicity 2, but only 1 or 3. If $\lambda = 1$ has multiplicity 3, then $Q = \mathrm{Id}$, which therefore is the only case where there can be a non-unique axis of rotation.

For any eigenvalue λ of Q, the inverse λ^{-1} will be an eigenvalue of $Q^{-1} = Q^T$ corresponding to the same eigenvector. Therefore, $Q\mathbf{w} = \mathbf{w}$ immediately implies that $\mathbf{w}$ is an element of the nullspace of $S = (Q - Q^T)$, i.e. $S\mathbf{w} = 0$. On the other hand if $\mathbf{z}$ is the unique axial vector of the skew matrix $\mathbf{S}$, then

$$0 = S\mathbf{w} = \mathbf{z} \times \mathbf{w}, \tag{5}$$

so that $\mathbf{z}$ and $\mathbf{w}$ are parallel, i.e. the axial vector of the skew matrix is parallel to the axis of rotation of Q.

c) If $\mathbf{v}$ is any vector orthogonal to the axis of rotation $\mathbf{w}$, then

$$\begin{aligned} Q\mathbf{v} \cdot \mathbf{w} &= \mathbf{v} \cdot Q^T\mathbf{w} \\ &= \mathbf{v} \cdot \mathbf{w} \\ &= 0 \end{aligned} \tag{6}$$

Furthermore from (1) we can conclude that if $\mathbf{v}$ is a unit vector, then so is $Q\mathbf{v}$.
On the one hand, the angle θ between $Q\mathbf{v}$ and $\mathbf{v}$ obeys

$$Q\mathbf{v} \cdot \mathbf{v} = \|Q\mathbf{v}\| \, \|\mathbf{v}\| \cos\theta \overset{(1)}{=} \cos\theta. \tag{7}$$

On the other hand the trace of a matrix is the sum of its eigenvalues[2]. Then (4) implies

$$\mathrm{tr}(Q) = 1 + 2\cos x. \tag{8}$$

In the special cases $x = 0$ and $x = \pi$, the proof is immediate since $\cos x = \pm 1$ and $\mathbf{v}$ is itself an eigenvector and $Q\mathbf{v} = \pm\,\mathbf{v}$ so that $\cos\theta = \pm 1$.
Otherwise, $x \in (0, \pi)$ and we will show that $\cos x = \cos\theta$ holds in general. Along the way, we will need a number of properties regarding the complex eigenvectors of Q. We list them hereunder with their proof. Once and for all, $x \in (0, \pi)$ and $\mathbf{z} \in \mathbb{C}^3$ is a norm 1 eigenvector of Q corresponding to the eigenvalue $e^{i\,x}$ from question 1.2. The hermitian product between complex vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{C}^3$ is noted $\langle \mathbf{a}, \mathbf{b} \rangle$. The scalar product between real vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$ is noted $\mathbf{x} \cdot \mathbf{y}$.

Propostion 1. *The conjugate $\overline{\mathbf{z}}$ of $\mathbf{z}$ is an eigenvector of Q with eigenvalue $e^{-i\,x}$.*

Proof. See exercise 1.2. □

Propostion 2. *If $x \in (0, \pi)$, the eigenvector $\mathbf{z}$ is such that $\langle \mathbf{z}, \overline{\mathbf{z}} \rangle = 0$.*

Proof. Compute

$$e^{-i\,x} \langle \mathbf{z}, \overline{\mathbf{z}} \rangle = \langle e^{i\,x}\,\mathbf{z}, \overline{\mathbf{z}} \rangle = \langle Q\,\mathbf{z}, \overline{\mathbf{z}} \rangle = \langle \mathbf{z}, Q^T\,\overline{\mathbf{z}} \rangle = \langle \mathbf{z}, e^{i\,x}\,\overline{\mathbf{z}} \rangle = e^{i\,x} \langle \mathbf{z}, \overline{\mathbf{z}} \rangle. \tag{9}$$

And whenever $x \in (0, \pi)$, Eq. (9) implies $\langle \mathbf{z}, \overline{\mathbf{z}} \rangle = 0$. □

[2] This comes from the fact that if $A \in \mathbb{R}^{n\times n}$ there exists $P \in SU(n)$ such that $P^{-1}AP$ is diagonal. Then $\mathrm{tr}(PAP^{-1})$ is the sum of the eigenvalues of A. But the trace is invariant under cyclic perturbations since $\mathrm{tr}(ABC) = A_{ij}B_{jk}C_{ki} = B_{jk}C_{ki}A_{ij} = \mathrm{tr}(BCA)$. So $\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(APP^{-1}) = \mathrm{tr}\,A$.

Propostion 3. *Let* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$ *be respectively the real and imaginary part of the eigenvector* $\mathbf{z} = \mathbf{x} + i\,\mathbf{y}$. *If* $x \in (0, \pi)$, *then* $\mathbf{x}$ *and* $\mathbf{y}$ *are orthogonal:* $\mathbf{x} \cdot \mathbf{y} = 0$. *Furthermore,*

$$\mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y} = 1/2. \tag{10}$$

Proof. On the one hand, from Proposition 2 we have

$$\begin{aligned} 0 = \langle \mathbf{z}, \overline{\mathbf{z}} \rangle = \langle \mathbf{x} + i\,\mathbf{y}, \mathbf{x} - i\,\mathbf{y} \rangle = \Big(\mathbf{x} \cdot \mathbf{x} - \mathbf{y} \cdot \mathbf{y}\Big) - 2\,i\,\Big(\mathbf{x} \cdot \mathbf{y}\Big), \\ \Rightarrow \mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y}, \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{y} = 0. \end{aligned} \tag{11}$$

On the other hand, $\mathbf{z}$ is of norm 1 so that

$$\langle \mathbf{z}, \mathbf{z} \rangle = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} = 1 \overset{(11)}{\Rightarrow} \mathbf{x} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{y} = \frac{1}{2}.$$

□

Now we are ready to proceed with the exercise. First we show that there exists a complex number $a \in \mathbb{C}$ such that $\mathbf{v} = a\,\mathbf{z} + \overline{a}\,\overline{\mathbf{z}}$. Indeed, since $\mathbf{z}$ and $\overline{\mathbf{z}}$ span the space orthogonal to $\mathbf{w}$ in $\mathbb{C}^3$, there exist complex numbers a, $b \in \mathbb{C}$ such that $\mathbf{v} = a\,\mathbf{z} + b\,\overline{\mathbf{z}}$. But then with $\mathbf{x}$ and $\mathbf{y}$ defined as in Proposition 3, the fact that $\mathbf{v}$ is a real vector implies

$$\big(\mathrm{Im}(a) + \mathrm{Im}(b)\big)\,\mathbf{x} + \big(\mathrm{Re}(a) - \mathrm{Re}(b)\big)\,\mathbf{y} = \mathbf{0} \Rightarrow b = \overline{a}, \tag{12}$$

where the implication is due to Proposition 3: $\mathbf{x}$ and $\mathbf{y}$ are orthogonal and of non zero norm so that both brackets must vanish independently.
Then the fact that $\mathbf{v}$ is a unit vector implies that $|a|^2 = 1/2$. Indeed, $1 = \langle \mathbf{v}, \mathbf{v} \rangle = \langle a\mathbf{z} + \overline{a\mathbf{z}}, a\mathbf{z} + \overline{a\mathbf{z}} \rangle = |a|^2 + |\overline{a}|^2 = 2|a|^2$, where the third equality is due to Proposition 2.
Finally, simply compute

$$\begin{aligned} \cos\theta &\overset{(7)}{=} Q\,\mathbf{v} \cdot \mathbf{v} = \langle Q\,(a\,\mathbf{z} + \overline{a}\,\overline{\mathbf{z}}), a\,\mathbf{z} + \overline{a}\,\overline{\mathbf{z}} \rangle = \langle a\,e^{i\,x}\,\mathbf{z} + \overline{a}\,e^{-i\,x}\,\overline{\mathbf{z}}, a\,\mathbf{z} + \overline{a}\,\overline{\mathbf{z}} \rangle, \\ &\overset{Prop.(3)}{=} |a|^2(e^{-i\,x} + e^{i\,x}) = \frac{e^{-i\,x} + e^{i\,x}}{2} = \cos x. \end{aligned} \tag{13}$$

d) For any $\mathbf{v} \in \mathbb{R}^3$, we have

$$\mathbf{n}^\times\,\mathbf{n}^\times\,\mathbf{v} = \mathbf{n}^\times\,(\mathbf{n}^\times\,\mathbf{v}) = \mathbf{n} \times (\mathbf{n} \times \mathbf{v}) = (\mathbf{n} \cdot \mathbf{v})\mathbf{n} - \mathbf{v} = (\mathbf{n} \otimes \mathbf{n} - \mathrm{Id})\,\mathbf{v}. \tag{14}$$

Since (14) holds for all $\mathbf{v}$ we have $\mathbf{n} \otimes \mathbf{n} = \mathrm{Id} + \mathbf{n}^\times\,\mathbf{n}^\times$ which can be substituted in Eq. (5) from the question sheet.

3. General form of the Darboux vector of an adapted framing of a given curve.

a) Compute

$$\frac{d}{ds}||\boldsymbol{\xi}||^2 = 2\boldsymbol{\xi}' \cdot \boldsymbol{\xi} = 2((\mathsf{u}_3\boldsymbol{r}' + \boldsymbol{r}' \wedge \boldsymbol{r}'') \wedge \xi) \cdot \xi = 0$$

Therefore $||\boldsymbol{\xi}(s)||^2$ is constant. From the initial value we get $||\boldsymbol{\xi}(s)||^2 = 1$ for all s.

b) Now we derive

$$\begin{aligned} \frac{d}{ds}(\boldsymbol{\xi} \cdot \mathbf{r}') &= ((\mathsf{u}_3(s)\mathbf{r}' + \mathbf{r}' \times \mathbf{r}'') \times \boldsymbol{\xi}) \cdot \mathbf{r}' + \boldsymbol{\xi} \cdot \mathbf{r}'' \\ &= ((\mathbf{r}' \times \mathbf{r}'') \times \boldsymbol{\xi}) \cdot \mathbf{r}' + \boldsymbol{\xi} \cdot \mathbf{r}'' \\ &= ((\mathbf{r}' \cdot \boldsymbol{\xi})\mathbf{r}'' - (\mathbf{r}'' \cdot \boldsymbol{\xi})\mathbf{r}') \cdot \mathbf{r}' + \boldsymbol{\xi} \cdot \mathbf{r}'' = (\mathbf{r}' \cdot \boldsymbol{\xi})(\mathbf{r}'' \cdot \mathbf{r}') = 0. \end{aligned}$$

The vector $\boldsymbol{\xi}(s)$ is perpendicular to $\mathbf{r}'(s)$ for all s.

c) Now by picking an initial value of $\boldsymbol{\xi}(0)$ we have an orthonormal frame $(\boldsymbol{\xi}, \mathbf{r}' \times \boldsymbol{\xi}, \mathbf{r}')$ of $\mathbf{r}(s)$.
The Darboux vector is $\mathsf{u}_3(s)\mathbf{r}' + \mathbf{r}' \times \mathbf{r}''$. This is the general form of a Darboux vector. To verify we calculate

$$\begin{aligned}
(\mathsf{u}_3\mathbf{r}' + \mathbf{r}' \times \mathbf{r}'') \times \mathbf{r}' &= (\mathbf{r}' \times \mathbf{r}'') \times \mathbf{r}' = \mathbf{r}'' \\
(\mathsf{u}_3\mathbf{r}' + \mathbf{r}' \times \mathbf{r}'') \times (\mathbf{r}' \times \boldsymbol{\xi}) &= -\mathsf{u}_3\boldsymbol{\xi} + (\mathbf{r}' \times \mathbf{r}'') \times (\mathbf{r}' \times \boldsymbol{\xi}) \\
&= -\mathsf{u}_3\boldsymbol{\xi} + \big[(\mathbf{r}' \times \mathbf{r}'') \cdot \boldsymbol{\xi}\big]\mathbf{r}' - \big[(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'\big]\boldsymbol{\xi} \\
&= -\mathsf{u}_3\boldsymbol{\xi} + \big[(\mathbf{r}'' \times \boldsymbol{\xi}) \cdot \mathbf{r}'\big]\mathbf{r}' \\
&= -\mathsf{u}_3\boldsymbol{\xi} + \mathbf{r}'' \times \boldsymbol{\xi} = (\mathbf{r}' \times \boldsymbol{\xi})'.
\end{aligned}$$

Note that the before last equality, obvious if $\mathbf{r}'' \times \boldsymbol{\xi} = \mathbf{0}$, comes from the fact that both $\mathbf{r}''$ and $\boldsymbol{\xi}$ are perpendicular to the unit vector $\mathbf{r}'$. Accordingly if $\mathbf{r}'' \times \boldsymbol{\xi} \neq \mathbf{0}$, then $\mathbf{r}' = \pm(\mathbf{r}'' \times \boldsymbol{\xi})/||\mathbf{r}'' \times \boldsymbol{\xi}||$ which can be substituted to conclude.

d) This is the fact that $\mathsf{u}_3(s)\mathbf{r}' + \mathbf{r}' \times \mathbf{r}''$ is the Darboux vector corresponding to the frame $(\boldsymbol{\xi}, \mathbf{r}' \times \boldsymbol{\xi}, \mathbf{r}')$. The concrete calculation is the same as in (c).

e) If $\mathbf{r}'' \neq 0$ then $\mathbf{r}' \times \mathbf{r}'' = \kappa\mathbf{b}$ and therefore the Darboux vector is

$$\mathsf{u}_3\mathbf{t} + \kappa\mathbf{b}$$

where $\mathbf{t} = \mathbf{r}'$, $\boldsymbol{n} = \frac{\mathbf{r}''}{||\mathbf{r}''||}, \kappa = ||\mathbf{r}''||$, $\mathbf{b} = \mathbf{t} \times \boldsymbol{n}$.

f) We simply compute

$$\begin{aligned}
(\tau\boldsymbol{r}' + \boldsymbol{r}' \wedge \boldsymbol{r}'') \wedge \boldsymbol{n} &= (\tau\boldsymbol{t} + \kappa\boldsymbol{b}) \wedge \boldsymbol{n} \\
&= \tau\boldsymbol{b} - \kappa\boldsymbol{t} \\
&= \boldsymbol{n}'
\end{aligned}$$

4. Frenet-Serret equations in $\mathbb{R}^n$.

Define the basis recursively: define $\widetilde{\mathbf{n}_{j+1}} = \mathbf{n}_j' + \kappa_j\,\mathbf{n}_{j-1}$. Then $\kappa_{j+1} = ||\widetilde{\mathbf{n}_{j+1}}||$ and $\mathbf{n}_{j+1} = \widetilde{\mathbf{n}_{j+1}}/\kappa_{j+1}$. Note that by construction, $\widetilde{\mathbf{n}_{j+1}}$ is a linear combination of the first $j+1$ derivatives of $\mathbf{r}$ and that therefore it can not vanish. The equation (3) in the question sheet is then verified by construction since we have

$$\mathbf{n}_j' = \kappa_{j+1}\,\mathbf{n}_{j+1} - \kappa_j\,\mathbf{n}_{j-1}. \tag{15}$$

All we need to check is that with the convention $\mathbf{t} = \mathbf{n}_0$, we have $\mathbf{n}_i \cdot \mathbf{n}_k = \delta_{ik}$ for all i and k. We do so recursively. It is blatantly true when $i \leq 1$ and $k \leq 1$. Then assume that it is true for all i and k strictly smaller than j. Then for any $i < j$, consider

$$\begin{aligned}
\mathbf{n}_i \cdot \mathbf{n}_j &= \mathbf{n}_i \cdot (\mathbf{n}_{j-1}' + \kappa_{j-1}\,\mathbf{n}_{j-2})/\kappa_j, \\
&= \frac{1}{\kappa_j}\mathbf{n}_i \cdot \mathbf{n}_{j-1}' + \frac{\kappa_{j-1}}{\kappa_j}\delta_{i(j-2)}, \\
&= \frac{-1}{\kappa_j}\mathbf{n}_i' \cdot \mathbf{n}_{j-1} + \frac{\kappa_{j-1}}{\kappa_j}\delta_{i(j-2)}, \\
&= \frac{-1}{\kappa_j}(\kappa_{i+1}\mathbf{n}_{i+1} - \kappa_i\,\mathbf{n}_{i-1}) \cdot \mathbf{n}_{j-1} + \frac{\kappa_{j-1}}{\kappa_j}\delta_{i(j-2)} = 0.
\end{aligned} \tag{16}$$

Finally, if the basis is reordered as $\{\mathbf{n}_1, \cdots, \mathbf{n}_{n-1}\}$, equation (3) in the question sheet becomes

$$\begin{pmatrix}\mathbf{n}_1 & \cdots & \mathbf{n}_n & \mathbf{t}\end{pmatrix}' = \begin{pmatrix}\mathbf{n}_1 & \cdots & \mathbf{n}_n & \mathbf{t}\end{pmatrix} \begin{pmatrix} 0 & -\kappa_2 & \dots & 0 & 0 & 0 & \kappa_1 \\ \kappa_2 & 0 & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & -\kappa_{n-2} & 0 & 0 \\ 0 & 0 & \cdots & \kappa_{n-2} & 0 & -\kappa_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \kappa_{n-1} & 0 & 0 \\ -\kappa_1 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$